CS/Math 240: Introduction to Discrete Mathematics 3/31/2011

Lecture 17 : Equivalence and Order Relations

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Last lecture we introduced the notion of a relation. A relation between sets A and B is a subset of their Cartesian product. We discussed a special case of relations, namely functions, where every element of A is related to at most one element of B. We also mentioned reflexive, symmetric, and transitive relations. Today we combine those three properties and discuss equivalence relations and order relations. After that, we mention graphs which are closely connected to relations.

17.1 Relations Continued

Last time we saw special types of relations. In the list below, R is a relation on A.

- R is reflexive if $(\forall a \in A)\ aRa$.
- R is antireflexive if $(\forall a \in A)\ \neg aRa$.
- R is symmetric if $(\forall a, b \in A)\ aRb \iff bRa$.
- R is antisymmetric if $(\forall a, b \in A)\ (aRb \wedge bRa) \Rightarrow (a = b)$.
- R is transitive if $(\forall a, b, c \in A)\ (aRb \wedge bRc) \Rightarrow aRc$.

We gave some examples of these properties. We summarize them below in Table 17.1. Recall that $a \iff b$ means a and b are propositional formulas with the same truth tables, and $a \equiv_3 b$ means a and b have the same remainder after division by 3.

relation	reflexive	symmetric	transitive
$\subseteq$	yes	anti	yes
$<$	anti	anti	yes
$\leq$	yes	anti	yes
$\mid$	yes	anti	yes
$\iff$	yes	yes	yes
$\equiv_3$	yes	yes	yes

Table 17.1: Properties of some relations.

17.1.1 Equivalence Relations

Definition 17.1. *A relation R on set A is an* equivalence relation *if it is reflexive, symmetric, and transitive.*

Example 17.1: From the relations in Table 17.1, only $\iff$ and $\equiv_3$ are equivalence relations. None of the other relations are symmetric, so they are not equivalence relations either. This may lead you to believe that every symmetric relation is an equivalence relation; however, this is not true. For example $\neq$ is symmetric, but it is not an equivalence relation because it is neither reflexive nor transitive. ⊠

An equivalence relation on A divides the elements of A into clusters of mutually related elements, and where no two elements of different clusters are related. This gives an alternative characterization of equivalence relations, which we explore next.

Definition 17.2. *A* partition *of A is a collection of subsets $A_1, A_2, \dots$ of A such that $\bigcup_i A_i = A$ and $A_i \cap A_j = \emptyset$ whenever $i \neq j$.*

In Definition 17.2, we could also say that the sets $A_1, A_2, \dots$ are *pairwise disjoint*. We call the individual sets A_i *clusters*, *partition classes*, or *equivalence classes*.

We will now show that a partition of A induces an equivalence relation on A, and an equivalence relation on A induces a partition of A. Let's start with an example.

Example 17.2: Consider the set $A = \{1, 2, 3, 4, 5, 6, 7\}$. The subsets $A_1 = \{1, 4, 7\}$, $A_2 = \{2, 5\}$ and $A_3 = \{3, 6\}$ are a partition of A.

Observe that $a \equiv_3 b$ implies a and b are in the same partition class. This is what we have in mind when we say that the partition of A into A_1, A_2 and A_3 is induced by the equivalence relation $\equiv_3$ on A. ⊠

Now we describe the correspondence between partitions of A and equivalence relations on A formally.

Proposition 17.3. *Given a partition $A_1, A_2, \dots$ of A, define R such that aRb if and only if a and b belong to the same partition class A_i. Then R is an equivalence relation.*

Proof. First note that a belongs to the same partition class as itself, so R is reflexive. If a belongs to the same partition class as b, then b belongs to the same partition class as a too, so R is symmetric. Finally, if a is in the same partition class as b and b is in the same partition class as c, then a is in the same partition class as c, so R is also transitive. It follows that R is an equivalence relation. □

Proposition 17.4. *Let R be an equivalence relation on a set A. For each $a \in A$, define the set $[a] = \{b \in A \mid aRb\}$. The collection $P = \{[a] \mid a \in A\}$ is a partition of A.*

In Proposition 17.4, we call $[a]$ the *equivalence class of a*. By saying "collection of sets $[a]$" we mean that we drop duplicates.

There is no reason to believe that the various equivalence classes $[a]$ don't overlap in all sorts of arbitrary ways. Let's start with an example to give ourselves some confidence that the equivalence classes overlap in a very structured way.

