CS/Math 240: Intro to Discrete Math 2/17/2011

Solutions to Homework 3

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Problem 1

We use induction to prove that $(\forall n \geq 0)\, P(n)$, where $P(n)$ denotes

$$\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2. \tag{1}$$

For the base case we have $n = 0$, so the left-hand side of (1) is $\sum_{i=0}^{0} i^3 = 0^3 = 0$, and the right-hand side is $(0(0-1)/2)^2 = 0$. Thus, $P(0)$ holds.

Now suppose $P(n)$ holds for $n \geq 0$. We want to show that $P(n+1)$ holds, i.e., that

$$\sum_{i=0}^{n+1} i^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2. \tag{2}$$

We start with the left-hand side (LHS) of (2), $\sum_{i=0}^{n+1} i^3$, and split the sum into $\sum_{i=0}^{n} i^3$ and $(n+1)^3$. The former equals $(n(n+1)/2)^2$ by the induction hypothesis, so we have

$$\begin{aligned}
\sum_{i=0}^{n+1} i^3 &= \left(0^3 + 1^3 + \cdots + n^3\right) + (n+1)^3 \\
&= \left(\sum_{i=0}^{n} i^3\right) + (n+1)^3 \\
&= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\
&= (n+1)^2 \cdot \left(\left(\frac{n}{2}\right)^2 + (n+1)\right) \\
&= (n+1)^2 \cdot \frac{n^2+4n+4}{4} \\
&= \frac{(n+1)^2 \cdot (n+2)^2}{4} \\
&= \left(\frac{(n+1)(n+2)}{4}\right)^2
\end{aligned}$$

This completes the induction step of the proof, and the proof by induction that (1) holds for all nonnegative integers n.

Problem 2

Let $P(n)$ denote the proposition that for all finite sets $A_1, A_2, \ldots, A_n$,

$$|\cup_{i=1}^n A_i| = \sum_{\phi \neq I \subseteq [n]} (-1)^{|I|-1} |\cap_{i \in I} A_i|. \tag{3}$$

We will use induction on n to prove that $P(n)$ holds true for all integers $n \geq 1$.

For the base case, $n = 1$, the left-hand side (LHS) of $P(1)$ evaluates to $|A_1|$. The right-hand side (RHS) of $P(1)$ is a sum over all the nonempty subsets of [1]. The only nonempty subset of [1] is 1. Therefore, the RHS evaluates to $(-1)^{1-1}|A_1| = |A_1|$, which is same as the LHS. Hence, the base case holds.

For the inductive step, we need to argue that $P(n) \Rightarrow P(n+1)$ where $n \geq 1$. So, let's assume that $P(n)$ holds. Consider the LHS of $P(n+1)$ –

$$\begin{aligned} |\cup_{i=1}^{n+1} A_i| &= |(\cup_{i=1}^n A_i) \cup A_{n+1}| \qquad (4) \\ &= |(\cup_{i=1}^n A_i)| + |A_{n+1}| - |(\cup_{i=1}^n A_i) \cap A_{n+1}| \end{aligned}$$

Note that we made use of the result proved in problem $1(a)$ of homework 2 to rewrite (4). Now,

$$|(\cup_{i=1}^n A_i)| \tag{5}$$

can be rewritten using the induction hypothesis. But, $|(\cup_{i=1}^n A_i) \cap A_{n+1}|$ cannot – at least not right away – as it includes "$\cap A_{n+1}$" which doesn't match the induction hypothesis. However, we can rewrite (5) using one of the identities from class, namely that for all sets X, Y, and Z,

$$(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z). \tag{6}$$

More precisely, we can show by induction that $(\forall n \geq 1)\, Q(n)$, where $Q(n)$ denotes the predicate that for all sets A and $A_1, A_2, \ldots A_n$,

$$(\cup_{i=1}^n A_i) \cap A = \cup_{i=1}^n (A_i \cap A).$$

For the base case, $n = 1$. The LHS and the RHS of $Q(1)$ both evaluate to $A_1 \cap A$. Hence, the base case holds.

