

## Lecture 3 : Predicates and Sets

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## DRAFT

Last time we discussed propositions, which are statements that are either true or false. Today we extend them to predicates, which allow us to capture properties of multiple objects at once.

### 3.1 Predicates

We begin with a definition.

**Definition 3.1.** A predicate is associated with some underlying domain  $D$ , and we define it as a mapping from  $D$  to propositions.

We can also view a predicate as a parametrized proposition, where the parameter ranges over the domain  $D$ . We now give some examples of predicates.

*Example 3.1:*

Over  $D = \{0, 1, 2, \dots\}$ , consider the predicate  $\text{Even}(x)$ :  $x$  is even.

- $\text{Even}(2)$  means “2 is even”, which is a true statement.
- $\text{Even}(3)$  means “3 is even”, which is a false statement.

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*Example 3.2:*

Still over  $D = \{0, 1, 2, \dots\}$ , consider the predicate  $\text{Prime}(x)$ :  $x$  is prime.

This predicate is true if  $x$  is prime, and false otherwise. That is,  $\text{Prime}(3)$  is true, and  $\text{Prime}(4)$  is false.

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#### 3.1.1 Predicates and Sets

There is a relationship between subsets of  $D$  and predicates on  $D$ . For a predicate  $P$  on  $D$ , the set  $S = \{x \in D \mid P(x)\}$  is the set of elements  $x$  from the domain  $D$  for which the predicate  $P$  is true. In the other direction, suppose we have a subset  $T \subseteq D$ . Then we can construct the predicate  $P(x)$ :  $x \in T$ .

#### 3.1.2 New Predicates From Old

We can compound predicates the same way we put together propositions. We can use the operators  $\neg$ ,  $\vee$ ,  $\wedge$ ,  $\Rightarrow$ , and  $\iff$  to form more complex predicates. There is also an additional operation defined on predicates called *quantification*. Quantification allows us to express concepts such as “all elements of  $D$ ” or “some elements of  $D$ ”. The former is called *universal quantification* and the latter is called *existential quantification*.

### 3.1.3 Universal Quantification

For *universal quantification*, we use the notation  $(\forall x)P(x)$  to mean “for all  $x$ ,  $P(x)$  holds”. The statement  $(\forall x)P(x)$  is true if  $P(x)$  is a true proposition for every  $x$  in the domain. It is false if there is at least one  $x$  in the domain for which  $P(x)$  is a false proposition.

Let’s see an example and make a few remarks.

*Example 3.3:*

$(\forall x)\text{Even}(x)$  is a false proposition. To see this, we just need to find one  $x$  for which  $\text{Even}(x)$  is false. One counterexample is  $x = 1$ .

$(\forall x)\text{Even}(x) \iff \neg\text{Even}(x + 1)$  is a true proposition. If  $x$  is even, then  $x + 1$  is odd, and if  $x$  is odd, then  $x + 1$  is even. Thus, the equivalence holds for every  $x$ .  $\square$

What if our domain  $D$  is empty? Is the statement  $(\forall x)P(x)$  true? We cannot find an element of the domain for which  $P$  doesn’t hold because  $D$  is empty. Therefore, the statement is true.

In the examples above, we omitted the domain  $D$ . If we want to state the domain explicitly, we can say  $(\forall x \in D)$  instead of just  $(\forall x)$ . It is common in practice to omit the domain when it is clear from the context what the domain is; however, in order to be precise, we should always specify the domain.

### 3.1.4 Existential Quantification

For *existential quantification*, we write  $(\exists x)P(x)$  to mean “there exists an  $x$  for which  $P$  holds”. The statement  $(\exists x)P(x)$  is true if  $P(x)$  is a true proposition for at least one  $x$  in the domain. It is false if there is no  $x$  in the domain for which  $P(x)$  is a true proposition.

Again, we illustrate this on an example.

*Example 3.4:*

$(\exists x)\text{Even}(x)$  is a true proposition. To see this, we just need to find one  $x$  for which  $\text{Even}(x)$  holds. For example  $x = 0$  is an example of an  $x$  for which  $\text{Even}(x)$  holds.

$(\exists x)\text{Even}(x) \wedge \text{Prime}(x) \wedge x > 2$  is false because all primes larger than 2 are odd.  $\square$

Here we should also warn the reader that the use of the word “any” in English is ambiguous and we have to rely on the context in order to understand its meaning. Sometimes, this word is equivalent to “some”, whereas in other situations it could be equivalent to “all”. For example, consider the sentence: “If you can solve any homework problem, you will get an A.” Almost certainly, “any” here means “all” and not “at least one”, although most students would wish the opposite were true. When speaking in the language of mathematics, we should avoid using such words, and use unambiguous expressions such as “for all” or “there exists” instead.

