Solutions to Homework 7

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Problem 1

We prove two implications to prove the equivalence

$$f \text{ is injective} \iff (\forall X, Y \subseteq A)\ \ f(X) \cap f(Y) = f(X \cap Y).$$

First assume that f is injective, and pick any $b \in B$. Since f is injective, the inverse of f is a function, which means that for every $b \in f(A)$ there is a unique $a \in A$ such that $f(a) = b$. Furthermore, if $b \in f(S)$ for any $S \subseteq A$, this means $a \in S$. With this observation in hand, we have the following chain of equivalences that shows $f(X) \cap f(Y) = f(X \cap Y)$.

$$\begin{aligned} b \in f(X) \wedge b \in f(Y) &\iff a \in X \wedge a \in Y \\ &\iff a \in X \cap Y \\ &\iff b \in f(X \cap Y) \end{aligned}$$

Now assume that for any two subsets $X, Y \subseteq A$ we have $f(X) \cap f(Y) = f(X \cap Y)$. Suppose $f(a_1) = f(a_2) = b$. Our goal is to show that $a_1 = a_2$. Observe that $f(\{a_1\}) = f(\{a_2\}) = \{b\}$. But our assumption, $f(\{a_1\}) \cap f(\{a_2\}) = f(\{a_1\} \cap \{a_2\}) = \{b\} \cap \{b\} = \{b\}$, so $\{b\} = f(\{a_1\} \cap \{a_2\})$. If $a_1 \neq a_2$, $\{a_1\} \cap \{a_2\}$ would be empty, and $f(\emptyset) = \emptyset \neq \{b\}$. Hence, we have $a_1 = a_2$, so f is injective.

Problem 2

Table 1 summarizes all the results for this problem. We give equivalence classes and argue whether a relation is a total order later.

Relation	**Equivalence relation?**	**Order relation?**
R_1	no	no
R_2	no	yes
R_3	no	no
R_4	yes	yes
R_5	yes	no

Table 1: A summary of answers for this question.

Recall the following definitions. In the list below, R is a relation on a set A. For this problem, A is the power set of D.

- R is reflexive if $(\forall a \in A)\ \ aRa$.
- R is symmetric if $(\forall a, b \in A)\ \ aRb \iff bRa$.

- R is antisymmetric if $(\forall a, b \in A) \ (aRb \wedge bRa) \Rightarrow (a = b)$.
- R is transitive if $(\forall a, b, c \in A) \ (aRb \wedge bRc) \Rightarrow aRc$.
- R is an equivalence relation if it is reflexive, symmetric, and transitive.
- R is an order relation if it is antisymmetric and transitive.

Now that we have reviewed the terminology, let's start analyzing our five relations. Below, we use X, Y, and Z for sets (they will be subsets of D). Since $|D| \geq 2$, there are at least two different elements in D, and we call them a and b.

Let's start with $R_1 = \{(X, Y) \mid X \cap Y = \emptyset\}$.

This is a symmetric relation because $X \cap Y = Y \cap X$, so $X \cap Y = \emptyset \iff Y \cap X = \emptyset$. Since $X \cap \overline{X} = \emptyset$ and $X \neq \overline{X}$, R_1 is not antisymmetric, and therefore R_1 is not an order relation.

Consider the sets $X = Z = \{a\}$ and $Y = \{b\}$. Then $(X, Y) \in R_1$ and $(Y, Z) \in R_1$ because $X \cap Y = \emptyset$ and $Y \cap Z = \emptyset$. However, $X = Z$, so $X \cap Z = X \neq \emptyset$, which means $(X, Z) \notin R_1$, and R_1 is not transitive. Therefore, R_1 is not an equivalence relation.

Now consider $R_2 = \{(X, Y) \mid X \cup Y = \emptyset\}$.