For the inductive step, we need to argue that $Q(n) \Rightarrow Q(n+1)$ where $n \geq 1$. So, let's assume that $Q(n)$ holds. Consider the LHS of $Q(n+1)$. Applying the identity (6) from class with $X = \cup_{i=1}^n A_i$, $Y = A_{n+1}$, and $Z = A$, we have that

$$\begin{aligned} (\cup_{i=1}^{n+1} A_i) \cap A &= ((\cup_{i=1}^n A_i) \cup A_{n+1}) \cap A \qquad (7) \\ &= ((\cup_{i=1}^n A_i) \cap A) \cup (A_{n+1} \cap A) \\ &= (\cup_{i=1}^n (A_i \cap A)) \cup (A_{n+1} \cup A) \\ &= \cup_{i=1}^{n+1} (A_i \cap A). \qquad (8) \end{aligned}$$

This finishes the proof that $Q(n)$ holds for all positive integers n.

We now continue with the proof of the induction step $P(n) \Rightarrow P(n+1)$ for $n \geq 1$. Applying our inductive hypothesis to both $\cup_{i=1}^{n} A_i$ and to $\cup_{i=1}^{n}(A_i \cap A_{n+1})$ and plugging those equations into (4) we obtain

$$\begin{aligned} |\cup_{i=1}^{n+1} A_i| &= |(\cup_{i=1}^{n} A_i)| + |A_{n+1}| + |(\cup_{i=1}^{n}(A_i \cap A_{n+1})| \\ &= \sum_{\phi \neq I \subseteq [n]} (-1)^{|I|-1} |(\cap_{i \in I} A_i| \\ &+ |A_{n+1}| \\ &- \sum_{\phi \neq I \subseteq [n]} (-1)^{|I|-1} |(\cap_{i \in I}(A_i \cap A_{n+1})| \end{aligned} \tag{9}$$

Now we will combine all the three terms together. Note that we would like the final expression to be a summation over all the non-empty subsets of $[n+1]$. Any subset of $[n+1]$ either has $n+1$ or not.

- If $n+1$ is not present in a subset, then that subset can be seen as a subset of $[n]$. The first term represents the sum over all the non-empty subsets of $[n+1]$ that do not contain $n+1$.
- If $n+1$ is present in a subset, then that subset can be viewed as a subset of $[n]$ with an additional element in form of $n+1$. The second and third term together account for all such subsets.

Rewriting (9), we get

$$\begin{aligned} |\cup_{i=1}^{n+1} A_i| &= \sum_{\phi \neq I \subseteq [n+1], n+1 \notin I} (-1)^{|I|-1} | \cap_{i \in I} A_i)| \\ &+ |A_{n+1}| \\ &+ \sum_{\phi \neq I \subseteq [n], J = I \cup \{n+1\}} (-1)^{|J|-1} |(\cap_{i \in J} A_i| \\ &= \sum_{\phi \neq I \subseteq [n+1], n+1 \notin I} (-1)^{|I|-1} | \cap_{i \in I} A_i| \\ &+ \sum_{I \subseteq [n], J = I \cup \{n+1\}} (-1)^{|J|-1} | \cap_{i \in J} A_i| \\ &= \sum_{\phi \neq I \subseteq [n+1]} (-1)^{|I|-1} | \cap_{i \in I} A_i|. \end{aligned} \tag{10}$$

This proves that $P(n) \Rightarrow P(n+1)$ for every $n \geq 1$. Since $P(1)$ also holds, this proves that $P(n)$ holds for every integer $n \geq 1$.

Problem 3

The flaw in this proof occurs at the inductive step for $P(1)$, i.e., in the proof that $P(0) \Rightarrow P(1)$. Let's analyze that step. By the induction hypothesis, $a^k = 1$ for all $k \in \mathbb{N}$ such that $k \leq 0$. This only says that $a^0 = 1$ for every nonzero a. But then the inductive argument reads

$$a^1 = \frac{a^0 * a^0}{a^{-1}},$$

where -1 is not a number for which the induction hypothesis applies. Our result is in fact $\frac{1*1}{\frac{1}{a}} = a \neq 1$ for $a \neq 1$. So our proof fails.

Problem 4

We define two sequences, $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$. The terms a_n and b_n represent the numbers of A-lings and B-lings in the i-th generation, respectively. We have $a_0 = 200$ and $b_0 = 800$. Our goal is to show that the population of A-lings in any generation is at most twice the population of B-lings.