### 3.1.5 Universal and Existential Quantifiers Are Related

There is a duality between existential and universal quantification. Consider the sentences: “Not everyone likes sunshine,” and “Some people don’t like sunshine.” These two sentences say the same thing. To see that, let’s write them in mathematical notation. The domain is the set of all people, and we have the predicate  $S(x)$ :  $x$  likes sunshine. The first sentence would then be

$$\neg(\forall x)S(x), \tag{3.1}$$

and the second sentence would be

$$(\exists x)\neg S(x). \tag{3.2}$$

In fact, no matter what the domain is and what  $S$  is, the statements (3.1) and (3.2) are logically equivalent. If it’s not true that a proposition holds for every element of the domain, there must be

an element of the domain for which the proposition doesn't hold. The converse of the last sentence is also true, i.e., if there is an element of the domain for which a proposition doesn't hold, it is not the case that the proposition holds for all elements of the domain. The statements  $\neg(\exists x)S(x)$  and  $(\forall x)\neg S(x)$  are logically equivalent for the same reason.

In the case of a statement with an existential quantifier, proving that it's false is harder than proving that it is true. In order to prove that such a statement is false, we need to show that the existentially quantified statement is false for every element of the domain, whereas to prove that it's true, we just need to find an element of the domain for which the existentially quantified statement holds. Observe that the exact opposite is true about universally quantified statements. Again, we see there is a duality.

There is a similar duality between conjunctions and disjunctions. We could write  $\forall$  as a “big AND” and an  $\exists$  as a “big OR”. For example, we could rewrite the statement  $(\forall x)\text{Prime}(x)$  as “2 is prime, and 3 is prime, and 4 is prime, and 5 is prime, ...” Putting a “not” in front of the sentence would give us a sentence of the form (3.1). We could then rewrite it in the form (3.2) as “Either 2 is not prime, or 3 is not prime, or 4 is not prime, or 5 is not prime, ...”

Another instance of the duality between conjunctions and disjunctions is *DeMorgan's law*, which states the following two logical equivalences.

- $\neg(P \wedge Q)$  is logically equivalent to  $\neg P \vee \neg Q$ .
- $\neg(P \vee Q)$  is logically equivalent to  $\neg P \wedge \neg Q$ .

### 3.1.6 Multiple Quantifiers and Domains

Recall Goldbach's conjecture from last lecture: “Every even integer greater than 2 can be written as a sum of two primes”. There are actually two domains involved in the statement of Goldbach's conjecture, namely the even integers greater than 2 and the prime numbers. Moreover, there are multiple quantifiers present in the statement, which becomes apparent if we rephrase it as “*for all* even numbers  $x$  greater than two *there exist* two primes that add up to  $x$ ”.

Let us now rewrite the statement of Goldbach's conjecture using predicate notation. First, we need to give names to our domains. Let  $D_1 = \{x \mid \text{Even}(x) \wedge x > 2\}$  be the set of even integers larger than 2, and  $D_2 = \{x \mid x \text{ is an integer such that } x \text{ is prime}\}$  the set of primes. Now we can write Goldbach's conjecture as follows:

$$(\forall x \in D_1)(\exists y \in D_2)(\exists z \in D_2)x = y + z \quad (3.3)$$

Now let's write (3.3) using only one domain,  $D = \{0, 1, 2, \dots\}$ , and our predicates Even and Prime. We first eliminate  $D_1$ . Start with the predicate

$$(\forall x \in D_1)P(x) \quad \text{where} \quad P(x) : (\exists y \in D_2)(\exists z \in D_2)x = y + z. \quad (3.4)$$

Note that an integer  $x$  is in  $D_1$  if  $x$  is even and greater than 2, i.e., if  $\text{Even}(x) \wedge x > 2$ , so we can rewrite (3.4) as

$$(\forall x \in D)(\text{Even}(x) \wedge x > 2) \Rightarrow P(x) \quad (3.5)$$

Note that we have increased the size of the domain. Thus, we may worry that our new statement (3.5) is not the same as the one we started with, so let's see that (3.4) and (3.5) are logically equivalent. Recall that an implication can only be false if the premise is true and the consequence is false. In our case, only if  $x$  is an even integer greater than 2 that does not satisfy  $P$ . We have increased the domain from  $D_1$  to  $D$ , but the implication will hold for all  $x$  that are in  $D$  but not in

$D_1$  by definition. Therefore, only an  $x$  from  $D_1$  can invalidate the truth of (3.5), and such  $x$  would also invalidate the truth of (3.4). Conversely, an  $x$  that invalidates (3.4) would also invalidate (3.5), so we can conclude that (3.4) and (3.5) are logically equivalent.

