CS/Math 240: Intro to Discrete Math 5/5/2011

Solutions to Homework 11

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Problem 1

Part (a)

There is a bijection with sequences of the form:

$$(\text{value of pair}, \text{suits of pair}, \text{value of other three cards}, \text{suits of other three cards}).$$

Thus, the number of hands with a single pair is:

$$13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3 = 1,098,240.$$

Alternatively, there is also a 3!-to-1 mapping to the tuple:

$$\begin{aligned}&(\text{value of pair}, \text{suits of pair},\\ &\text{value 3rd card}, \text{suit 3rd card}, \text{value 4th card}, \text{suit 4th card}, \text{value 5th card}, \text{suit 5th card}).\end{aligned}$$

Thus, the number of hands with a single pair is:

$$\frac{13 \cdot \binom{4}{2} \cdot 12 \cdot 4 \cdot 11 \cdot 4 \cdot 10 \cdot 4}{3!} = 1,098,240.$$

Part (b)

This is the set of all hands minus the hands with either no kings or one king:

$$\binom{52}{5} - \binom{48}{5} - 4 \cdot \binom{48}{4} = 108,336.$$

Alternatively, this is also the set of all hands of two, three, or four kings:

$$\binom{48}{3}\binom{4}{2} + \binom{48}{2}\binom{4}{3} + \binom{48}{1}\binom{4}{4} = 108,336.$$

Part (c)

There are $\binom{51}{4}$ hands containing the ace of spades, an equal number containing the ace of clubs and $\binom{50}{3}$ containing both. By the inclusion-exclusion formula, the hands containing one or the other or both, equals

$$\binom{51}{4} + \binom{51}{4} - \binom{50}{3} = 480,200.$$

Part (d)

There is a bijection from the solutions of the equation to the binary strings containing n zeros and k ones, where x_0 is the number of 0s preceding the first 1, x_k is the number of 0s following the last 1, and x_i is the number of 0s between the ith and $(i+1)$st 1 for $0 < i < k$. Thus, the number of solutions is

$$\binom{n+k}{k}.$$

Part (e)

There is a bijection from the solutions of

$$\sum_{i=0}^{k} x_i \leq n$$

where each x_i, $0 \leq i \leq k$, is a nonnegative integer, to the solutions of

$$\sum_{i=0}^{k+1} x_i = n,$$

where each x_i, $0 \leq i \leq k+1$, is a nonnegative integer, namely the mapping

$$(x_0, x_1, \ldots, x_k) \to (x_0, x_1, \ldots, x_k, n - \sum_{i=0}^{k} x_i).$$

Therefore, by part (d), the number of solutions is

$$\binom{n+k+1}{k+1}.$$

Part (f)

Pair up students by the following procedure. Line up the students and pair the first and second, the third and fourth, the fifth and sixth, etc. The students can be lined up in $(2n)!$ ways. However, this overcounts by a factor of 2^n, because we would get the same pairing if the first and second students were swapped, the third and fourth were swapped, etc. Furthermore, we are still overcounting by a factor of $n!$, because we would get the same pairing even if pairs of students were permuted, e.g., the first and second were swapped with the ninth and tenth. Therefore, the number of pairings is:

$$\frac{(2n)!}{n! \cdot 2^n}.$$

Part (g)

There are $\binom{n}{2}$ potential edges between distinct vertices and n selfloops, each of which may or may not appear in a given simple graph. Therefore, the number of simple undirected graphs on the vertex set $\{1, 2, \ldots, n\}$ is

$$2^{\binom{n}{2}+n}.$$

Problem 2

Part (a)

Lets first work on counting the permutations of $\{1, 2, \dots, n\}$ that are not derangements. Let us denote that set by S. Let S_i be the set of all permutations $(x_1, x_2, \dots, x_n)$ of the set $\{1, 2, \dots, n\}$ such that $x_i = i$. Then $S = \cup_{i=1}^{n} S_i$.

By the inclusion-exclusion principle, we have that

$$\begin{aligned} |S| &= \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|+1} |\cap_{i \in I} I_i| \qquad (1) \\ &= \sum_i |S_i| - \sum_{i,j} |S_i \cap S_j| + \sum_{i,j,k} |S_i \cap S_j \cap S_k| - \dots + (-1)^{n+1} |S_1 \cap S_2 \cap \dots \cap S_n|, \end{aligned}$$

where in each summation of the latter formula, the subscripts are distinct elements of $\{1, 2, \dots, n\}$.

What is $|S_i|$? There is a bijection between permutations of $\{1, 2, \dots, n\}$ with i in the ith position and unrestricted permutations of $\{1, 2, \dots, n\} - \{i\}$. Therefore, $|S_i| = (n-1)!$.

How about $|S_i \cap S_j|$ where i and j are distinct? The set $S_i \cap S_j$ consists of all permutations with i in the ith position and j in the jth position. Thus, there is a bijection with permutations of $\{1, 2, \dots, n\} - \{i, j\}$, and so $|S_i \cap S_j| = (n-2)!$.

More generally, if $i_1, i_2, \dots, i_k$ are all distinct, then the same argument gives that $|S_{i_1} \cap S_{i_2} \cap \dots \cap S_{i_k}| = (n-k)!$. In (1) there is one such term for each k-element subset of the n-element set $\{1, 2, \dots, n\}$, i.e., there are $\binom{n}{k}$ such terms. Thus, (1) becomes

$$|S| = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)! = n! \cdot \sum_{k=1}^{n} (-1)^{k+1}/k!.$$

Thus, the number of derangements of length n equals

$$n! - |S| = n! \cdot \sum_{k=0}^{n} (-1)^k / k!. \qquad (2)$$

Part (b)

Since there are $n!$ permutations of length n, by (2) the fraction of permutations of length n that are derangements equals $\sum_{k=0}^{n} (-1)^k / k!$. Since $e^x = \sum_{k=0}^{\infty} x^k / k!$, the fraction converges to e^x for $x = -1$, i.e., the fraction is asymptotically equivalent to $1/e$.

Problem 3

Partition the $n \times n$ square into n^2 unit squares. Each of the $n^2 + 1$ points lies in one of these n^2 unit squares. (If a point lies on the boundary between squares, assign it to a square arbitrarily.) Therefore, by the pigeonhole principle, there exist two points in the same unit square. And the distance between those two points can be at most $\sqrt{2}$.

Problem 4

Let S denotes the set of all length-n sequences of 0's, 1's and a single *.

Let $P = \{0, \ldots, n-1\} \times \{0,1\}^{n-1}$. On the one hand, there is a bijection from P to S by mapping (k, x) to the word obtained by inserting a * just after the kth bit in the length-$(n-1)$ binary word, x. So

$$|S| = |P| = n2^{n-1} \tag{3}$$

by the product rule.

On the other hand, every sequence in S contains between 1 and n nonzero entries since the $*$, at least, is nonzero. The mapping from a sequence in S with exactly k nonzero entries to a pair consisting of the set of positions of the nonzero entries and the position of the * among these entries is a bijection, and the number of such pairs is $\binom{n}{k}k$ by the generalized product rule. Thus, by the sum rule:

$$|S| = \sum_{k=1}^{n} k \binom{n}{k}$$

Equating this expression and the expression (3) for $|S|$ proves the identity.