Let's try to write a proof by induction right away. Define the predicate $P(n)$: $a_n \leq 2b_n$, which says that "the number of A-lings in generation n is at most twice the number of B-lings in generation n".

The base case $P(0)$ holds because $a_0 = 200$ and $b_0 = 800$. We see that $200 = a_0 \leq 2b_0 = 1600$.

Now assume that $P(n)$ holds, that is, $a_n \leq 2b_n$. Our goal is to show that $a_{n+1} \leq 2b_{n+1}$. To do so, we need expressions for a_{n+1} and b_{n+1} in terms of a_n and b_n. Those depend on which population is bigger in generation n, so we proceed by cases.

Case 1: $b_n \leq a_n \leq 2b_n$. In this case, there is an excess of A-lings. Every B-ling pairs up with some A-ling, and each pair produces one A-ling and one B-ling for a total of b_n A-lings and b_n B-lings. The remaining $a_b - b_n$ A-lings form pairs and each pair produces 3 A-lings for a total of $3\lfloor(a_n - b_n)/2\rfloor$ A-lings and no B-lings. Here we use the notation $\lfloor x\rfloor$ to denote the real number x rounded down to an integer. Thus, we have $b_{n+1} = b_n$ and $a_{n+1} = b_n + 3\lfloor(a_n - b_n)/2\rfloor$. But if $a_n = 2b_n$ (which can happen in this case) we have $a_{n+1} = b_n + 3\lfloor b_n/2\rfloor$, which is roughly $\frac{5}{2}b_n = \frac{5}{2}b_{n+1}$, so we have too many A-lings. We failed to prove the implication $P(n) \Rightarrow P(n+1)$ in this case. In fact, one can show that if $a_m \geq b_m + 2$ for some m, then the population of A-lings will grow uncontrollably. Intuitively, this makes sense because every pair of A-lings produces three A-lings and pairing A-lings with B-lings doesn't grow the B-ling population.

It looks like we are stuck in our proof attempt because $a_n \leq 2b_n$ doesn't always imply $a_{n+1} \leq 2b_{n+1}$. But, just like in the tiling problem from class, we can get the induction scheme to work by strenthening the statement.

Recall that the initial populations are $a_0 = 200$ and $b_0 = 800$, so the initial populations do not fall in Case 1. Let's look what happens in the case when $a_n \leq b_n$. Then there is an excess of B-lings. Every A-ling pairs up with some B-ling, and each such pair produces one A-ling and one B-ling for a total of a_n A-lings and a_n B-lings. The remaining $b_n - a_n$ B-lings form pairs and each pair produces one A-ling and two B-lings for a total of $\lfloor(b_n - a_n)/2\rfloor$ A-lings and $2\lfloor(b_n - a_n)/2\rfloor$ B-lings. Thus, we have $a_{n+1} = a_n + \lfloor(b_n - a_n)/2\rfloor$ and $b_{n+1} = b_n + 2\lfloor(b_n - a_n)/2\rfloor$. Now $a_n \leq b_n$ and $\lfloor(b_n - a_n)/2\rfloor \leq 2\lfloor(b_n - a_n)/2\rfloor$, which implies that $a_{n+1} \leq b_{n+1}$. Hence, we have shown that $(\forall n \geq 0)\, Q(n) \Rightarrow Q(n+1)$, where $Q(n)$ denotes the statement that $a_n \leq b_n$. Since $Q(0)$ also holds, we have shown by induction that $(\forall n \geq 0)Q(n)$ holds. That is, we proved the following proposition.

Proposition 1. *Starting from a population of* 200 *A-lings and* 800 *B-lings, the A-lings will never outnumber the B-lings.*

In symbols this means that $(\forall n)\quad a_n \leq b_n$ when the initial populations are $a_0 = 200$ and $b_0 = 800$. It follows that with such starting populations, $(\forall n)\ \ a_n \leq 2b_n$, which is what we wanted to show.