Example 17.3: Consider the congruence modulo 3 ($\equiv_3$) relation on $A = \{1, 2, 3, 4, 5, 6, 7\}$. The equivalence classes are:

$$\begin{aligned}
[1] &= \{1, 4, 7\} & & \\
[2] &= \{2, 5\} & & \\
[3] &= \{3, 6\} & & \\
[4] &= \{1, 4, 7\} & &= [1] \\
[5] &= \{2, 5\} & &= [2] \\
[6] &= \{3, 6\} & &= [3] \\
[7] &= \{1, 4, 7\} & &= [1]
\end{aligned}$$

⊠

We make the following observations about Example 17.3.

Observation 17.5. *Let R be an equivalence relation on A. Then the following are true.*

(i) $(\forall a \in A)\ \ a \in [a]$

(ii) $(\forall a, b \in A)\ \ aRb \Rightarrow ([a] = [b])$

(iii) $(\forall a, b \in A)\ \ \neg aRb \Rightarrow ([a] \cap [b] = \emptyset)$

Observation 17.5 is actually true in general. Before we prove it, let's see how it helps us in proving Proposition 17.4.

Proof of Proposition 17.4. Part (i) of Observation 17.5 tells us that the union of all equivalence classes $[a]$, $\bigcup_{a \in A}[a]$, is all of A.

Parts (ii) and (iii) of Observation 17.5 tell us that two equivalence classes don't overlap in an "arbitrary way". Suppose $a, b \in A$. There are two cases to consider.

Case 1: aRb. In this case, $[a] = [b]$ by part (ii) of Observation 17.5. Since P is a set, $[a]$ and $[b]$ are actually the same element of P, and not two different elements of P that have a nonempty intersection. Thus, a and b don't make P violate the definition of a partition.

Case 2: $\neg aRb$. In this case $[a] \cap [b] = \emptyset$, so a and b such that $\neg aRb$ also cannot cause P to violate the definition of a partition. □

Proof of Observation 17.5. Since R is reflexive, aRa, so $a \in [a]$. This proves part (i).

Let aRb and pick $x \in [a]$. Then aRx by definition of $[a]$. Since R is symmetric, we have bRa. Now we have bRa and aRx, so bRx because R is transitive. It follows that $x \in [b]$, and $[a] \subseteq [b]$. Now switch the roles of a and b in this argument to get $[b] \subseteq [a]$ and $[a] = [b]$. This proves part (ii).

We conclude the proof by proving the contrapositive of part (iii). That is, if $[a] \cap [b] \neq \emptyset$, then aRb. If $[a] \cap [b] \neq \emptyset$, there is some $x \in A$ such that $x \in [a]$ and $x \in [b]$. Hence, aRx and bRx. Since R is symmetric, xRb as well. Now we have aRx and xRb, so aRb. This completes the proof. □

We conclude our discussion of equivalence relations with a remark about equivalence classes for the equivalence relations from Example 17.1.

We have only discussed $\equiv_3$ on the domain $A = \{1, 2, 3, 4, 5, 6, 7\}$. It turns out that we can extend this relation to all of $\mathbb{Z}$ and get the following equivalence classes.

$[1] = \{x \in \mathbb{Z} \mid$ The remainder of x after division by 3 is 1

$[2] = \{x \in \mathbb{Z} \mid$ The remainder of x after division by 3 is 2

$[3] = \{x \in \mathbb{Z} \mid$ The remainder of x after division by 3 is 0

The equivalence classes for logical equivalence ($\iff$) are the sets of propositional formulas whose truth tables are the same.

17.1.2 Order Relations

Definition 17.6. *A relation on a set A is an* order relation *if it is antisymmetric and transitive. An order relation is a* strict order relation *if it is antireflexive. An order relation is a* total order relation *(usually shorthanded as* total order*) if $(\forall a, b \in A)\ \ x \neq y \Rightarrow (xRy \vee yRx)$.*

Example 17.4: We deduce from Table 17.1 that $\subseteq$, $<$, $\leq$, and $\mid$ are order relations. The relation $<$ is also strict, whereas $\leq$ and $\mid$ are not strict. The relations $<$ and $\leq$ are total orders. To show that $\subseteq$ is not a total order, pick $X = \{1, 2\}$ and $Y = \{2, 3\}$, and observe that $X \not\subseteq Y$ and $Y \not\subseteq X$. For the divisibility relation, notice that $3 \nmid 7$ and $7 \nmid 3$, which shows that $\mid$ is not a total order. ⊠

While an order relation R need not be a total order, one can construct another order relation $\tilde{R}$ that is a total order and that satisfies $xRy \Rightarrow x\tilde{R}y$. If $\tilde{R}$ is as such, we say $\tilde{R}$ is an *extension* of

R. We do not prove this now, and only give an example. We will prove a more general result when we talk about directed graphs.

