CS/Math 240: Intro to Discrete Math 2/10/2011

Solutions to Homework 2

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Problem 1

(a) First consider the case where the sets A and B are disjoint. In that case the number of elements in the union $A \cup B$ is simply the sum of the number of elements in A and the number of elements in B: $|A \cup B| = |A| + |B|$. If A and B overlap, then the latter formula does not hold because we are counting the elements in the intersection $A \cap B$ twice. Compensating for that leads to the given formula: $|A \cup B| = |A| + |B| - |A \cap B|$.

Let's now prove that formula rigorously. In order to do so, we break up $A \cup B$ into several disjoint parts. Once we've done that, we can apply our simple rule that the cardinality of a disjoint union is the sum of the cardinalities. As can be seen in figure 1, $A \cup B$ can be expressed

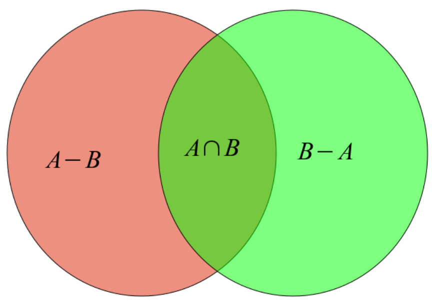


Figure 1: Union of Sets

as union of three disjoint sets:

- the set of elements present only in A. This set can be expressed as the set difference $A - B$.
- the set of elements present only in B. This set can be expressed as the set difference $B - A$.
- the set of elements present in both A and B. This is the set $A \cap B$.

Therefore, we have

$$A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$$

Equating the cardinalities of the two sides of the above equation, and using the simple rule for the cardinality of a disjoint union, we have

$$|A \cup B| = |A - B| + |B - A| + |A \cap B| \quad (1)$$

Since A is the disjoint union of $A - B$ and $A \cap B$, we also have that $|A| = |A - B| + |A \cap B|$, so $|A - B| = |A| - |A \cap B|$. Similarly, we have that $|B - A| = |B| - |A \cap B|$. Plugging these equations into the right-hand side of (1), we obtain

$$\begin{aligned} |A \cup B| &= |A| - |A \cap B| + |B| - |A \cap B| + |A \cap B| \\ &= |A| + |B| - |A \cap B| \end{aligned}$$

□

(b) We will use the equation from 1(a) to derive the equation for cardinality of union of three sets. We can view $A \cup B \cup C$ as $A \cup (B \cup C)$. Now we can use the equation derived in 1(a) on sets A and $B \cup C$. We get the following.

$$\begin{aligned} |A \cup B \cup C| &= |A \cup (B \cup C)| \\ &= |A| + |B \cup C| - |A \cap (B \cup C)| \end{aligned}$$

The term $|B \cup C|$ can be further broken down using 1(a), namely $|B \cup C| = |B| + |C| - |B \cap C|$. Combined we get

$$|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |A \cap (B \cup C)| \tag{2}$$

Now, we are left with the term $|A \cap (B \cup C)|$ that needs to be simplified. To do that, recall one of the propositions proved in Lecture 4.

Proposition. *Let A, B, and C be sets. Then*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

The proposition implies that $|A \cap (B \cup C)| = |(A \cap B) \cup (A \cap C)|$. By applying 1(a) to the right-hand side and using the fact that $(A \cap B) \cap (A \cap C) = A \cap B \cap C$, we get that

$$|A \cap (B \cup C)| = |A \cap B| + |A \cap C| - |A \cap B \cap C|$$

Plugging the last equation into (2) gives

$$|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|.$$

Problem 2

In order to prove the equality of two sets, we will show that they are subsets of each other; $[(A \subseteq B) \land (B \subseteq A)] \Leftrightarrow (A = B)$. To show that two sets are not equal, it is sufficient to show a member of one set that does not belong to the other.

(a) The two sets are equal for all choices of a, b. To show this, we will take the steps described above.

- $M_a \cap M_b \subseteq M_{lcm(a,b)}$

  The smallest member of $M_a \cap M_b$ is by definition $lcm(a,b)$. Imagine a member $C \in M_a \cap M_b$ that is not a multiple of $lcm(a,b)$. Then there must a factor of $lcm(a,b)$ that is not a factor of C. However, every factor of $lcm(a,b)$ is composed entirely either of factors of a, factors of b, or both [otherwise, it would not be as small as possible]. So there is a factor of either a or b that is not a factor of C; so C cannot be in $M_a \cap M_b$, a contradiction. So all members of $M_a \cap M_b \subseteq M_{lcm(a,b)}$.

