Invariant finite Borel measures for rational functions on the Riemann sphere

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Abstract

To study finite Borel measures on the Riemann sphere invariant under a rational function R of degree greater than one, we decompose them in an R-invariant component measure supported on the Julia set and a finite number of mutually singular R-invariant component measures vanishing on the Julia set. The latter ones can be described easily. For a characterization of the former one, we use a general approach based on a weight function for R on the Riemann sphere. We investigate the relation between weight functions for R and R-invariant Borel probability measures on the Riemann sphere in both directions and discuss how such a measure can be constructed, given a weight function for R.

1 Introduction

In iteration theory, it is known for more than 20 years [22, theorem 2.2 p. 27] that the iterates of a nonlinear complex polynomial P (up to an additive constant) are orthogonal to each other w.r.t. a hermitian bilinear form based on the logarithmic equilibrium measure on the Julia set of P [1]. This property also holds w.r.t. the corresponding symmetric bilinear form [7]. The underlying measure also being characterizable as the unique balancedly P-invariant Borel probability measure on the complex plane [2], it has been extended to symmetric bilinear forms based on other P-invariant measures [8, 19, 20, 6, 5, 24].
On the other hand, balancedly invariant measures for rational functions of degree greater than one have been studied meanwhile [16, 4].

In this paper, we shall study finite Borel measures on the Riemann sphere invariant under a rational function R of degree greater than one in their own right.
Before doing so, we shall define our terms in a preliminary section, mainly concerning Julia sets and Fatou sets. For a good review of this matter, including classical references such as [15, 12, 13, 10], we refer to [9].
In section 3, we shall decompose any finite R-invariant Borel measure into a finite number of R-invariant component measures, each vanishing outside some R^{-1}-invariant subset of the Riemann sphere. The component measures associated with (subsets of) the Fatou set, will be analyzed. They have a simple structure and hence are easily described. On the contrary, the component measure supported on the Julia set, is of a much more

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complicated nature.
To describe it, we need the general approach of section 4, using a weight function for R. This weight function is a condensed form of the weight functions used in [19, 20, 6, 5]. We prefer to use it, because it leads to a characterization that is independent of the choice of a specific complete set of branches of R^{-1}, and allows a simple continuity condition to guarantee the existence of an R-invariant Borel probability measure characterized by it. We prove that a Borel measurable weight function can be associated with any R-invariant Borel probability measure, without the restriction used in [19, 5] that the critical points do not support any discrete measure. We also notice, that for a rigorous development, a complete set of *measurable* branches of R^{-1} is needed. We shall show that such a set always exists. Conversely, we establish that an R-invariant Borel probability measure can be associated with any continuous weight function. Uniqueness questions will be addressed.
In the last section, we shall see how an R-invariant Borel probability measure associated with a given weight function, can be constructed. In case of certain choices of weight functions for polynomials, the moment generating function can be obtained by using a technique developed in [8, 19, 20, 6, 5].

2 Preliminaries

In this section, we introduce the notation we shall be using in the sequel.

2.1 Concerning the Riemann sphere

By $\overline{\mathbb{C}}$ we denote the Riemann sphere, i.e. the one-point compactification of the complex plane $\mathbb{C}$ with the point at infinity ∞. We use the topology induced by the modulus norm on $\mathbb{C}$ and the one induced by the spherical metric on $\overline{\mathbb{C}}$.
C_R symbols the circle in $\mathbb{C}$ centred at the origin with radius $R \in \mathbb{R}_\circ^+$. For any closed Jordan curve $J \subset \mathbb{C}$, we use the notation $\mathrm{int}(J)$ for the interior of J.