Let's also give some intuitive reasoning behind Proposition 1. Pairing an A-ling and a B-ling doesn't change the sizes of the two populations. Also note that every pair of B-lings produces two B-lings, so the population of B-lings doesn't change. A new A-ling is born for each pair of B-lings that got paired together due to the lack of A-lings. Thus, the difference between the two populations is halved. As the A-ling population increases, more and more B-lings get paired with A-lings, which slows down the growth of the A-ling population. When the two populations become the same size, no more changes in the populations occur because all B-lings are paired with A-lings and vice versa.

Problem 5

Part a

We will start by examining small values of n. When $n = 1$, the first player obviously loses since he/she must take the remaining stick. For $n = 2, 3, 4$, however, the first player can choose exactly enough sticks so that there is one remaining for the second player; in other words, the first player has a winning strategy. From this, and looking at possible strategies for $n = 5$, we make the following conjecture $P(n)$: If n is of the form $4k + 1$ for some integer k, then the second player has a winning strategy; othewise the first player has a winning strategy.

We show that $(\forall n \geq 1)\, P(n)$ holds by strong induction.

Our base case is $P(1)$. As we argued above, then the first player has a winning strategy, which is consistent with the fact that 1 can be written as $1 = 4 \cdot 0 + 1$.

For the strong induction step, we assume that $P(m)$ holds for all integers m in the range $1 \leq m \leq n$, and we want to argue that $P(n + 1)$ holds. Let us call the first player for the game with $n + 1$ sticks Alice, and the second player Bob. We consider four cases.

1. Case $n + 1$ is of the form $4k$. If Alice picks 3 sticks, Bob then sees $4(k1) + 1$ sticks, so Alice has a winning strategy, consistent with our conjecture $P(n + 1)$.

2. Case $n + 1$ is of the form $4k + 1$. Since $n \geq 1$, we know that $n + 1 \geq 5$. Alice can then chose to remove 1, 2 , or 3 sticks. If she removes one stick, the remaining number of sticks is $n = 4k$. By the strong induction hypothesis, the player who plays first at this point has a winning strategy. That player is Bob, so Bob has a winning strategy. Similarly, if Alice removes two sticks, the remaining number is $4(k1) + 3$. Again, Bob has a winning strategy, by the same reasoning. Similarly, if Alice removes 3 sticks, Bob has a winning strategy. So, however Alice moves, Bob has a winning strategy for the subsequent rounds. So, Bob has a winning strategy. This proves our conjecture $P(n + 1)$ in this case.

3. Case $n + 1$ is of the form $n + 1 = 4k + 2$. If Alice removes 1 stick, Bob is left with $4k + 1$, so Alice has a winning strategy, consistent with our conjecture $P(n + 1)$.

4. Case $n + 1$ is of the form $4k + 3$. If Alice picks 2 sticks, Bob is left with $4k + 1$ sticks, so Alice has a winning strategy, consistent with $P(n + 1)$.

So in any case, $P(n + 1)$ holds, so by strong induction we concluce that $P(n)$ holds for all integers $n \geq 1$.

Part b

We first note that the concept of a winning strategy for a two-player game can be expressed succinctly using a logical expression. Breaking our winning strategy into a series of moves for both players, we note that while the player employing the strategy has complete freedom to choose his or her own moves, that player must be able to select these moves in such a way that it defeats all possible counter-moves by the opposing player. So for the first player to have a winning strategy, for example, there must exist a first move that player can take, such that for all second moves by the opponent, the first player can still win. So there exists a third move the first player can take that follows in the same fashion until the game ends, in a finite number of steps, with a victory for the first player. More formally, this can be expressed as follows:

$$(\exists x_1)(\forall x_2)(\exists x_3)\dots(Qx_n)\text{The sequence of moves } (x_1, x_2, ...x_n) \text{ leads to a win for player one,}$$

where $Q = \begin{cases} \exists & \text{if } n \text{ is odd} \\ \forall & \text{otherwise} \end{cases}$

We note that if we take the negation of this statement, we can push that negation past our quantifiers [Recall: $\neg\exists x(P(x)) = \forall x(\neg P(x))$] to get the following statement:

$$(\forall x_1)(\exists x_2)(\forall x_3)\dots(Qx_n)\text{The sequence of moves } (x_1, x_2, ...x_n) \text{ does not lead to a win for player one,}$$

where $Q = \begin{cases} \forall & \text{if } n \text{ is odd} \\ \exists & \text{otherwise} \end{cases}$

Where (assuming no ties) the above translates into a winning strategy for the second player. For the second player to win, for any initial move the first player takes, there must exist a second counter-move by the second player that is a part of a winning strategy; that is, for any third move by the first player...etc. Clearly either the first statement or its negation must be true in any situation; so we can conclude that for our two-player game, there is either a winning strategy for the first player, or a winning strategy for the second.