  An alternate way to come to this contradiction is to decompose C as $k \cdot lcm(a,b) + r$ where k is some positive integer and r is a remainder $0 < r < lcm(a,b)$ [$r \neq 0$ follows from C not being a multiple of $lcm(a,b)$]. Since both C and $k \cdot lcm(a,b)$ are common multiples of a and b, r must be as well; but then r is a common multiple less than the least common multiple. This is a contradiction. Hence, there can be no $C \in M_a \cap M_b$ that is not a multiple of $lcm(a,b)$.

- $M_{lcm(a,b)} \subseteq M_a \cap M_b$

  By definition, $lcm(a,b)$ is a common multiple of a and b. Any multiple of this number will clearly be a common multiple of a and b as well. So having proved both directions, our proof is complete.

(b) The two sets are not equal; a satisfactory counterexample is $a = 2, b = 3$. $gcd(2,3) = 1$, which cannot be a multiple of either 2 or 3. So $M_{gcd(2,3)} \not\subseteq M_2 \cup M_3$.

Problem 3

We start by proving a lemma which we use throughout the solution to this problem.

Lemma 1. *If S and T are countable sets, then so is $S \cup T$.*

Proof. We prove this lemma directly.

Let $s_1, s_2, s_3, \ldots$ and $t_1, t_2, t_3, \ldots$ be enumerations of the sets S and T, respectively. Then consider the list formed by interleaving the enumerations of S and T above, that is, the list $s_1, t_1, s_2, t_2, s_3, t_3, \ldots$.

We claim that this is a list of all elements of $S \cup T$. We give a proof by cases.

Suppose $x \in S \cup T$. Then either $x \in S$ or $x \in T$.

Case 1: If $x \in S$, there is some integer i such that $x = s_i$ in our enumeration of S. Then x is at position $2i - 1$ of the list we constructred.

Case 2: Similarly, if $x \in T$, there is some integer j such that $x = t_j$ in our enumeration of T. In that case, x is at position $2j$ in the list we constructed.

Since x falls in one of our two cases, we have shown that all elements of $S \cup T$ appear somewhere in our list. Moreover, since we constructed our list only from elements of S and elements of T, there are no elements from outside of $S \cup T$ in our list, which means that we indeed have a list of all elements of $S \cup T$.

To get an enumeration of all elements of $S \cup T$ from our list, it suffices to remove all duplicates from our list.

It follows that $S \cup T$ is countable. □

We will also make use of the next result that follows from Lemma 1

Corollary 1. *Suppose U is an uncountable set. Also assume S and T are subsets of U where S is countable and such that $S \cup T = U$. Then T is uncountable.*

Proof. We argue by contradiction.

Assume that T is countable. Since S is also countable, it follows by Lemma 1 that $S \cup T$ is countable. But $S \cup T = U$, so U is countable. This contradicts the fact that U is uncountable, so the assumption that T is countable is incorrect.

Therefore, T is uncountable. □

Also recall that the set $\mathbb{R}$ of all real numbers is uncountable. As we develop the solution to this problem, we will also show that all of the following subsets of $\mathbb{R}$ are uncountable. Notice the zero subscript in the names of the first two sets.

$$\mathbb{R}_0^+ = \{x \in \mathbb{R} \mid x \geq 0\}$$
$$\mathbb{R}_0^- = \{x \in \mathbb{R} \mid x \leq 0\}$$
$$\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$$
$$\mathbb{R}^- = \{x \in \mathbb{R} \mid x < 0\}$$

Part a

We now use Lemma 1 to show that there are no uncountable sets A and B such that both $A - B$ and $A \cap B$ are countable. That is, we show that if A and B are not countable, then $A - B$ is not countable or $A \cap B$ is not countable.

We prove the contrapositive. We show that if $A - B$ and $A \cap B$ are both countable, then A is countable or B is countable.

Suppose that $A - B$ and $A \cap B$ are both countable. Notice that $A = (A - B) \cup (A \cap B)$. Then A is countable by Lemma 1 with $S = A - B$ and $T = A \cap B$. A fortiori, A or B are countable.

Part b

We first show that the sets $\mathbb{R}_0^+ = \{x \in \mathbb{R} \mid x \geq 0\}$ and $\mathbb{R}_0^- = \{x \in \mathbb{R} \mid x \leq 0\}$ are both uncountable.

From a purported enumeration of $\mathbb{R}_0^+$ we can obtain an enumeration of $\mathbb{R}_0^-$ by negating every element in the enumeration. The converse also holds. So, either both $\mathbb{R}_0^+$ and $\mathbb{R}_0^-$ are countable, or they are both uncountable. Notice that $\mathbb{R} = \mathbb{R}_0^+ \cup \mathbb{R}_0^-$, so if $\mathbb{R}_0^+$ and $\mathbb{R}_0^-$ were both countable, $\mathbb{R}$ would be countable too by Lemma 1, which is a contradiction. Thus, both $\mathbb{R}_0^+$ and $\mathbb{R}_0^-$ are uncountable.

