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Lecture 26 : Permutations and Combinations

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Last time we started talking about counting. The general setting is that we are given a description of a finite set, and our goal is to find the set's cardinality. This has applications for example in program analysis and discrete probability theory.

We saw some basic counting strategies, namely the bijection rule, the union rule, and the product rule. We also saw extensions of the bijection rule and the product rule. Today we describe a general kind of a problem that captures all the counting problems we discussed last time, and give an example where we count the number of ways to get various hands in poker. After that, we discuss the binomial theorem which is an important ingredient in many counting proofs. At the end of lecture, we discuss a generalization of the sum rule from last time and use the binomial theorem to prove its correctness.

26.1 Permutations and Combinations

We can characterize the counting problems we discussed last time in the context of the following problem. We have a bowl with n distinct balls, and we draw a ball from the bowl k times. We want to count the number of ways we can draw the balls, subject to the following kinds of restrictions.

1. *Replacement*: In some variations, we can draw the same ball twice, which we think of as putting the ball back in the bowl after we draw it, i.e., replacing it. In other variations, the ball is discarded after it is drawn.

2. *Ordering*: In some settings, the order in which we pick the balls matters, whereas in other settings it does not matter.

We summarize the number of ways to draw the balls in each of the four variations of the problem in Table 26.1.

	With replacement?	
Ordering matters?	Yes	No
Yes	n^k	$\frac{n!}{(n-k)!}$
No	$\binom{n+k-1}{k-1}$	$\binom{n}{k}$

Table 26.1: Number of ways to pick one of n balls k times with different kinds of rules.

First consider the situation when the order in which we draw the balls matters, and when we replace a ball after it is drawn. This way, there are n balls in the bowl for each draw, so there are n options for each draw. It follows that there are a total of n^k ways to draw the balls. You can show this more rigorously by combining the bijection rule with the product rule if you wish.

Now consider the situation when order matters but when we do not replace a ball that was drawn. In that case, there are n balls to choose from in the first draw, $n-1$ balls to choose from in the second draw, and in general there are $n-i$ options after i draws. Thus, the total number of ways to draw k balls in this case is $n\cdot(n-1)\cdots(n-k+1)$. We can multiply this by $1 = \frac{(n-k)(n-k-1)\cdots 2\cdot 1}{(n-k)(n-k-1)\cdots 2\cdot 1}$ to get an expression in terms of factorials:

$$n\cdot(n-1)\cdots(n-k+1) = n\cdot(n-1)\cdots(n-k+1)\cdot\frac{(n-k)(n-k-1)\cdots 2\cdot 1}{(n-k)(n-k-1)\cdots 2\cdot 1} = \frac{n!}{(n-k)!}.$$

If we do not replace the balls and disregard the ordering, we are picking a k-element subset of the set of n balls, and we saw last time that the number of ways to do this is $\binom{n}{k}$. Another way to think about it is that there is a $(k!)$-to-1 relationship between the number of ways to pick k balls in the setting when order matters (all $k!$ ways of picking k balls when order matters correspond to the same subset consisting of k balls), and our generalized bijection rule tells us that, in that case, the number of ways to pick k balls when order doesn't matter is $(n!/(n-k)!)/k!$, and this is exactly what $\binom{n}{k}$ is.

Here we also remark that there are many names for the quantity $\binom{n}{k}$. Some of the names are *combination*, *binomial*, or *binomial coefficient*.

Finally, consider the situation when we replace the balls after being drawn, and when ordering does not matter. This is exactly the situation we were facing in our first example in Lecture 25. There, we had to pick 12 bagels out of 5 kinds (so $n = 5$ and $k = 12$). Replacing each bagel after it is drawn means that we can pick multiple bagels of the same kind.

