CS/Math 240: Intro to Discrete Math	4/14/2011

Solutions to Homework 8

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Problem 1

To prove our if and only if statement (expressible as $P \iff Q$) , it is necessary to show both directions hold. We will first show that if a graph contains any cycles of odd length, it is not bipartite ($\neg Q \Rightarrow \neg P$). Then, we will show that if a graph contains no cycles of odd length, it must be bipartite ($Q \Rightarrow P$).

Part One: $\neg Q \Rightarrow \neg P$

We first show that a graph is not bipartite if it contains any cycles of odd length. Starting at an arbitrary vertex u in this cycle, we note that there is a path of length $2k + 1$ from u to v. We also note that to be bipartite, we must alternate placing vertices on this path into the partitions L and R. However, if we place u in L, then, we see that after following the path along $2k + 1$ edges, we must place u in R, a contradiction. A symmetric argument can be shown for placing u in R initially.

Now, we must also show that if a graph contains no cycles of odd length, it will be bipartite. We do so through an inductive proof on the number of vertices in the graph.

Part Two: $Q \Rightarrow P$

If a graph has one vertex, it is clearly bipartite assuming no self loops (which would be a cycle of odd length). This vertex can be labeled either L or R.

Now, assume that any graph with n vertices that contains no cycles of odd length can be labeled as bipartite. We must show that a graph with n + 1 vertices that contains no cycles of odd length can be labeled bipartite as well.

Taking any vertex v in the graph, we will remove v and all edges incident to v from our G to attain the minor G'. Since G' has n vertices and no cycles of odd length (removing v can only remove cycles from G entirely, not decrease their length), G' is bipartite by our inductive hypothesis. In order to add v and its edges back into G while maintaining the bipartite property, we must have all neighbors of v belonging to the same partition. Consider all pairs of vertices connected to v in the original graph G. We will denote one such pair as (u, w). Note that if no such pair exists ($|Neighbor(v)| < 2$), all neighbors of v necessarily belong to the same partition.

If u and w belong to the same connected component in the minor G', then there is at least one path between u and w which does not involve our selected vertex. We note that a path of length two necessarily exists between u and w in the original graph G, namely the path $u \to v \to w$. These two paths can be combined create a cycle in G. Since the total length of a cycle in G must even from our assumption, any path between u and w in G' will necessarily be even as well.

Then, u and w will have at least one even path connecting them in G'. This even path forces u and w to belong to the same partition (either L or R) when the resulting n-vertex graph is labeled as bipartite.

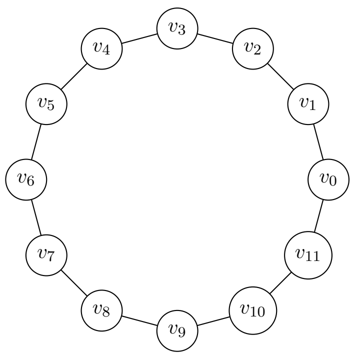
If u and w do not belong to the same connected component in the minor G', we note that each of the connected components u and w belong to can be labeled as bipartite independently of the other. In this case, we can label each subgraph in a way such that u and w belong to the same partition (either L or R).

So, all pairs of vertices connected to v can be put into the same partition (either L or R); this means that all vertices connected to v can be put into the same partition. Then, we can add v back into G by placing v into the opposite partition; this gives us a bipartite graph of size n+1. We have proved both directions of the implication, and our proof is complete.

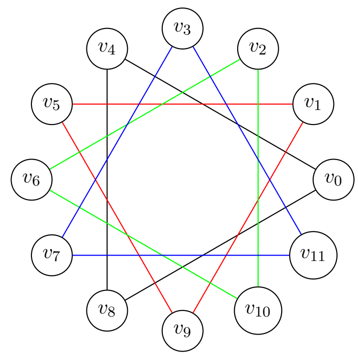
Problem 2

We start with some examples and show how to use them to come up with a solution to this problem.