Let $A = \mathbb{R}_0^+$ and $B = \mathbb{R}_0^-$. The intersection of A and B is the set $\{0\}$, which is finite, and, therefore, countable. It follows by Corollary 1 with $U = A$, $S = \{0\}$, and $T = A - B = \{x \in \mathbb{R} \mid x > 0\}$ that $A - B$ is uncountable. This also makes sense intuitively because taking away one element from an uncountable set should not suddenly make it countable.

Also notice that this shows that $\mathbb{R}^+$ is uncountable because $\mathbb{R}^+ = A - B$. By the above negation argument, this means that the set $\mathbb{R}^-$ is uncountable, too.

Part c

Let $A = \mathbb{R}_0^+$ and $B = \mathbb{R}^+$. We showed in part (b) that both of these sets are uncountable.

Notice that the set $A - B = \{0\}$, which is finite, and, therefore, countable. Since $A \cap B = B$ and B is uncountable, we have found two uncountable sets A and B such that $A - B$ is countable and $A \cap B$ is uncountable.

Part d

Now let $A = \mathbb{R}$ and $B = \mathbb{R}_0^+$. Observe that $A - B = \mathbb{R}^-$ and $A \cap B = B$. We saw earlier that the sets $\mathbb{R}$, $\mathbb{R}_0^+$, and $\mathbb{R}^-$ are all uncountable, which means that we have found uncountable sets A and B for which $A - B$ and $A \cap B$ are both uncountable.

Part e

We show that for any mapping S from a countable set A to countable sets, the union $\bigcup_{x \in A} S(x)$ is countable. To clarify, if the map S gets an element $x \in A$ as input, it outputs the countable set $S(x)$.

We define some notation. Define $Y = \bigcup_{x \in A} S(x)$. Let $x_0, x_1, x_2, x_3, \ldots$ be an enumeration of A. Then let $Y_i = S(x_i)$ for $i \in \mathbb{N}$. Since S maps into countable sets, Y_i is countable, so there is an enumeration $y_{i1}, y_{i2}, y_{i3}, \ldots$ of Y_i. Finally, notice that $Y = \bigcup_{k \in \mathbb{N}} Y_k$.

We modify the proof we used in class to show that $\mathbb{Q}$ is countable to prove that Y is countable. Consider a table whose entry in the i-th row and j-th column is y_{ij}. We show that every element of Y is somewhere in this table. Let $y \in Y$. Since $Y = \bigcup_{k \in \mathbb{N}} Y_k$, there is some integer i such that $y \in Y_i$. Furthermore, since Y_i is countable, there is an integer j such that $y = y_{ij}$. Then y is in row i and column j of our table.

It is also easy to see that our table doesn't contain any elements outside of Y by construction.

Now we traverse our table diagonal by diagonal, and form a list of elements of Y the same way we did in order to enumerate all rational numbers. As a reminder, this means that the first few terms of our list are $y_{01}, y_{02}, y_{11}, y_{03}, y_{12}, y_{21}, y_{04}, \ldots$. Eliminating duplicates gives an enumeration of Y.

Part f

Now we drop the requirement from part (e) that A is countable, and describe a mapping S from an uncountable set A to countable sets such that $\bigcup_{x \in A} S(x)$ is not countable.

Consider the mapping S that takes elements of A to singleton subsets of A, i.e., $S(x) = \{x\}$. Since singleton sets are finite, they are countable, so S is indeed a mapping from A to countable sets. Now $\bigcup_{x \in A} S(x) = \bigcup_{x \in A} \{x\} = A$. Thus, if A is not countable, neither is $\bigcup_{x \in A} S(x)$.

Thus, if A is uncountable, there is a mapping S of A to countable sets, for which $\bigcup_{x \in A} S(x)$ is uncountable, which implies that the given statement does not hold in general.

Problem 4

Part a

Our goal is to show that $\sqrt[3]{2}$ is not root-rational. We argue the same way we did in class to show that $\sqrt{2}$ is irrational. The proof goes by contradiction.

Suppose that $\sqrt[3]{2}$ is root-rational. Then there exist positive integers a and b such that

$$\sqrt[3]{2} = \sqrt{a/b}. \tag{3}$$

In (3), we can assume without loss of generality that $\gcd(a, b) = 1$.

Now raise both sides of (3) to the sixth power to get $4 = a^3/b^3$, and rearrange to obtain

$$4b^3 = a^3. \tag{4}$$

It follows that a^3 is even. Therefore, a is even as well, and we can write $a = 2c$ for some integer c. Substituting into (4) then yields $4b^3 = (2c)^3 = 8c^3$, so $4b^3 = 8c^3$, and $b^3 = 2c^3$. This means that b^3 is even, so b is even too. But now a and b are both even, which contradicts the assumption that $\gcd(a, b) = 1$. Thus, the assumption that $\sqrt[3]{2}$ is root-rational is wrong, and it follows that $\sqrt[3]{2}$ is not root-rational.