This is a symmetric relation because $X \cup Y = Y \cup X$, so $X \cup Y = \emptyset \iff Y \cup X = \emptyset$. Also observe that if $X \cup Y = \emptyset$ and $Y \cup X = \emptyset$, then, in fact, both X and Y are the empty set, so $X = Y$. Hence, R_2 is also antisymmetric. This relation is an example of a relation that is both symmetric and antisymmetric.

For any nonempty subset $X \subseteq D$, we have $X \cup X = X \neq \emptyset$. Thus, R_2 is not reflexive, and, therefore, not an equivalence relation.

Finally, R_2 is transitive. If $X \cup Y = \emptyset$ and $Y \cup Z = \emptyset$, an earlier argument implies $X = Y = Z = \emptyset$, so $X \cup Z = \emptyset$.

Since R_2 is antisymmetric and transitive, it is an order relation. However, it is not a total order. For example, consider the sets $X = \{a\}$ and $Y = \emptyset$. Then we have neither $X \cup Y = \emptyset$ nor $Y \cup X = \emptyset$.

Let's focus on $R_3 = \{(X, Y) \mid |X| \leq |Y|\}$ next. We show that this relation is neither symmetric nor antisymmetric, thus showing that it's neither an order relation nor an equivalence relation.

Consider the sets $X = \{a\}$, $Y = \{b\}$, and $Z = \{a, b\}$. Then $(X, Y) \in R_3$ and $(Y, X) \in R_3$, but $X \neq Y$, so R_3 is not antisymmetric. Also note $(X, Z) \in R_3$ but $(Z, X) \notin R_3$, so R_3 is not symmetric either.

Next up is $R_4 = \{(X, Y) \mid X \cap Y = X \cup Y\}$.

We find an alternative statement for R_4. Let X and Y be any subsets of D. Note that $X \subseteq X \cup Y$ and $X \cap Y \subseteq Y$. Thus, if $X \cup Y = X \cap Y$, we get $X \subseteq Y$. By switching the roles of X and Y, we also get $Y \subseteq X$ and $X = Y$. Conversely, if $X = Y$, then certainly $X \cap Y = X \cup Y$ because both the intersection and the union is just X. Hence, we can rewrite $R_4 = \{(X, Y) \mid X = Y\}$.

Equality is easily seen to be reflexive, symmetric, and transitive, so R_4 is an equivalence relation. Every set is equal to itself, so R_4 is reflexive. If X is the same as Y, then also Y is the same as X, so R_4 is symmetric. Finally, if X is the same as Y and Y is the same as Z, then also X is the same as Z, so R_4 is transitive.

Every subset of D is in its own equivalence class.

Equality is also antisymmetric. If $X = Y$ and $Y = X$, then $X = Y$ follows because it's part of the assumption. Thus, equality is also an order relation.

Equality is not a total order. For example, pick $X = \{a\}$ and $Y = \emptyset$. Then we have neither $X = Y$ nor $Y = X$.

Finally, let's consider $R_5 = \{(X, Y) \mid \text{there is a total function } f : X \to Y \text{ that is a bijection.}\}$

Let $X \subseteq D$. Then the identity map $f : X \to X$ defined by $f(a) = a$ is a bijection because for any $a \in X$, only a maps to a, so f is injective, and a is the image of a under f, so f is surjective. The identity map is also total. So, R_5 is reflexive.

Now suppose $X, Y \subseteq D$ such that $(X, Y) \in R_5$. Let f be a total function from X to Y that is a bijection, and consider the inverse relation f^{-1}. It's a function because f is injective, and it's total because f is surjective. We need to show that f^{-1} is a bijection from Y to X. Because f is total, every $a \in X$ maps to some $b \in Y$. Then $f^{-1}(b) = a$, so f is surjective. Finally, suppose $f^{-1}(b_1) = f^{-1}(b_2)$. Then there exists $a \in X$ such that $f(a) = b_1$ and $f(a) = b_2$. But f is a function, so $b_1 = b_2$, and f^{-1} is injective. It follows that f^{-1} is a bijection. So, R_5 is symmetric.