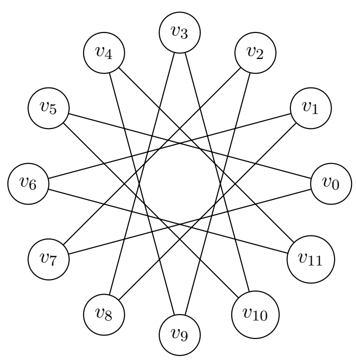
Figure 1 shows the graphs for $n = 12$ and various values of d. We can see varying numbers of connected components. In some case the graph is connected, and in some cases it is not.



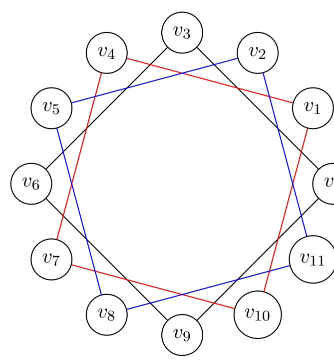
(a) $n = 12$, $d = 1$ or $d = 11$, connected.



(b) $n = 12$, $d = 4$ or $d = 8$, not connected.



(c) $n = 12$, $d = 5$ or $d = 7$, connected.



(d) $n = 12$, $d = 3$ or $d = 9$, not connected.

Figure 1: Some examples with $n = 12$.

As expected, when $d = 1$, the edges go around the circle, and the graph is connected in that case. We show this in Figure 1a.

On the other hand, when $d = 4$, we see in Figure 1b that the graph has multiple connected components (shown in different colors). In particular, each component is a cycle of length 3 (and looks like an equilateral triangle), and there are four of them. You would also find that there are two disjoint cycles of length 6 when $d = 2$, and three disjoint cycles of length 4 when $k = 3$. Note that 2, 3 and 4 all divide 12, so we suspect that when d divides n, we get a graph that is not connected.

Suppose $n = kd$. Label the vertices from v_0 to v_{n-1} as one goes around the circle. The vertices v_0 and v_d form an angle of $\frac{2\pi}{n} \cdot d$. Hence, they have an edge between them. Similarly, vertices v_d and v_{2d} are connected by an edge. Continuing this process, we end with an edge between $v_{(k-1)d}$ and v_{kd}. But $n = kd$, so v_{kd} is actually v_0. We have obtained a cycle of length $k = n/d$. Note that each vertex has as at most two edges incident on it: one connecting it to a vertex that appears d points clockwise from it on the circle, and one connecting it to a vertex that appears d points counterclockwise from it on the circle. Therefore, there is no path from the vertices on the cycle we constructed to any vertex that is not on that cycle, and the graph is not connected in this case. It follows that if $d > 1$, the cycle we constructed does not contain all vertices of the graph, and the graph is disconnected.

So our first guess could be that the graph is not connected when d divides n, and connected otherwise. The cases $d = 5$, $d = 6$ and $d = 7$ may reinforce that belief. We show the case $d = 5$ in Figure 1c. Note that the graph is connected and that 5 does not divide 12. But then we come to the cases $d = 8$ (the graph looks like the one in Figure 1b) and $d = 9$ (the graph is in Figure 1d). Neither 8 nor 9 divide 12, but the graphs are not connected in those two cases. So the situation is a little more complicated.

What is the difference between the case $d = 7$ and $d = 9$, then? We see that $\gcd(7, 12) = 1$ and the graph is connected, whereas $\gcd(9, 12) = 3$ and the graph is not connected. It looks like all the other values of d follow a similar pattern. If $\gcd(d, n) = 1$, the graph is connected, and the graph is not connected otherwise. We now prove that this amended conjecture is correct.