Part b

Proposition 1. *For every positive integer n, $\sqrt[3]{n}$ is root-rational if and only if n is a perfect cube, i.e., there exists an integer m such that $n = m^3$.*

Proof. Let n be a positive integer. We prove two implications. First, we show that if n is a perfect cube, then $\sqrt[3]{n}$ is root-rational. Second, we show that if $\sqrt[3]{n}$ is root-rational, then n is a perfect cube.

Let's first assume that n is a perfect cube. Then there is a positive integer m such that $n = m^3$. Thus, $\sqrt[3]{n} = \sqrt[3]{m^3} = m = \sqrt{m^2/1}$. Since m^2 and 1 are both positive integers, this shows that $\sqrt[3]{n}$ is root-rational.

Now assume that $\sqrt[3]{n}$ is root-rational. Then there exist positive integers a and b such that $\sqrt[3]{n} = \sqrt{a/b}$. Taking the sixth power of the last equality yields

$$n^2 = \frac{a^3}{b^3}. \tag{5}$$

We can assume without loss of generality that $\gcd(a, b) = 1$. This means that $\gcd(a^3, b^3) = 1$ as well, so the fraction a^3/b^3 is in reduced form. Notice that a fraction in reduced form is a natural number only if the denominator is 1. Since n^2 is a positive integer, (5) implies that a^3/b^3 is also a natural number, so $b^3 = 1$, and, therefore, $b = 1$. Then we can rewrite (5) as $n^2 = a^3$.

Now consider the prime factorizations of a and n. Let $p_1, p_2, \ldots, p_r$ be distinct primes and $e_i, f_i \in \mathbb{N}$ for $i \in \{1, \ldots, r\}$ such that

$$a = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$$
$$n = p_1^{f_1} p_2^{f_2} \cdots p_r^{f_r}$$

Since $n^2 = a^3$, we have

$$p_1^{2f_1} p_2^{2f_2} \cdots p_r^{2f_r} = p_1^{3e_1} p_2^{3e_2} \cdots p_r^{3e_r} \tag{6}$$

Note that two integers are equal if and only if they have the same prime factorizations, which means that the left-hand side and the right-hand side of (6) are equal if and only if $2f_i = 3e_i$ for all $i \in \{1, \ldots, r\}$. This means that 3 divides $2f_i$. But then 3 divides f_i. This means that there exist integers $d_1, d_2, \ldots, d_r$ such that $f_i = 3d_i$ for all $i \in \{1, \ldots r\}$, and we can rewrite the prime factorization of n as

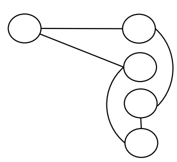
$$n = p_1^{3d_1} p_2^{3d_2} \cdots p_r^{3d_r} = \left(p_1^{d_1} p_2^{d_2} \cdots p_r^{d_r}\right)^3.$$

We see that n is a perfect cube because it's the cube of $p_1^{d_1} p_2^{d_2} \cdots p_r^{d_r}$, which is an integer.

It follows that $\sqrt[3]{n}$ is root-rational if and only if n is a perfect cube. □

Problem 5

(a) There exist configurations where the above property does not hold; this is when each person knows exactly two people [and does not know two others by definition]. An example is shown below.



It is not possible for there to be a group of 3 mutual friends if all people know exactly two people; assuming the existence of this group, there will be two people A and B not belonging to it. A and B each need to know exactly two others; but if they know a person C in the friends group, then C will know three people, a contradiction of our assumption. So A and B can at most know one other person; but this also contradicts our assumption.

A similar argument can be made for why it is not possible to have a group of 3 mutual strangers if all people know exactly two people. If such a situation occured, these three strangers A, B, and C would each have to know D and E [since they cannot know each other]. But then D and E know three people, a contradiction of our assumption.

Note that when interpreted as a graph, the above situation can be more succinctly described as a cycle of size 5. In such a cycle, each node (person) has edges to (knows) exactly two other nodes by definition.

(b) It can be argued similiarly to the case with 6 people that if a single person A knows 3 or more people, the property will hold; if these 3+ people do not know each other at all, they are a group of mutual strangers, and if any of them know each other, those two and A are mutual friends. Likewise, if a single person A knows 1 or less people, then there is a group of 3+

people who A does not know. If any two of them do not know each other, they and A are mutual strangers; but if they alll know each other, they are mutual friends.

The only remaining case is when all people know exactly two people. This is the case described in (a), a cycle of size 5. So the property holds for all cases other than a 5-cycle.

Extra Credit Problem

The solution will be described on the next assignment.