2.2 Concerning rational functions

$\mathbb{C}[z]$ is the set of polynomial functions on $\overline{\mathbb{C}}$ over $\mathbb{C}$ and $\mathbb{C}(z)$ the set of rational functions on $\overline{\mathbb{C}}$ over $\mathbb{C}$. If $R(z) = \frac{P(z)}{Q(z)}$ with $P(z), Q(z) \in \mathbb{C}[z]$, then the degree of the rational function R is $\deg R \triangleq \max\{\deg P, \deg Q\}$, provided that P and Q are relatively prime.
Let $R(z) \in \mathbb{C}(z)$ be of degree greater than one. The iterates of R are defined in the usual way: $R^{\circ 0}$ is the identity transformation on $\overline{\mathbb{C}}$ and $R^{\circ(k+1)} \triangleq R \circ R^{\circ k}$ for any $k \in \mathbb{N}$. The Fatou set $F(R)$ of R is the set of all points $z_o \in \overline{\mathbb{C}}$ for which the iterates $\{R^{\circ k}\}_{k=0}^{\infty}$ form a normal family on a sufficiently small neighbourhood $N(z_o)$ of z_o. For the concept of a normal family of functions on $\overline{\mathbb{C}}$, we refer to [14]. The Julia set $J(R)$ is the complement of $F(R)$. It is a nonempty perfect subset of $\overline{\mathbb{C}}$ and can also be defined as the closure of the set of repulsive periodic points of R.
A periodic point $z_o \in \overline{\mathbb{C}}$ of R is a fixed point of some iterate $R^{\circ k}$ of order $k > 0$. It is classified according to its eigenvalue $\lambda(z_o)$ defined as $\lambda(z_o) \triangleq \frac{d}{dz} R^{\circ p}(z)|_{z=z_o}$ if $z_o \in \mathbb{C}$ and $\lambda(z_o) \triangleq \lim_{z\to\infty} \frac{z}{R^{\circ p}(z)}$ if $z_o = \infty$, p representing the period of z_o, i.e. the smallest $n \in \mathbb{N}_\circ$ such that $R^{\circ n}(z_o) = z_o$. It is called superattractive iff $\lambda(z_o) = 0$, attractive

iff $0 < |\lambda(z_o)| < 1$, indifferent iff $|\lambda(z_o)| = 1$ and repulsive iff $|\lambda(z_o)| > 1$. If z_o is an indifferent periodic point, it is called rationally indifferent iff $\lambda(z_o) = e^{2\pi i\alpha}$ for some $\alpha \in \mathbb{Q}$, and irrationally indifferent otherwise.
The Fatou set $F(R)$ consists of a number of disjoint components, i.e. maximal connected subsets. The set of components of $F(R)$ being closed under R, R induces a transformation on them. The periodic components of $F(R)$ can be classified into five categories, namely superattractive domains, attractive domains, parabolic domains, Siegel disks and Herman rings.

- A superattractive domain D of period p contains a superattractive p-periodic point z_o. There exists a neighbourhood $N(z_o) \subseteq D$ of z_o and a conformal homeomorphism $\phi : N(z_o) \to \mathrm{int}(C_r)$ for some $r \in \mathbb{R}_o^+$ such that

$$R^{\circ p}|_{N(z_o)} = \phi^{-1} \circ \phi^k, \tag{1}$$

  $k \geqslant 2$ representing the multiplicity of z_o as a root of the equation $R^{\circ p}(z) = R^{\circ p}(z_o)$ over $\overline{\mathbb{C}}$. All points of D converge to z_o under $R^{\circ p}$.

- An attractive domain D of period p contains an attractive p-periodic point z_o. There exists a neighbourhood $N(z_o) \subseteq D$ of z_o and a conformal homeomorphism $\phi : N(z_o) \to \mathrm{int}(C_r)$ for some $r \in \mathbb{R}_o^+$ such that

$$R^{\circ p}|_{N(z_o)} = \phi^{-1} \circ (\lambda(z_o) \cdot \phi) . \tag{2}$$

  All points of D converge to z_o under $R^{\circ p}$.

- A parabolic domain D of period p has a rationally indifferent periodic point $z_o \in J(R)$ as a boundary point, the period of which divides p. All points of D converge to z_o under $R^{\circ p}$.

- A Siegel disk D of period p is a component of $F(R)$ such that $R^{\circ p}|_D$ is conformally conjugate to a rotation of the disk $\mathrm{int}(C_r)$ for some $r \in \mathbb{R}_o^+$, i.e. there exists a conformal homeomorphism $\phi : D \to \mathrm{int}(C_r)$ and an angle $\theta_o \in \mathbb{R}$, such that

$$R^{\circ p}|_D = \phi^{-1} \circ \left(e^{i\theta_o} \cdot \phi\right) . \tag{3}$$

  $\frac{\theta_o}{2\pi}$ is always irrational, hence $\phi^{-1}(0)$ is an irrationally indifferent p-periodic point contained in D.