Let's figure out when are two vertices connected. Consider vertices v_i and v_j such that $i < j$. In that case they form an angle of $\frac{2\pi}{n} \cdot s$ where $s = j - i$. Each edge connects vertices that form an angle of $\frac{2\pi}{n} \cdot d$. Then if we start at v_i and following k such edges, we end in a vertex that forms an angle of $\theta = k \cdot \frac{2\pi}{n} \cdot d$ with the starting vertex. If we want to end in v_j after k steps, this angle should be the same as the angle $\frac{2\pi}{n} \cdot s$, which means it should have the form $\frac{2\pi}{n} \cdot s + 2\pi r$ for some integer r. (It is not sufficient to say the angle should be $\frac{2\pi}{n} \cdot s$ because the angle θ could be more than 2π. For example, the angle covered by a path of length 3 from v_0 to v_3 in Figure 1c is $3 \cdot \frac{10\pi}{12} > 2\pi$.) So the angle between v_i and v_j satisfies

$$k \cdot \frac{2\pi}{n} \cdot d = \frac{2\pi}{n} \cdot s + 2\pi r.$$

We can divide by $\frac{2\pi}{n}$ and rearrange the equation to get

$$s = kd - rn. \tag{1}$$

Since $\gcd(d, n)$ divides d and n, it divides the right-hand side of (1). Hence, s is divisible by $\gcd(d, n)$ as well, and we have proved the following claim.

Claim 1. *There is a path from vertex v_i to vertex v_j only if $s = j - i$ is a multiple of* $\gcd(d, n)$.

In case when $\gcd(d, n) > 1$, Claim 1 tells us that the graph is disconnected. For example, vertices v_0 and v_1 cannot be connected by a path in this case.

To complete the proof of our conjecture, we show that if s is divisible by $\gcd(d,n)$, there is a path of length k from v_i to v_j for some k. This will prove that the graph is connected when $\gcd(d,n) = 1$.

Recall the following result about the greatest common divisor which we proved in Problem 4 of Homework 4.

Lemma 1. *Let $a, b \in \mathbb{N}$. Then there exist integers u and v such that $\gcd(a,b) = u \cdot a + v \cdot b$.*

Lemma 1 implies that there exist integers u and v such that $\gcd(d,n) = ud + vn$. Hence, if $s = t \cdot \gcd(d,n)$, we have $t \cdot \gcd(d,n) = tud + tvn$, and there is a path of length $k = tu$ from v_i to v_j. This proves the converse of Claim 1.

Claim 2. *Suppose i and j with $i < j$ are such that $\gcd(d,n)$ divides $s = j - i$. Then there is a path from vertex v_i to vertex v_j.*

Note that $s \in \{0, \ldots, n-1\}$. If $\gcd(d,n) = 1$, every possible value of s is a multiple of $\gcd(d,n)$, and, therefore, the graph is connected by Claim 2. On the other hand, if $\gcd(d,n) > 1$, Claim 1 implies that vertices v_0 and v_1 are not connected because $1 = 1 - 0$ is not divisible by $\gcd(d,n)$.

For the sake of completeness, let's restate our answer: The graph is connected if and only if $\gcd(d,n) = 1$. (Upon further inspection, we see that the graph is split into $\gcd(d,n)$ cycles of length $n/\gcd(d,n)$, but we don't need that for this problem.)

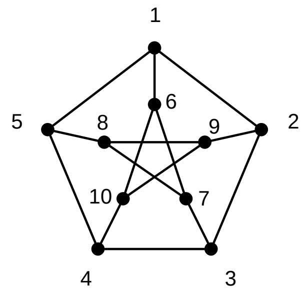
Problem 3

Part One

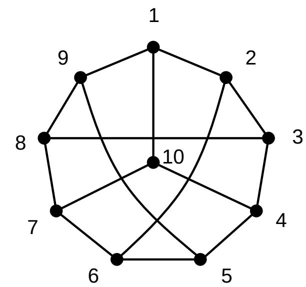
Recall the definition of isomorphism; two graphs G and H are isomorphic if and only if there is a one-to-one function f between the vertices of G and H such that $\forall u, v \in V : (u,v) \in G \iff (f(u), f(v)) \in H$. To show that two graphs are isomorphic, we can provide the function f. To show that two graphs are non-isomorphic, we make use of the fact that two isomorphic graphs will share the same structural properties. Examples of these properties are number of edges/vertices, number of strongly connected components, whether the graph is planar, etc.

We first note that graph G_2 has 16 edges, in contrast to graphs G_1, G_3, and G_4 which all have 15. We know that in order for graphs to be isomorphic, they must have an equal number of vertices and edges; so straightaway we know that G_2 is not isomorphic to any of the other graphs. Alternately, G_2 has two vertices of degree four (those being 8 and 10); there are no such counterparts in any of the other three graphs.