Recall that we found a bijection between the ways to pick the bagels and the set of sequences of binary strings of length 16 with exactly 4 ones in them. The number of zeros between each pair of ones (or between an end of a string and the 1 nearest to that end) indicated how many bagels of some kind we picked. This idea generalizes. There is a bijection between the number of ways to pick k balls out of n with replacement and without caring about the order in which the balls are picked, and the number of binary strings of length $n+k-1$ with exactly $n-1$ ones. The number of such strings is the same as the number of $(n-1)$-element subsets of a domain of size $n+k-1$, i.e., $\binom{n+k-1}{n-1}$ by the bijection rule.

26.1.1 Example: Counting Poker Hands

Let's get some more practice with our counting rules. Consider a deck of cards that's used for playing poker. The cards have 4 different suits: spades, clubs, hearts, and diamonds. For each suit, there are 13 cards, one for each of the values 2 through 10, jack (J), queen (Q), king (K), and ace (A). This means that there are a total of $4\cdot 13 = 52$ cards.

A poker hand consists of 5 of those 52 cards. Since there is only one card for each suit-value pair, the dealer gives a player those 5 cards by drawing from the deck without replacement. The order in which the cards are drawn does not matter, so there are $\binom{52}{5} = 2598960$ different possible hands.

Example 26.1: A *four-of-a-kind* is a hand with four cards with the same value, one for each suit, and an additional card.

There are 13 ways to pick which value will be present in all suits. Since order doesn't matter, this fully determines four of the five cards in the hand. We can pick the last card any way we want. Bu the generalized product rule, there are now 12 choices for the value and 4 values for the suit. Thus, the total number of four-of-a-kind hands is $13\cdot 12\cdot 4 = 624$. ⊠

We are not going to discuss discrete probability in this course, but let's at least mention it now. The discrete probability of an event with a certain property happening is the number of events that have the property divided by the total number of possible events. There are 624 events (hands) of the type four-of-a-kind, and there are a total of 2598960 possible events (hands). Thus, the probability of a four-of-a-kind is $624/2598960 \approx 0.0002$ (where the symbol $\approx$ means "approximately equal to"). This says that about 1 out of 5000 hands is a four-of-a-kind.

Example 26.2: A *full house* consists of three cards with the same value and different suits and two other cards with the same value (that is different from the first one) and two different suits. We use the generalized product rule to count the number of full houses.

There are 13 ways to pick the value shared by 3 cards. The 3 cards can each have 1 possible suit out of 4 and the suits are drawn with replacement, which means there are $\binom{4}{3} = 4$ ways to pick the suits for the three cards. Another way to think about this is that we pick which suit is not present among the three cards, and there are four ways of picking that suit. Thus, the 3 cards can be picked $13 \cdot 4 = 52$ ways.

The two remaining cards must have the same value, and there are 12 options for the value because we cannot pick the value the three other cards have. The two cards will have different suits, and we can pick the two suits $\binom{4}{2} = 6$ ways. Thus, given the first three cards, there are $12 \cdot 6 = 72$ ways to pick the last two cards, and it follows that the total number of different full houses is $52 \cdot 72 = 3744$. ⊠

Example 26.3: Now let's count the number of hands with *two pairs*. That is, there are two cards with the same value and different suit, two other cards with the same value (but different from the first two cards) and different suit, and one additional card.

We can pick the value for the first pair 13 ways, and the suit $\binom{4}{2}$ ways. When the first pair is picked, we can pick the value for the second pair 12 ways, and we have $\binom{4}{2}$ options for the suits of the two cards with that value. Finally, there are 11 values to choose from for the last card, and any one out of the four suits is fair game. This gives a total of $13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 11 \cdot 4 = 247104$.