- A Herman ring D of period p is a component of $F(R)$ such that $R^{\circ p}|_D$ is conformally conjugate to a rotation of the annulus $\{z \in \mathbb{C} \,|\, r_1 < |z| < r_2\}$ for some $r_1, r_2 \in \mathbb{R}_o^+$ with $r_1 < r_2$, i.e. there exists a conformal homeomorphism $\phi : D \to \{z \in \mathbb{C} \,|\, r_1 < |z| < r_2\}$ and an angle $\theta_o \in \mathbb{R}$, such that

$$R^{\circ p}|_D = \phi^{-1} \circ \left(e^{i\theta_o} \cdot \phi\right) . \tag{4}$$

  Once again $\frac{\theta_o}{2\pi}$ is always irrational, but there is no corresponding periodic point in this case.

We shall call superattractive, attractive and parabolic domains attraction domains, and Siegel disks and Herman rings rotation domains.
If D is a p-periodic component of $F(R)$, then each of the components $R^{\circ i}(D)$ for $i = 1(1)p-1$ is of the same category as D. In case of a rotation domain D_1, a conformal homeomorphism ϕ_i satisfying (3) or (4) for the rotation domain $D = R^{\circ(i-1)}(D_1)$ of the same cycle, is given by $\phi_i = \phi \circ R^{\circ(p-i)}$, ϕ satisfying (3) or (4) for the rotation domain $D = R^{\circ(p-1)}(D_1)$.
There is only a finite number of periodic components of $F(R)$, and all components of $F(R)$ are eventually mapped on a periodic cycle. The union of all components of $F(R)$ eventually mapped on a periodic cycle of components, will be called the attraction region of that cycle.

2.3 Concerning measures

By a measure, we shall always understand a finite positive Borel measure, having $\overline{\mathbb{C}}$ as universe if no further specification is given. The set of all finite positive Borel measures on a topological space $\mathcal{T}$ will be denoted by $\mathcal{M}(\mathcal{T})$ and those having support on a subset T of $\mathcal{T}$ by $\mathcal{M}_T(\mathcal{T})$. Analogously, by a probability measure, we shall always understand a Borel probability measure, having $\overline{\mathbb{C}}$ as universe if no further specification is given. The set of all Borel probability measures on a topological space $\mathcal{T}$ will be denoted by $\mathcal{P}(\mathcal{T})$ and those having support on a subset T of $\mathcal{T}$ by $\mathcal{P}_T(\mathcal{T})$.
A Dirac measure supported by the point $t \in \mathcal{T}$ is symbolled by δ_t.
As to the measurability of transformations, we use the term 'measurable' as a shorthand for 'Borel measurable'. The set of complex functions integrable w.r.t. a measure μ is denoted by $\mathcal{L}^1(\mu)$. The set of Borel subsets of $\mathcal{T}$ is symbolled by $\mathcal{B}(\mathcal{T})$.
A measure μ is called invariant w.r.t. the measurable transformation T iff $\mu T^{-1} = \mu$, i.e. iff $\mu(T^{-1}(B)) = \mu(B)$ for any measurable set B. If T is a transformation with domain D and A is any set, we define $T(A) \triangleq \{T(d) \mid d \in A \cap D\}$ and $T^{-1}(A) \triangleq \{d \in D \mid T(d) \in A\}$. The set A is called T-closed iff $T(A) \subseteq A$, and T-invariant iff $T(A) = A$; it is called T^{-1}-closed iff $T^{-1}(A) \subseteq A$, and T^{-1}-invariant iff $T^{-1}(A) = A$.