A quick check shows that graphs G_1 and G_3 are isomorphic. Assigning vertices starting with 1:1, and seeking to preserve symmetry in both graphs, we end up with



(a) G_1



(b) G_3

	$G_1(v)$	$G_3(v)$
1	1	
6	10	
10	7	
7	4	
4	6	
3	5	
5	2	
2	9	
8	3	
9	8	

Comparing the edges in the two graphs, we have

$G_1(u, v)$	$G_3(u, v)$
$1, 2$	$1, 9$
$1, 5$	$1, 2$
$1, 6$	$1, 10$
$2, 3$	$9, 5$
$2, 9$	$9, 8$
$3, 4$	$5, 6$
$3, 7$	$5, 4$
$4, 5$	$6, 2$
$4, 10$	$6, 7$
$5, 8$	$2, 3$
$6, 7$	$10, 4$
$6, 10$	$10, 7$
$7, 8$	$4, 3$
$8, 9$	$3, 8$
$9, 10$	$8, 7$

So G_1 and G_3 are isomorphic.

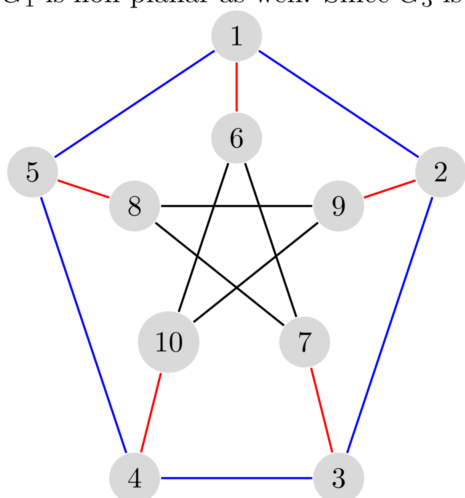
Comparing G_1 and G_4, we note that if all edges are preserved between graphs, all cycles must be preserved as well Note, however, that in G_4, there are a number of cycles of length 4 (for example, 1, 6, 9, 5). It can be quickly verified that no cycle of this length exists in G_1. So we can conclude that G_1 and G_4 are not isomorphic.

Since isomorphism is a transitive relation, we can further conclude that G_3 and G_4 are not isomorphic. We have compared all pairs, and thus are finished.

Part Two

We can see immediately that G_4 is a planar graph; its representation as given has no intersecting edges.

For G_1, we note that if we contract the edges (1, 6), (2, 9), (3, 7), (4, 10), and (5, 8), this will cause the minor G_1' as seen below. Then, G_1' is isomorphic to K_5, which we know to be non-planar. So G_1 is non-planar as well. Since G_3 is isomorphic to G_1, by the same argument G_3 is non-planar.



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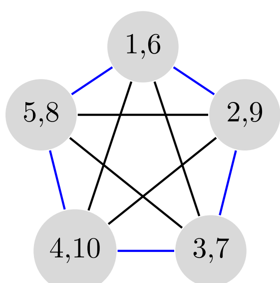
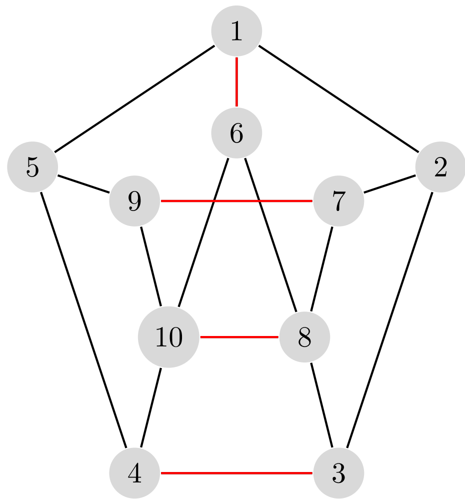


An alternate proof that G_1 is non planar goes as follows: we assume for the moment that G_1 is planar. Note that the smallest cycle in G_1 is of length 5. Considering the set of edge-face pairs $B = \{(e, f) \in E \times F \mid e \text{ is on the border of } f\}$, we see that $|B| \geq 5\,|F|$. However, we also know that every individual edge can only border two faces; so $|B| \leq 2\,|E|$. However, applying Euler's formula (which states that $|V| - |E| + |F| = 2$), we see that if G_1 is a planar graph, $|F| = 7$. Since $|E| = 15$, this gives the inequality $35 \leq |B| \leq 30$, a contradiction. So G_1 cannot be planar.