Now we need to rewrite  $P(x)$  which still has  $D_2$  in it. We can write  $P(x)$  as

$$(\exists y \in D_2)Q(x, y) \quad \text{where} \quad Q(x, y) : (\exists z \in D_2)x = y + z$$

An integer  $y$  is prime if  $\text{Prime}(y)$  is true. We are only interested in primes that add up to  $x$ , so we “and” that requirement with  $Q(x, y)$ . We do this instead of using an implication because if we used an implication, a number that is not prime would give us a false premise, thus making the implication true, which would make the existentially quantified statement true as well. But we only want primes to make  $P(x)$  true, so we rewrite  $P(x)$  using the domain  $D$  as

$$P(x) : (\exists y \in D)\text{Prime}(y) \wedge Q(x, y). \quad (3.6)$$

We would use the same strategy to eliminate the use of  $D_2$  in the expression for  $Q(x, y)$  to get

$$Q(x, y) : (\exists z \in D)\text{Prime}(z) \wedge x = y + z. \quad (3.7)$$

So now let’s combine all that we just did to rewrite Goldbach’s conjecture using a single domain  $D = \{0, 1, 2, \dots\}$ . We start with (3.4):

$$(\forall x \in D_1)(\exists y \in D_2)(\exists z \in D_2)x = y + z.$$

Now apply (3.5) to get

$$(\forall x \in D)(\text{Even}(x) \wedge x > 2) \Rightarrow ((\exists y \in D_2)(\exists z \in D_2)x = y + z).$$

Now let’s start getting rid of  $D_2$ , so we apply (3.6) and get

$$(\forall x \in D)(\text{Even}(x) \wedge x > 2) \Rightarrow ((\exists y \in D)\text{Prime}(y) \wedge ((\exists z \in D_2)x = y + z)).$$

Finally, apply (3.7) to get an expression that uses only a single domain.

$$(\forall x \in D)(\text{Even}(x) \wedge x > 2) \Rightarrow [(\exists y \in D)\text{Prime}(y) \wedge ((\exists z \in D)\text{Prime}(z) \wedge x = y + z)].$$

An interesting operation is to move all the quantifiers to the front. Consider the statement

$$(\forall x \in D)(x \in D_1 \Rightarrow (\exists y \in D_2)Q(x, y)), \quad (3.8)$$

which is a rewording of Goldbach’s conjecture. We pull out the existential quantifier to get

$$(\forall x \in D_2)(\exists y \in D)x \in D_1 \Rightarrow P(x, y). \quad (3.9)$$

We argue that (3.8) and (3.9) are logically equivalent. In (3.8), we pick the  $y$  in the consequence of the implication. Picking the  $y$  beforehand corresponds to (3.9). If the premise in the implication in (3.8) is true, we need to pick a  $y$  that satisfies the consequence. Note that such a  $y$  only depends only on  $x$ , and  $x$  is already known at the point in (3.9) where we pick  $y$ , so we can pick the same  $y$  beforehand and the consequence of the implication will still hold. Conversely, if we pick a good

$y$  (i.e., one that makes the consequence of the implication true) beforehand, we can pick that same  $y$  it later (inside the consequence). Hence, (3.8) and (3.9) are logically equivalent.

Now say we move the existential quantifier even further left and obtain

$$(\exists y \in D_2)(\forall x \in D)x \in D_1 \Rightarrow Q(x, y). \quad (3.10)$$

We may ask whether this additional move yields a predicate that is logically equivalent to the previous one. Let's express the new predicate (3.10) in English. It says that "there exists a prime,  $y$ , such that for all even integers  $x$  greater than 2, there exists a prime  $z$  such that  $x = y + z$ ." This is a false statement. To see that, let  $y$  be any prime. We now argue that with this fixed  $y$ , the proposition

$$(\forall x \in D)x \in D_1 \Rightarrow Q(x, y)$$

is false. To do so, we find an even integer  $x$  such that the difference  $z = x - y$  is not a prime. For example, pick  $x = 4 \cdot y$ . That is an even integer because it's a product of an even number with  $y$ . Then  $x - y = 4y - y = 3y$ , but that is not prime because 3 divides  $3y$  and  $3y \neq 3$ .