But we are overcounting. For example, suppose we pick the hand $4\heartsuit, 4\diamondsuit, A\spadesuit, A\heartsuit, J\clubsuit$. That is, the first pair is a pair of fours (hearts and diamonds), the second pair is a pair of aces (spades and hearts), and the last card is a jack of clubs. But we could pick the aces first and the fours second. That is, the hand $A\spadesuit, A\heartsuit, 4\heartsuit, 4\diamondsuit, J\clubsuit$ is the same hand, but we count it as a separate hand. The order in which we pick the pairs does not matter, so there is a 2-to-1 correspondence between the ways of picking a the hand using our method and the set of hands that consist of two pairs. By the generalized bijection rule, this means that the number of hands that have two pairs is actually $247104/2 = 123552$. ⊠

There is another approach one can use to count the number of hands with two pairs. First, pick the two values for the pairs. This is done by picking 2 out of the 13 possible values, and there are $\binom{13}{2}$ ways for this. Now that the values for the pairs have been picked, we can pick the suits for each of the pairs. The suits for each pair are independent of each other, and there are $\binom{4}{2}$ ways to pick the suits for each pair. Finally, we can pick the remaining card $11 \cdot 4$ ways like before. Thus, the total number of ways to pick a hand with two pairs is $\binom{13}{2} \cdot \binom{4}{2}^2 \cdot 11 \cdot 4$. This looks exactly the same as our expression in Example 26.3, except the division by two is now accounted for by the binomial coefficient $\binom{13}{2} = 13 \cdot 12/2$.

Example 26.4: Now let's discuss a hand that is mostly not useful in poker. Suppose we have at least one card from each suit.

We can pick one value of each suit, and the choices are independent. Thus, we can do this 13^4 ways by the product rule. Afterwards, we can pick one suit, and for each suit there are 12 cards left to choose from, which gives us 48 options for the last card. Using the generalized product rule, we multiply the two values together to get $13^4 \cdot 48 = 1370928$.

But now we are overcounting. For example, suppose we pick the hand $2\spadesuit, 3\clubsuit, 4\heartsuit, 4\diamondsuit, 2\heartsuit$. We get the same hand if we pick $2\spadesuit, 3\clubsuit, 2\heartsuit, 4\diamondsuit, 4\heartsuit$, that is, we pick the 2 of hearts as the one heart card and the 4 of hearts as the additional fifth card, instead of picking them the other way around. There are no other ways to get this hand, so there is a 2-to-1 mapping between the ways to pick hands and the actual hands, which means we need to divide the number we obtained in the previous paragraph by 2. It follows that the number of hands that contain at least one card of every suit is $1370928/2 = 685464$. ⊠

26.2 Binomial Theorem

Now we look at a basic theorem that has many applications in counting. Let's start with a motivating example.

Example 26.5: We want to evaluate $(x+y)^4$ by turning it into a sum of products of different powers of x and y. That is, we want to turn $(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$ into a sum of the form $a_4x^4 + a_3x^3y + a_2x^2y^2 + a_1xy^3 + a_0y^4$. We can evaluate the product as

$$\begin{aligned}(x+y)^4 &= xxxx+ \\ &+ xxxy + xxyx + xyxx + yxxx+ \\ &+ xxyy + xyxy + xyyx + yxxy + yxyx + yyxx+ \\ &+ xyyy + yxyy + yyxy + yyyx+ \\ &+ yyyy \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + t^4\end{aligned}$$

For example, observe that the terms $xxxy$ and $xyxx$ are the same, and both contribute to the term with x^3y in it. We get a contribution to the term x^3y if we pick x in three out of the four terms in the product $(x+y)(x+y)(x+y)(x+y)$, and y from the remaining term. Thus, there are $\binom{4}{3} = 4$ ways to get a contribution towards the term with x^3y. Similarly, to find the coefficient in front of the term x^2y^2 we count the number of ways we can pick x and y from the four terms so that both x and y are picked twice. This number is $\binom{4}{2} = 6$ because once we choose which terms to pick x from, the terms which we pick y from are determined. ⊠

In fact, the observation we made in Example 26.5 holds in general. We want to express $(x+y)^n$ as a sum of *monomials* (a monomial is a constant times some product of variables; for example $6x^2y^2$ is a monomial). The monomials have the form $a_kx^ky^{n-k}$ for $k \in \{0, \dots n\}$, where $x^0 = y^0 = 1$ as usual. To obtain x^ky^{n-k}, we pick x from k of the terms in the product $(x+y)^n$, and pick y from all the remaining terms. This can be done $\binom{n}{k}$ ways. Thus, we get the following.