Finally, for G_2, we note that if we contract the edges (1, 6), (3, 4), (7, 9) and (8, 10), the resulting graph G_2' will contain only the edges (5, 1), (5, 7), (5, 3), (2, 1), (2, 7), (2, 3), and (8, 1), (8, 7), (8, 3). Then as seen below, this minor G_2' is isomorphic to $K_{3,3}$, which we know to be non-planar. So G_2 is non-planar as well.



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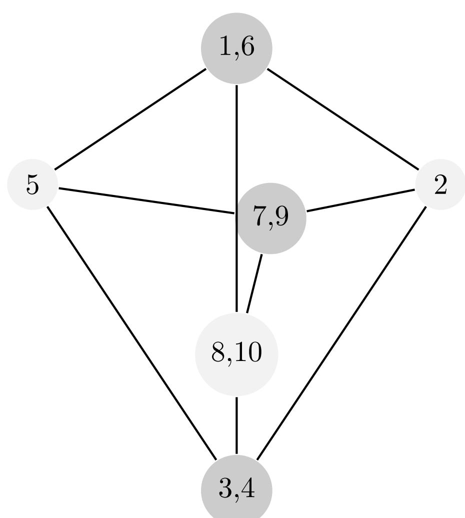

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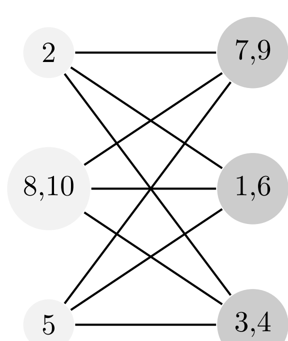

Alternately, we can prove that G_2 is nonplanar by contradiction; assuming that G_2 is planar, we see that according to Euler's formula $|F| = 2 + 16 - 10 = 8$. We note that in this graph, there exists only one cycle of length 3; further, vertex 6 is involved in three cycles of length 5. Considering the set of vertex-face pairs $B = \{(v, f) \in V \times F \mid v \text{ is on the border of } f\}$, we see that $|B| \geq 3 \cdot 5 + 1 \cdot 3 + 4 \cdot 4 = 34$ (Three faces with five vertices, one face with three vertices, and the remaining faces with more than three vertices each). However, we also note that any individual vertex can at most be connected to a number of faces equal to its degree. This gives $|B| \leq 32$. But now $34 \leq |B| \leq 32$, a contradiction. So G_2 cannot be planar.

Problem 4

Part a

Let $G = (V, E)$ be a simple planar graph without self-loops. We show that every connected component contains a vertex of degree 5 or less. Assume without loss of generality that G is connected and let F be the set of its faces.

We argue by contradiction. Suppose that all vertices in G have degree at least 6. We show that this causes a violation of Euler's formula:

$$|V| - |E| + |F| = 2. \tag{2}$$

We proved in Lecture 18 that $\sum_{v \in V} \deg(v) = 2|E|$, and by our assumption that every vertex has degree at least 6 we have $\sum_{v \in V} \deg(v) \geq \sum_{v \in V} 6 = 6|V|$. It follows that $2|E| \geq 6|V|$, and $|E| \geq 3|V|$. We rewrite this as

$$|V| \leq \frac{|E|}{3}. \tag{3}$$

Next, G is simple and doesn't have self-loops, which means that every face has at least three edges on its border. Having just one edge on the border of a face would imply that edge is a self-loop, while G doesn't have any self-loops by assumption, and having only two edges on the border of a face would imply the two edges are between the same pair of vertices, thus contradicting the assumption that G is simple.