3 Decomposition

Let $R(z) \in \mathbb{C}(z)$ be of degree greater than one and μ be an R-invariant measure. To analyze μ, we want to decompose it into a finite sum of $n+1$ mutually singular R-invariant measures μ_i. This can be achieved by using a finite partition $\{C_i\}_{i=0}^n$ of $\overline{\mathbb{C}}$, consisting of R^{-1}-closed Borel sets, and defining each of the component measures μ_i as the measure μ of the intersection with the partition set C_i. Provided the Fatou-set $F(R)$ is not empty, $\{J(R), F(R)\}$ is such a partition. The behaviour of R on $J(R)$ being chaotic, $J(R)$ can hardly be decomposed any further in a finite number of R^{-1}-closed Borel subsets. $F(R)$ on the contrary, can be partitioned into the attraction regions of the periodic cycles of components of $F(R)$, yielding a disjoint union of a finite number, say n, of R^{-1}-closed open sets $\cup_{k\in\mathbb{N}}(R^{\circ k})^{-1}(D_i)$, each consisting of the components of $F(R)$ eventually mapped on a fixed periodic cycle of components, one of which is D_i $(i = 1(1)n)$. All this leads to the following theorem.

Theorem 1 *Let $R(z) \in \mathbb{C}(z)$ be of degree greater than one, $C_0 = J(R)$, $\{C_i\}_{i=1}^n$ be the partition of $F(R)$ into attraction regions of periodic cycles of components of $F(R)$, $\mu \in \mathcal{M}(\overline{\mathbb{C}})$ and $\mu_i : \mathcal{B}(\overline{\mathbb{C}}) \to \mathbb{R}^+ : B \to \mu(B \cap C_i)$ for $i = 1(1)n$,*
then $\mu = \sum_{i=0}^n \mu_i$ and μ is R-invariant iff all of the μ_i's are R-invariant.

Proof Trivial. □

So any R-invariant measure μ can be written as a sum of R-invariant measures with support on $J(R)$ or vanishing outside one of the attraction regions of periodic cycles of components of $F(R)$, and vice versa. Therefore, we shall consider each of these R-invariant measures separately in the sequel. For a characterization of the one with support on $J(R)$, we need the general approach of the next section using a weight function on $\overline{\mathbb{C}}$. In this section, we shall be able to characterize completely those associated with $F(R)$.

A first result is the following theorem, which implies that an R-invariant measure vanishing outside the attraction region of a periodic cycle of components of $F(R)$, actually vanishes outside the periodic cycle itself.

Theorem 2 *Let $R(z) \in \mathbb{C}(z)$ be of degree greater than one and $\mu \in \mathcal{M}(\overline{\mathbb{C}})$ be R-invariant. The nonperiodic components of $F(R)$ have measure zero and the periodic components belonging to the same cycle, have the same measure. Analogously, nonperiodic points do not support any discrete measure and periodic points belonging to the same cycle, support the same discrete measure.*

Proof

- Let D be any nonperiodic component of $F(R)$, then we know that for any $i, j \in \mathbb{N}$ such that $i \neq j$
$$(R^{\circ i})^{-1}(D) \cap (R^{\circ j})^{-1}(D) = \varnothing.$$
Indeed, suppose that $z_o \in (R^{\circ i})^{-1}(D) \cap (R^{\circ j})^{-1}(D)$ and $i < j$. Let E be the component of $F(R)$ containing z_o. Because the set of components of $F(R)$ is closed under R, $R^{\circ i}(E) = R^{\circ j}(E) = D$, so
$$D = R^{\circ j}(E) = R^{\circ (j-i)}(R^{\circ i}(E)) = R^{\circ (j-i)}(D).$$
Hence D would be periodic, contrary to our hypothesis. On the other hand, because of the R-invariance of μ, we know that $\mu((R^{\circ i})^{-1}(D)) = \mu(D)$ for any $i \in \mathbb{N}$, so $((R^{\circ i})^{-1}(D))_{i=1}^{\infty}$ is a countable sequence of disjoint sets all having the same measure $\mu(D)$. μ being bounded, this implies $\mu(D) = 0$, proving the first assertion.

- Let C be any periodic component of $F(R)$ with period p. Because of the R-invariance of μ, we know that for any $n \in \mathbb{N}_o$
$$\mu(R^{\circ (n-1)}(C)) \leqslant \mu\left(R^{-1}(R^{\circ n}(C))\right) = \mu(R^{\circ n}(C)).$$
Since $R^{\circ 0}(C) = C = R^{\circ p}(C)$, this implies that $\mu(R^{\circ n}(C)) = \mu(C)$ for $n = 1(1)p-1$, proving the second assertion.