Note that (3.10) is false. If (3.9) and (3.10) were logically equivalent, we would disprove Goldbach's conjecture. But there has been almost no progress on proving or disproving Goldbach's conjecture since it was stated in 1742, which suggests that (3.9) and (3.10) are not logically equivalent. Somebody in the last 350 years must have thought about proving that (3.9) and (3.10) are logically equivalent in an attempt to disprove Goldbach's conjecture, and the attempt most likely did not work.

Therefore, let's analyze the relationship between (i):  $(\forall x)(\exists y)P(x, y)$  and (ii):  $(\exists y)(\forall x)P(x, y)$ . Note that if (ii) is true, it implies that (i) is true as well, but implication doesn't hold in the other direction. It holds in some cases, such as  $P(x, y)$ : true. On the other hand, take  $P(x, y)$ :  $x = y$ . Then although  $(\forall x)(\exists y)P(x, y)$  is true, we see that  $(\exists y)(\forall x)P(x, y)$  is false. The lesson is that we can move a universal quantifier to the left of an existential quantifier without turning a true statement into a false one. We remark that this comes at a price. Such an operation loses information because now we're saying that for every  $x$  we can find a  $y$  as opposed to having a  $y$  that works for all  $x$ .

Returning back to the fact that (i) being true doesn't imply (ii) is true, consider the contrapositive of that statement. This says that if a statement of the form (ii) is false, we cannot conclude that the corresponding statement of the form (i) is false. Observe that (3.10) has the form (ii) and is false, so we cannot conclude from it that Goldbach's conjecture, which has the form (i), is false.

### 3.1.7 Turning Sentences into Predicates

Let's get some more experience going between predicates and English sentences. In Task 3.1, we convert English sentences into predicates.

The solutions in Task 3.1 use some shortcuts in the notation for the sake of readability. In particular, some parentheses are omitted. Strictly speaking, we never defined the meanings of these notational shortcuts, but using them is common practice, just like it is common practice to omit domains in quantified expression. When using these shortcuts, we just have to make sure that we can parse the final expression in a unique way.

Here are some common notational shortcuts.

- We don't put parentheses around parts of an "and" of multiple statements, that is, we write  $A \wedge B \wedge C$  instead of  $(A \wedge B) \wedge C$  or  $A \wedge (B \wedge C)$ .

- We also make use of priority rules similar to operator precedence in programming languages, which also allows us to omit some parentheses. An expression in parentheses has highest precedence, followed by an “and” or an “or”. When all those are evaluated, we evaluate all implications. Finally, when everything else is done, we evaluate all equivalences.

### Task 3.1: Turning Sentences into Predicates

#### Problem Statement

Let  $D$  be the set of all people. We have the following predicates:

- $F(x)$ :  $x$  is female
- $M(x)$ :  $x$  is male
- $P(x, y)$ :  $x$  is a parent of  $y$
- $W(x, y)$ :  $x$  is married to  $y$

Express the following English statements using the predicates above:

1. Everyone has a father and a mother.
2. Some children's parents are not married.
3. Not every married couple has children.
4. No one is married to an uncle.

#### Solution

1. The translation of the first sentence, “Everyone has a father and a mother,” is

$$(\forall x) [(\exists y)M(y) \wedge P(y, x)] \wedge [(\exists z)F(z) \wedge P(z, x)].$$

Let's see how one could arrive at this solution. The sentence says something about a person, so we need a variable representing that person. Let  $x$  be that variable.

Now  $x$  has a father and  $x$  has a mother. This is a statement of the form  $A \wedge B$  where  $A$  is “ $x$  has a father” and  $B$  is “ $x$  has a mother”. Since *every*  $x$  is supposed to have a father and a mother, we arrive at the statement

$$(\forall x)A \wedge B. \tag{3.11}$$

Next, let's express  $A$  in terms of our predicates. Since  $x$  has a father, that means there exists some person who is  $x$ 's father. We need a variable for this person, say  $y$ . So  $y$  is  $x$ 's father, which means that it's male and that it's a parent of  $x$ . We express that as  $M(y) \wedge P(y, x)$ . Thus,  $A = (\exists y)M(y) \wedge P(y, x)$ . Using the same argument, we write  $B = (\exists z)F(z) \wedge P(z, x)$ .