So consider the set $B = \{(e, f) \in E \times F \mid e \text{ is on the boundary of } f\}$. We bound its size in two different ways.

First, every edge can be on the boundary of at most two faces, so

$$|B| = \sum_{e \in E} (\text{number of faces with } e \text{ on their border}) \leq \sum_{e \in E} 2 = 2|E|.$$

Second, since every face has at least 3 edges on its border, we have

$$|B| = \sum_{f \in F} (\text{number of edges on the border of } f) \geq \sum_{f \in F} 3 = 3|F|.$$

Combining these two observations yields $3|F| \leq |B| \leq 2|E|$, so

$$|F| \leq \frac{2|E|}{3}. \tag{4}$$

Substituting (3) and (4) into the left-hand side of (2) yields $|V| - |E| + |F| \leq \frac{1}{3}|E| - |E| + \frac{2}{3}|E| = 0$. But the right-hand side of (2) is 2, so we have a contradiction with (2). It follows that every simple planar graph without self-loops has at least one vertex of degree at most 5.

Part b

We prove that we can color the vertices of a planar graph G without self-loops using six colors by induction on the number of vertices. The main idea is that if a vertex v has degree 5 or less, we can defer assigning a color to it until the very end. This is because its neighbors can have at most five different colors in any coloring, so there will be one color left for v. We now proceed with the proof.

For the base case, consider a simple planar graph with 1 vertex. We can assign an arbitrary color to this vertex to get a six-coloring.

Now assume that we can color any simple planar graph H on n vertices without self-loops using at most six colors. Consider a simple planar graph G on $n + 1$ vertices without self-loops.

By part (a), G has a vertex of degree at most 5. Let v be one such vertex. Consider the subgraph $G' = (V', E')$ of G formed by all vertices of G except for v, and all edges of G except for those incident on v. The resulting subgraph G' is a simple planar graph on n vertices without self-loops because it's a subgraph of a simple planar graph without self-loops. Thus, six colors are sufficient for coloring the vertices of G' by the induction hypothesis. Since all edges between two of vertices in V' that appear in G also appear in G', we get a valid coloring of all vertices of G except for v. To complete the six-coloring of G, just color v with a color that is not used to color any of its (at most five) neighbors in G. This completes the proof.

Problem 5

Each person can shake hands at most six times. Since Alice received seven different replies, therefore these replies were the seven different numbers in the range [0,6]. Let us label everyone, except Alice, as P_i where i is the number of times that person shook hands.

Consider the person P_6. P_6 shook hands six times. Since a person cannot shake hands with their spouse, therefore P_6 shook hands with everyone else at the party except her spouse. On the other hand, P_0 did not shake hands with any one at the party. Apart from P_0 and P_6, there were only six other people at the party. So P_6 shook hands with all of them. Hence, P_0 and P_6 are a couple and that Alice shook hands with P_6. Note that Alice and P_0 did not shake hands since P_0 did not shake hands with any one.

If we remove P_0 and P_6 from the context , we would left with people $P_1, ..., P_5$ and Alice. Also, we need to discount the handshakes P_0 and P_6 were involved in. P_0 did not shake hands with any one, while P_6 shook hands with every one. So, the remaining P_i's shook hands $i - 1$ times with the people left. Now consider P_5. P_5 shook hands four times with people P_1 to P_4 and Alice. Also, P_5 did not shake hands with her spouse. We also know that P_1 did not shake hands with any of the remaining persons. This means that P_5 shook hands with P_2, P_3, P_4 and Alice. Therefore, P_5 and P_1 are a couple. We can also infer that Alice shook hands with P_5 and not with P_1.

We now reduce the problem further to that involving four persons – P_2, P_3, P_4 and Alice. We have the following information – P_2 shook hands with none of the remaining persons, P_3 shook hand of one person, P_4 shook hands with two of the remaining persons. Similar as above, we can reason that P_4 and P_2 are a couple and that Alice shook hands with P_4 and that she did not shake hands with P_2. Since we are now left with only P_3 and Alice, therefore P_3 has to be Bob.

Hence, Alice and Bob both shook hands three times each.