- Both assertions concerning the measure of singletons can be established analogously, considering a singleton instead of a component of $F(R)$.

□

Now we shall investigate in detail R-invariant measures that vanish outside a periodic cycle of components of $F(R)$. We shall distinguish between cycles of (super)attractive domains, of parabolic domains and of rotation domains.

3.1 Cycles of (super)attractive domains

R-invariant measures that vanish outside a cycle of superattractive or attractive domains, are balanced discrete measures supported by the periodic points of the cycle.

Theorem 3 *Let $R(z) \in \mathbb{C}(z)$ be of degree greater than one, C be the union of the superattractive or attractive domains belonging to the superattractive respectively attractive p-periodic cycle $(z_i)_{i=1}^p$ of R and $\mu \in \mathcal{M}(\overline{\mathbb{C}})$ vanishing outside C.*

$$\mu R^{-1} = \mu \iff \mu = \frac{\mu(C)}{p} \sum_{i=1}^{p} \delta_{z_i}. \tag{5}$$

Proof

The implication $\Leftarrow$ being trivial, we confine ourselves to proving the other implication.

- There is a neighbourhood $N(z_1)$ of z_1, such that μ vanishes on the reduced neighbourhood $N^\star(z_1) \triangleq N(z_1) \setminus \{z_1\}$. Indeed, because of the existence of the homeomorphism ϕ satisfying (1) or (2) in case $(z_i)_{i=1}^p$ is superattractive of attractive respectively, for any point $z_o \neq z_1$ in a sufficiently small neighbourhood $N(z_1)$ of z_1, we can choose $R_1, R_2 \in \mathbb{R}_o^+$, such that $B \subset A$, z_o is interior to $A \setminus B$ and $(R^{\circ p})^{-1}(A) \subset B$, where $A = \overline{\mathbb{C}} \setminus \phi^{-1}(\mathrm{int}(C_{R_1}))$ and $B = \overline{\mathbb{C}} \setminus \phi^{-1}(\mathrm{int}(C_{R_2}))$. Consequently,
$$\mu\left((R^{\circ p})^{-1}(A)\right) \leqslant \mu(B) \leqslant \mu(B) + \mu(A \setminus B) = \mu(A).$$
Because of the R-invariance of μ, this implies that $\mu(A \setminus B) = 0$ and since z_o is interior to $A \setminus B$, that $z_o \notin \mathrm{supp}(\mu)$. So $N^\star(z_1)$ contains no points of $\mathrm{supp}(\mu)$ and $\mu(N^\star(z_1)) = 0$.

- It is clear that
$$C \setminus \bigcup_{k\in\mathbb{N}} (R^{\circ k})^{-1}(z_1) \subseteq \bigcup_{k\in\mathbb{N}} (R^{\circ k})^{-1}(N^\star(z_1)). \tag{6}$$
Moreover, because of the R-invariance of μ, the right-hand side of (6) has measure zero under μ, so the left-hand side too. As the set of preimages $P \triangleq \cup_{k\in\mathbb{N}}(R^{\circ k})^{-1}(z_1)$ of z_1 is countable, this yields that $\mu = \sum_{z\in P} \mu(\{z\}) \cdot \delta_z$. Because of the second part of theorem 2, this sum reduces to the right-hand term of the right-hand side of (5).

□

3.2 Cycles of parabolic domains

R-invariant measures that vanish outside a cycle of parabolic domains, vanish everywhere.

Theorem 4 *Let $R(z) \in \mathbb{C}(z)$ be of degree greater than one, C be the union of the parabolic domains belonging to the rationally indifferent p-periodic cycle $(z_i)_{i=1}^{p}$ of R and $\mu \in \mathcal{M}(\overline{\mathbb{C}})$ vanishing outside C. μ is R-invariant iff $\mu \equiv 0$.*

Proof

The implication $\Leftarrow$ being trivial, we confine ourselves once again to proving the other implication.