Finally, suppose $X, Y, Z \subseteq D$ are such that $(X, Y) \in R_5$ and $(Y, Z) \in R_5$. Then there exist total bijective functions $f : X \to Y$ and $g : Y \to Z$. Consider the function $h : X \to Z$ defined by $h(a) = g(f(a))$. This is a total function because f is defined on all of X, and g is defined on all of Y, which means it's defined on all of $f(X) \subseteq Y$.

Now we show that h is a bijection. First suppose $h(a_1) = h(a_2)$. This means that $g(f(a_1)) = g(f(a_2))$. Since g is injective, $f(a_1) = f(a_2)$. It follows that $a_1 = a_2$ because f is injective as well. Now suppose $b \in Z$. Since g is surjective, there is some $c \in Y$ such that $b = g(c)$. Furthermore, f is surjective, so there is some $a \in X$ such that $c = f(a)$. Then $b = g(c) = g(f(a)) = h(a)$, and we see that h is surjective. Thus, $(X, Z) \in R_5$, and R_5 is transitive.

It follows that R_5 is an equivalence relation.

We said in Lecture 16 that if f is a total bijective function from X to Y, then $|X| = |Y|$. Thus, X and Y can be in the same equivalence class only if $|X| = |Y|$. We prove that the equivalence classes are the sets $\{X \subseteq D \mid |X| = r\}$ for $r \in \{0, 1, \ldots, |D|\}$. In particular, we construct a bijection from X to Y for any $X, Y \subseteq D$ such that $|X| = |Y| = r$. Let $a_1, a_2, \ldots, a_r$ be an enumeration of X, and $b_1, b_2, \ldots, b_r$ an enumeration of Y. Then the function $f : X \to Y$ defined by $f(a_i) = b_i$ is a total bijective function from X to Y.

Finally, we argue that R_5 is not antisymmetric, thus showing it is not an order relation. For example, consider $X = \{a\}$ and $Y = \{b\}$. There is a total bijection f from X to Y (just set $f(a) = b$), and its inverse is a total bijection from Y to X (we have $f^{-1}(b) = a$). Thus, $(X, Y) \in R_5$ and $(Y, X) \in R_5$, but $X \neq Y$. It follows that R_5 is not antisymmetric. Therefore, it is not an order relation.

Problem 3

Part One

To show that R_L is an equivalence relation, we must show that it is reflexive, symmetric, and transitive. We will prove these in order.

Reflexive: We note that $xz = xz$. So, the statement $(xz \in L \iff xz \in L)$ evaluates to either $T \iff T$ or $F \iff F$ for all z. In either case, the expression is true. As this holds for every x and z, R_L is reflexive.

Symmetric: We note that $\iff$ is a symmetric relationship. That is, if $P \iff Q$ holds, then $Q \iff P$ also holds. Assume we have $(x, y) \in R_L$ then $(\forall z) xz \in L \iff yz \in L$, so $(\forall z) yz \in L \iff xz \in L$, which means that $(y, x) \in R_L$.

Transitive: Finally, we note that $\iff$ is a transitive relationship. That is, if $P \iff Q$ and $Q \iff R$ holds, then $P \iff R$ holds. This can be shown directly; $P \iff Q$ forces P and Q to the same truth values, and $Q \iff R$ does the same for Q and R. Thus, P and R must have the same truth values as well.

Now assume we have $(x, y) \in R_L$ and $(y, w) \in R_L$. Then, for any z, $xz \in L \iff yz \in L$ and $yz \in L \iff wz \in L$. Letting P denote $xz \in L$, Q denote $yz \in L$, and R denote $wz \in L$, we see that $xz \in L \iff wz \in L$. As the latter holds for every z, we have that $(x, z) \in R_L$.

Having shown that R_L is reflexive, symmetric, and transitive, we have shown that R_L is an equivalence relation.