Theorem 26.1 (Binomial theorem)**.** *For all x, y, and n,*

$$(x+y) = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

You can give a formal proof by induction. We leave it to you as an exercise. The binomial theorem also explains why we call the values $\binom{n}{k}$ binomial coefficients. The word binomial comes from the fact that we take a sum of two terms (in the case of the binomial theorem the two terms are x and y).

26.3 Inclusion-Exclusion

We have already generalized the bijection rule and the product rule for counting. We now have all the tools necessary for the generalization of the sum rule too.

Recall that for the sum rule, we have sets $S_1, S_2, \ldots, S_s$ that are pairwise disjoint. In that case,

$$\left|\bigcup_{i=1}^{s} S_i\right| = \sum_{i=1}^{s} |S_i| .$$

This rule fails when there are intersections. In fact, you saw this on your second homework, where we asked you to prove the following two formulas

$$|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2| \tag{26.1}$$

$$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3| \tag{26.2}$$

On the third homework, you had to use induction to prove a generalization of the two formulas above. We show the formula below. Recall that $[s]$ stands for $\{1, \ldots, s\}$ and I is a nonempty subset of $[s]$.

$$\left|\bigcup_{i=1}^{s} S_i\right| = \sum_{\emptyset \neq I \subseteq [s]} (-1)^{|I|+1} \left|\bigcap_{i \in I} S_i\right| \tag{26.3}$$

Equation (26.3) is known as the *inclusion-exclusion* formula, and this is also the formula for the generalized sum rule.

Let's now explain the term "inclusion-exclusion" and give an alternative proof of (26.3) using the binomial theorem. Notice that on the right-hand side of (26.2), all terms involving 1 or 3 sets is added, whereas all terms involving 2 sets are subtracted. The same also happens in the general case of (26.3). The term $|\bigcap_{i \in I} S_i|$ is added if $|I|$ is odd (since in that case $(-1)^{|I|+1}$ is even, and is subtracted otherwise. Thus, for consecutive cardinalities of I, we switch between adding (including) and subtracting (excluding) the term on the right-hand side that corresponds to I. This explains the term inclusion-exclusion.

Now we prove (26.3). First, using the observations in the previous paragraph, we rewrite the right-hand side as

$$\sum_{k=1}^{s} (-1)^k \sum_{\substack{I \subseteq [s] \\ |I| = k}} \left|\bigcap_{i \in I} S_i\right| . \tag{26.4}$$

Consider an element $x \in \bigcup_{i=1}^{s} S_i$. It contributes 1 to the left-hand side of (26.3). Now suppose x is in n of the sets S_i, and without loss of generality assume it's in S_1 through S_n and is not in S_{n+1} through S_s. Thus, it contributes 1 to $|\bigcap_{i \in I} S_i|$ if and only if $I \subseteq [n]$, and for each such term the contribution of x towards (26.4) is 1 if $|I|$ is odd and -1 if $|I|$ is even. Thus, the total contribution of x to (26.4) is

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1}.$$

We pull out a factor of (-1) from the sum and also multiply each individual term by $1 = 1^{n-k}$ to get

$$(-1) \cdot \sum_{k=1}^{n} \binom{n}{k} (-1)^k 1^{n-k}$$

Now this looks almost like the binomial theorem with $x = -1$ and $y = 1$, except the sum starts at 1 and not 0. The term with zero would be $\binom{n}{0}(-1)^0 1^n = 1$ (note that $\binom{n}{0} = 1$ because there is exactly one way of picking a zero-element subset of a set of n elements, namely don't pick anything). Thus, if we add and subtract 1, we get

$$(-1)\left[-1 + \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k}\right] = 1 - (-1+1)^n = 1 - 0^n = 1.$$

Finally, we see that the contribution of x to (26.4) is 1. Thus, the contribution of x to the right-hand side of (26.3) is also 1. Since every element contributes the same to both sides of (26.3), this proves the binomial theorem.