Finally, we substitute our expressions for  $A$  and  $B$  into (3.11) to get the answer.

We could have actually used  $y$  in the second existential predicate (in the second set of square brackets) because the “scope” of the first  $y$  ends with the first closing square bracket. We could also express this using all quantifiers at the beginning as follows:

$$(\forall x)(\exists y)(\exists z)M(y) \wedge P(y, x) \wedge F(z) \wedge P(z, x)$$

Note that we cannot use  $y$  twice if we do that.

2. For the second sentence, “Some children's parents are not married,” the translation into a predicate is

$$(\exists x)(\exists y)(\exists z)P(y, x) \wedge P(z, x) \wedge \neg W(y, z) \wedge y \neq z.$$

There are three people mentioned in this sentence: a child and its two parents. Let's use  $x$  for the child and  $y$  and  $z$  for the parents. Now there are three conditions that must be true about these three people.

- (a)  $y$  is  $x$ 's parent. The translation of this is  $P(y, x)$ .
- (b)  $z$  is  $x$ 's parent. The translation of this is  $P(z, x)$ .
- (c) The parents are not married. The translation is  $\neg W(y, z)$ .

Since the three conditions must hold simultaneously, we “and” them together in our final answer. Now the statement says that *some* parents are not married. In other words, there exists a child and two other people who satisfy the three conditions above. Thus,  $x$ ,  $y$  and  $z$  are all existentially quantified and we obtain

$$(\exists x)(\exists y)(\exists z)P(y, x) \wedge P(z, x) \wedge \neg W(y, z). \quad (3.12)$$

That is almost what we want. Say that Bob is a child of Alice. Then we could pick  $x$  to be Bob, pick  $y$  to be Alice, and pick  $z$  to be Alice as well. Since Alice isn't married to herself, this choice would make the existentially quantified statement (3.12) true. But that's not what we are trying to capture with our sentence. Thus, to prevent this situation, we also require a fourth condition, namely that  $y$  and  $z$  are different.

3. The translation of the third sentence, “Not every married couple has children,” is

$$(\exists x)(\exists y)W(x, y) \wedge [(\forall z)\neg P(x, z)] \wedge [(\forall z)\neg P(y, z)]$$

We can rewrite our sentence as “There exists a married couple without children.” This is a statement about two people. We use  $x$  and  $y$  to denote them. We see from our rephrasing that  $x$  and  $y$  are existentially quantified.

Now  $x$  and  $y$  satisfy the following conditions:

- (a) They are married, which translates to  $W(x, y)$ .
- (b)  $x$  doesn't have children, which means that every person  $z$  is not a child of  $x$ . The translation is  $(\forall z)\neg P(x, z)$ .
- (c)  $y$  doesn't have children, which translates to  $(\forall z)\neg P(y, z)$ .

We “and” all these conditions together and get the final answer.

4. The translation of the last sentence is

$$(\forall x)(\forall u) [(\exists p)P(p, x) \wedge ((\exists y)P(y, x) \wedge P(y, u) \wedge p \neq u)] \Rightarrow \neg W(x, u).$$

This is a statement about two people who are married. Call the two people  $x$  and  $u$ . Since nobody is married to an uncle, this means that if  $x$  and  $u$  are married, then  $u$  is not  $x$ 's uncle. Thus, we have a universally quantified statement of the form  $A \Rightarrow B$  where  $A$  is “ $x$  and  $u$  are married” and  $B$  is “ $u$  is not  $x$ 's uncle”. Instead, we can express the contrapositive  $\neg B \Rightarrow \neg A$  which says that if  $u$  is  $x$ 's uncle, then  $x$  and  $u$  are not married.

To express that  $x$  and  $u$  are not married, we write  $\neg W(x, u)$ .

Now let's focus on the statement “ $u$  is  $x$ 's uncle.” If  $u$  is  $x$ 's uncle, one of  $x$ 's parents,  $p$ , is a sibling of  $u$ . Thus, we arrive at the statement

$$(\exists p)P(p, x) \wedge S(p, u)$$

where  $S(p, u)$  is the statement “ $p$  and  $u$  are siblings”.

Now  $p$  and  $u$  are siblings if they have a common parent and  $p \neq u$ , so  $S(p, u)$  is

$$(\exists y)P(y, p) \wedge P(y, u) \wedge p \neq u.$$

Finally, we combine all our partial answers into a complete answer.