Part Two

(i) First, we examine an arbitrary string x, say the empty string $x = \lambda$, and determine its equivalence class $[x]$, i.e., the set of strings y such that xR_Ly. We note that if we append any string z to λ, we obtain z, so λz is in L if and only if z has an even number of 1s. Looking at the definition of R_L, we need to determine for which strings y the following holds: For every string z, z has an even number of 1s iff yz has an even number of 1s. Since the number of 1s in yz is the sum of the number of ones in y and the number of 1s in z, the latter holds iff y has an even number of 1s, i.e., iff $y \in L$. Thus, $[\lambda] = L$.

The latter implies that for every $x \in L$, $[x] = L$. Thus, we have determined the equivalence class of every element in L.

Next, we consider an arbitrary element for which we haven't determined the equivalence class yet, say $x = 1$. Now xz is in L iff z has an odd number of ones. Thus, $[1]$ consists of all strings y for which the following holds: For every string z, z has an odd number of 1s iff yz has an even number of 1s. By a similar reasoning as above, the latter holds iff y has an odd number of 1s, i.e., iff $y \in \overline{L}$. Thus, $[1] = \overline{L}$.

Since L and $\overline{L}$ cover all of D, we have determined all the equivalence classes of R_L, namely L and $\overline{L}$.

(ii) Again, we start by determining the equivalence class of an arbitrary string, say $x = 01$. We note that for any string z with $k - 1$ 1s, xz is in L; the same is true for any string with a number of 1s equal to $2k - 1$, $3k - 1$, etc.... Any string z that does not fit this format, i.e., for which the number of 1s is not one less than a multiple of k, xz is not in L.

Letting $k = 3$ for the moment, we note that the string 10111 has the same behavior as 01. In fact, all strings with 1, 4, 7, ... total 1s will do so. So for $k = 3$, one equivalence class is the set of strings whose number of 1s modulo 3 is 1.

More generally, for any positive integer k, the equivalence classes of R_L are $X_i = \{$ all strings such that the number of 1s taken modulo k is equal to $i\}$, for $i = 0....k - 1$. The argument for an arbitrary X_i being an equivalence class is as follows:

Take any $x \in X_i$, and any string z. Then $xz \in L$ iff z has a number of 1s equal to $k - i$ modulo k, since then and only then xz has a number of 1s equal to 0 modulo k (in other words, a number of 1s equal to a multiple of k). Then, $yz \in L$ iff y has a number of 1s equal i modulo k, i.e., iff $y \in X_i$.

Since $\bigcup X_i = D$, the X_i's are all the equivalence classes for any positive integer k.

For $k = 0$, L consists of all strings with no 1s. In this case, xz is in L iff both x and z are in L. The above reasoning shows that the equivalence classes are L and $\overline{L}$.

(iii) We note that this is similar to (ii), in that strings that have the same number of 1s will fall into the same equivalence class. For example, 01 will be in the same class as 100. Since L contains strings with an exact number of 1s, rather than a multiple, we also note that if $01z \in L$ for some z, the *only* other strings y where $yz \in L$ will be those where y has exactly one 1.

However, note that there are some strings x fro which there is no z such that $xz \in L$. For example, if $k = 3$ and $x = 1111$, the string xz will never have 3 1s no matter what z is. For any two such strings x, the if and only if condition for R_L will evaluate to $F \iff F$. So these strings are all in the same equivalence class.

In this case, then, the equivalence classes of R_L are $X_i = \{$ all strings such that the number of 1s is equal to i$\}$, $i = 0, ..., k$, and $Y = \{$ all strings with more than k 1s$\}$. The argument for an arbitrary X_i is as follows:

Take any $x \in X_i$, and any string z. Then $xz \in L$ iff z has a number of 1s equal to $k - i$ in order for xz to have k 1s. Then $yz \in L$ iff y has exactly i 1s, i.e., iff $y \in X_i$.