We can formalize the above argument as a proof by induction on the number of steps n: For every integer $n \geq 0$, in a n-step game between two people there exists a winning strategy for either the first player or for the second. $P(0)$ correspond to the no-step game; allowing no ties, the game is an automatic win for one of the two players. So for that player, the 'winning strategy' is to do nothing.

Now, assuming that $P(n)$ holds [that is, for all n-step games between two people, there is a winning strategy for one player], we must show that $P(n+1)$ holds as well. To do this, we will break our $(n+1)$ step game into two separate parts:

- The first player selects a move.
- The two players begin an n-step game whose starting position is the result of the previous move, with the second player taking first action in this 'sub-game.'

From $P(n)$, the second stage of our $(n+1)$ step game has a winning strategy for one player. Examining the first stage of the game, we see there are two possible situations for the first player. This player can either select a move x_1 in such a way that the second stage of the game has a winning strategy for him/her, or he cannot. In the first case, we have shown that the first player has a winning strategy for the $(n+1)$ step game; namely {choose x_1, follow the winning strategy for n-step game}. In the second case, we see that the second player has a winning strategy for our $(n + 1)$ step game, which is {do nothing; follow the winning strategy for n-step game}. So given that $P(n)$ holds, $P(n + 1)$ must hold. Combined with our base case, this proves the original statement.

Extra Credit Problem

Consider the case when there is only a single Venusian, Alice, with a mark on her forehead. Since she cannot observe any other Venusian with a mark, she would realize that she has a mark as soon as the Earthling proclaims that there is someone with a mark on the forehead. Therefore, Alice will die the morning of the second day.

Now, consider the case when there are two Venusians, Alice and Bob, with marks on their foreheads. Both of them can observe the other person's mark. So, when the Earthling makes his statement, it is consistent with Alice's knowledge that Bob is the only person with a mark on the forehead and that Alice herself does not have a mark on the forehead. So, Alice does not die the morning of the second day. By the same token, neither does Bob. However, when at the gathering the second day Alice notices that Bob is still alive, she realizes that she herself must also have a mark on her forehead: "Since Bob is still alive, Bob could observe a mark on someone's forehead yesterday. But since that mark is not visible to me, there must be a mark on my forehead." Hence, Alice dies the next morning, i.e., the morning of the third day. The same logic leads to Bob's death that morning.

This leads to the conjecture that if there are n Venusians with a mark on their forehead, then they all die on morning of the $(n + 1)$st day, i.e., the nth day after the Earthling's visit. Let us denote this statement by $P(n)$. We argue that $P(n)$ holds for all integers $n \geq 1$ by induction. We already argued the base case $n = 1$.

For the induction step, assume $P(n)$ holds for some $n \geq 1$. We need to show that $P(n+1)$ holds. Now consider the point of view of a person with a mark. Lets call her Alice. Alice can observe n Venusians with marks. She would reason that since $P(n)$ holds (by the induction hypothesis), if these n Venusians are the only ones with marks on their foreheads, they should die on the morning of the $(n+1)$st day. So, Alice cannot conclude that she has a mark on her forehead during the first n days, so she won't die during the first $n + 1$ days. But since every one else with a mark would reason the same, no one would die during the first $n + 1$ days. At the meeting on the $(n + 1)$st day, Alice would reason that since all the n Venusians with marks are alive, these Venusians can also observe n other Venusians with marks on their foreheads. Then, Alice would realize that there are $n + 1$ Venusians with marks, and that she also has a mark on her forehead. This would lead to the death of Alice the morning of the $(n + 2)$nd day. The same holds for the n other Venusians with a mark on their forehead. So, $P(n + 1)$ holds. This finishes the induction step of the proof, and the proof itself.