Example 17.5: We can extend the divisibility relation $|$ on positive integers to $\leq$ which is a total order. Observe that if $a \mid b$, then also $a \leq b$, so $\leq$ is indeed an extension of $|$. There are other extensions of $|$ to total orders, but the one we gave is quite natural. ⊠

17.1.3 k-ary Relations

So far we have only talked about binary relations. There are also k-ary relations which generalize the notion of a binary relation. Just like a binary relation is a subset of the Cartesian product of 2 sets, a k-ary relation is a subset of the Cartesian product of k sets. In other words, a k-ary relation is a collection of *k-tuples*. These come up for example in relational databases.

Note that if $k = 2$, we get the notion of a relation we've been discussing today and last time. For $k = 1$, we get regular sets.

Example 17.6: The set $\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ is a ternary (3-ary) relation on $\mathbb{R}^3$. It is the set of points on a sphere of radius 1 centered at the origin. ⊠

For the rest of this course, we will only consider binary relations.

17.2 Graphs

We have seen graphs before in this course. Now we take some time to define what a graph is more formally. In particular, we will discuss directed graphs, also known as digraphs.

The definition below uses the term multiset. A *multiset* is a collection very similar to a set, except duplicates of an element are allowed. Every set is also a multiset, but there are some multisets that are not sets. For example $\{1, 2, 3\}$ is a set and also a multiset. On the other hand, $\{1, 1, 2\}$ is a multiset, but not a set.

Definition 17.7. *A* digraph *G is a pair $G = (V, E)$, where V is a set of* vertices *and E is a multiset of elements of $V \times V$. Elements of E are called* edges.

We often use pictures to represent graphs. Each vertex is a labeled dot or a circle, and an edge is a line connecting two such dots, with an arrow indicating the direction of the edge.

Example 17.7: Consider the directed graph in Figure 17.1. We have

$$\begin{aligned} V &= \{a, b, c, d\} \\ E &= \{e_1, e_2, e_3, e_4, e_5, e_6\} \\ &= \{(a, b), (b, c), (c, d), (c, d), (d, b), (d, a)\} \end{aligned}$$

Notice that since E is a multiset, it is fine that it contains two copies of the pair (c, d). ⊠

We said graphs were a continuation of our study of binary relations. Indeed, if E is a set, E is a binary relation on V. When this is the case, we say $G = (V, E)$ is a *simple* graph.

17.2.1 Paths

Another important notion is a path in a graph. Informally speaking, a path is a sequence of edges that "connect together".

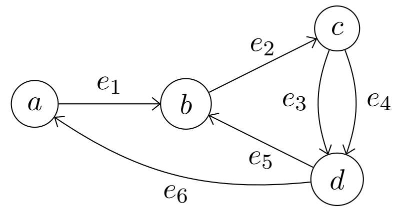

Figure 17.1: An example of a digraph.

Definition 17.8. *Let $G = (V, E)$ be a graph, and $u, v \in V$. A* path from u to v *is a sequence of edges such that the start vertex of the first edge is u, the end vertex of the last edge is v, and the start vertex of the $(i+1)$st edge is the same as the end vertex of the ith edge. The* length *of a path is the number of edges.*

Example 17.8: We list some paths in the graph from Figure 17.1 in Table 17.2 and point out a few things.

The path P_1 is the empty sequence. Such a path has length zero and has the same start and end vertex. Following this path corresponds to starting at a and staying put.

There can be multiple paths between two vertices. For example, P_2 and P_3 are both paths from a to c. A path can also repeat edges. For example, the edge e_2 appears twice on path P_3.

Finally, observe that the path P_4 starts and ends at a. We call any path that starts and ends at the same vertex a *cycle*. ⊠

Path	**Start vertex**	**End vertex**	**Sequence of edges**	**Length**
P_1	a	a	Empty sequence	0
P_2	a	c	e_1, e_2	2
P_3	a	c	e_1, e_2, e_3, e_4, e_2	5
P_4	a	a	$e_1, e_2, e_3, e_5, e_2, e_4, e_6$	7

Table 17.2: Some paths in the graph from Figure 17.1.

The notion of a path allows us to look at another relation. Consider the relation R^* on V where uR^*v if there is a path in G from u to v. This is a transitive relation, and is called the *transitive closure* of the graph G.