For the equivalence class Y, note that for any $x \in Y$, $(\forall z) xz \notin L$. Since the only strings x with the latter property are exactly those in Y, we have that Y is an equivalence class of R_L.

We see that $\bigcup X_i \cup Y = D$, so these are all the equivalence classes.

Problem 4

(a) Let x and y be any two integers, and z be their supremum. Then, as per the definition of the supremum, z should satisfy the following conditions: (i) $x \leq z$, (ii) $y \leq z$, and (iii) $(\forall z')(x \leq z' \wedge y \leq z' \Rightarrow z \leq z')$. From the first two conditions, we can conclude that z has to be at least as much as the greater of x and y. Let $z = \max(x, y)$. Now we claim that z satisfies the third condition as well, for if any z' is at least as much as both of x and y, then it is as much as z. Moreover, $\max(x, y)$ is the only value with these properties. Hence, $\sup(x, y) = \max(x, y)$.

Again consider two integers x and y, and let z be their infimum. Then, z should satisfy the following conditions: (i) $z \leq x$, (ii) $z \leq y$, and (iii) $(\forall z')(z' \leq x \wedge z' \leq y \Rightarrow z' \leq z)$. From the first two conditions, $z \leq \min(x, y)$. We claim that $z = \min(x, y)$ is the infimum of x and y. This is because if any z' satisfies the first two conditions, then z' would be at most z. Hence, $\inf(x, y) = \min(x, y)$.

(b) At first it might appear that a similar argument as above would work and that $\sup(x, y) = \max(x, y) + 1$, and $\inf(x, y) = \min(x, y) - 1$. But this is not correct. In fact, neither a supremum nor an infimum exist for any x and y. Indeed, consider any z satisfying the first two conditions. As we can choose $z' = z$ in the third condition and for that choice of z' the hypothesis $x < z' \wedge y < z'$ holds, the third condition requires that $z < z$. But this is impossible, so no integer can act as supremum of x and y. Similarly, we can show that no integer can act as infimum of x and y.

(c) The three conditions are: (i) $x|z$, (ii) $y|z$, and (iii) $(\forall z')(x|z' \wedge y|z' \Rightarrow z|z')$. By (i) and (ii), z is a multiple of both x and y. The least such natural number is the LCM of x and y. Let $z = \text{LCM}(x, y)$. It suffices to prove that any other natural number z' such that $x|z'$ and $y|z'$ satisfies $z|z'$. We have that $z \leq z'$ since z was assumed to be the least possible common multiple of x and y. Then, by using the division algorithm, we get that $z' = zq + r$ where $0 \leq r < z$. Since x divides z' and z, x has to divide r as well. Similarly, y has to divide r as well. But if both x and y divide r, then r has to be 0 as z is the LCM of x and y and $r < z$. Hence, $\sup(x, y) = \text{LCM}(x, y)$.

Now, let us work out the infimum of x and y. The three conditions are: (i) $z|x$, (ii) $z|y$, and (iii) $(\forall z')(z'|x \wedge z'|y \Rightarrow z'|z)$. By (i) and (ii), z is a divisor of both x and y. Hence, a possible candidate for $\inf(x, y)$ is $\text{GCD}(x, y)$. Now, we show that $z = \text{GCD}(x, y)$ satisfies (iii) as well, i.e., any divisor of x and y is a divisor of z as well. Consider any positive integer z' such that $z'|x$ and $z'|y$. Recall from part 4(c) of HW 4 that there exist integers u and v such that $z = \text{GCD}(x, y) = ux + vy$. Since z' divides both x and y, z' also divides $ux + vy$ and therefore divides z. Therefore, z does satisfy (iii) and is the only positive integer satisfying all three conditions, so $inf(x, y) = \text{GCD}(x, y)$.

(d) The three conditions in this case are: (i) $x \subseteq z$, (ii) $y \subseteq z$, and (iii) $(\forall z')(x \subseteq z' \wedge y \subseteq z' \Rightarrow z \subseteq z')$. Let $z = x \cup y$. Then, z satisfies the first two conditions. Now, we need to show that it also satisfies (iii). Consider any z' which satisfies the premises in (iii). Now consider any $\alpha \in z$. Then, $\alpha \in x \vee \alpha \in y$. This means that $\alpha \in z'$, which means that $z \subseteq z'$. Hence, $sup(x, y) = x \cup y$.

Now, let us work out the infimum of x and y. The three conditions in this case are: (i) $z \subseteq x$, (ii) $z \subseteq x$, and (iii) $(\forall z')(z' \subseteq x \wedge z' \subseteq y \Rightarrow z' \subseteq z)$. Let $z = x \cap y$. Then, z satisfies the first two conditions. Now, we need to show that it also satisfies (iii). Consider any z' which satisfies the premises in (iii). Now consider any $\alpha \in z'$. Then, $\alpha \in x \wedge \alpha \in y$. This means that $\alpha \in z$, which means that $z' \subseteq z$. Hence, $inf(x, y) = x \cap y$.

Problem 5

We will interpret this problem as a digraph, where the n people playing the game are vertices, and a directed edge e_{ij} exists from every person i to its nearest neighbor (and thus their water balloon target) j. We then see that a survivor in the graph is a vertex which has no directed edge ending at itself, i.e., a vertex with indegree 0.

We now make the following statement: The only possible cycles of positive length have length 2. There are no cycles of length 1 as there are no selfloops. We prove by contradiction that there are no cycles of length larger than 2.

Let us denote by $d(u, v)$ the distance between players u and v. Consider any three consecutive vertices u, v, and w on a cycle. If u and w are distinct, then the existence of the edge from v to w implies that $d(v, w) < d(v, u)$. Since $d(v, u) = d(u, v)$ and any three consecutive points on a cycle of length more than 2 are distinct, this means that the distance of two consecutive points on a cycle of length 3 strictly decreases as we go around the cycle. However, the latter is impossible because the graph is finite, so at some point we will reach the same pair (u, v) as we started from, so we'd obtain the contradiction that $d(u, v) < d(u, v)$.

Given the above statement, we present two ways to finish the proof that there is at least one survivor when the number of players is odd.

The first proof is by contradiction. Suppose there is no survivor. Then every vertex has indegree at least 1. By construction, every vertex has outdegree exactly one. Since the sum of the outdegrees and indegrees are the same, this implies that every vertex has indegree and outdegree 1. This means that the digraph is a collection of disjoint cycles. Since every cycle consists of exactly two players and the number of players is odd, this is impossible.

Alternately, we can give a proof by induction that every every integer $k \geq 0$, a game with $2k+1$ players has at least one survivor.

The base case $k = 0$ is trivial as there is only one player. For the induction step $P(k) \Rightarrow P(k+1)$, first note that the digraph for $2k + 3$ players has to contain a cycle. This is because every vertex has outdegree 1 and the digraph is finite, so if we start from an arbitrary vertex and keep following the outgoing edge, there has to be a point where we hit a vertex for the second time, and we have a cycle between the first time and the second time we hit that vertex.

Consider any such cycle. We know that the cycle has length 2. Thus there is a pair of vertices (u, v) which have directed edges to each other. We can see if we remove these two players, the number of survivors in the remaining $2k + 1$ player graph will be less than or equal to the number of survivors in the original $2k + 3$ player graph. This is because removing u and v from the graph does not remove any edges which end at any of the other $2k + 1$ players (but may add new such edges, since u and v may have had multiple edges ending at each of them in the original graph). From our inductive hypothesis, any $2k + 1$ player game has at least one survivor. So the $2k + 3$ game will have at least one survivor as well.

Thus, all games with an odd number of players have at least one survivor, and the proof is complete.