Choose any point $z_o \in C$. By the definition of C, we know that $\lim_{k\to\infty} R^{\circ kp}(z_o)$ exists and belongs to $\{z_i\}_{i=1}^{p}$. Let this limit be z_m $(1 \leqslant m \leqslant p)$. Consider any neighbourhood $N(z_o)$ of z_o sufficiently small to guarantee that $\overline{N(z_o)} \subset C$. C being an open set, this is always possible.

As $C \subset F(R)$, $\{R^{\circ k}\}_{k=0}^{\infty}$ is by the definition of $F(R)$ a normal family on C. Therefore, there is a subsequence $(R^{\circ k_j p})_{j=0}^{\infty}$ of $(R^{\circ kp})_{k=0}^{\infty}$ converging uniformly on the compact subset $\overline{N(z_o)}$. Since $\lim_{k\to\infty} R^{\circ kp}(z) = z_m$ for any $z \in \overline{N(z_o)}$, it converges to the constant function z_m.

Putting $R_{k_j} \triangleq \sup\left\{|z - z_m| \mid z \in R^{\circ k_j p}(\overline{N(z_o)})\right\}$, this means that $\lim_{j\to\infty} R_{k_j} = 0$. On the other hand $R^{\circ k_j p}(\overline{N(z_o)})$ is for any $j \in \mathbb{N}$ a closed set that does not contain z_m, since $z_m \in J(R)$. So $r_{k_j} \triangleq \inf\left\{|z - z_m| \mid z \in R^{\circ k_j p}(\overline{N(z_o)})\right\} > 0$ for any $j \in \mathbb{N}$. Consequently, there exists an increasing subsequence $(l_j)_{j=0}^{\infty}$ of $(k_j)_{j=0}^{\infty}$ satisfying $R_{l_{j+1}} < r_{l_j}$ for any $j \in \mathbb{N}$. So $\cup_{j\in\mathbb{N}} R^{\circ l_j p}(\overline{N(z_o)})$ is a countable disjoint union.

Because of the R-invariance of μ, $\mu\left(R^{\circ l_j p}(\overline{N(z_o)})\right) \geqslant \mu(\overline{N(z_o)})$ holds for any $j \in \mathbb{N}$. μ being bounded, the σ-additivity of μ applied to the countable disjoint union mentioned implies that $\mu(\overline{N(z_o)}) = 0$.

Since this conclusion holds for any sufficiently small neighbourhood $N(z_o)$ of z_o, $z_o \notin \operatorname{supp}(\mu)$. □

3.3 Cycles of rotation domains

R-invariant measures μ vanishing outside a periodic cycle of rotation domains, are a bit more complicated than those vanishing outside cycles of attraction domains. However, they still do not exhibit the full complexity of general R-invariant measures as described in the following section, because they can be characterized by a single measure on $\mathbb{R}^+$, or equivalently, by a single distribution function on $\mathbb{R}^+$. Indeed, because of the homeomorphism satisfying (3) or (4) in case of Siegel disks or Herman rings respectively, μ can be seen to be a (sum of) transformed measure(s) on $\mathbb{C}$ invariant under each rotation around the origin and hence characterizable by a measure on $\mathbb{R}^+$. This idea is formalized in theorem 5. To prove it, we need two lemmata.

The first one states that the set of all halfopen segments centred around the origin of $\mathbb{C}$, forms a semi-ring generating the σ-algebra of Borel subsets of $\mathbb{C}$.

Lemma 1 *Let*

$$S(r_1, r_2, \theta_1, \theta_2) \triangleq \left\{ re^{i\theta} \,|\, r_1 \leqslant r < r_2 \text{ and } \theta_1 \leqslant \theta < \theta_2 \right\}, \tag{7}$$

then

$$\mathcal{A} \triangleq \{S(r_1, r_2, \theta_1, \theta_2) \,|\, r_1, r_2 \in \mathbb{R}^+,\ \theta_1, \theta_2 \in \mathbb{R}, r_1 \leqslant r_2 \text{ and } 0 \leqslant \theta_2 - \theta_1 < 2\pi\} \tag{8}$$

is a semi-ring generating the σ-algebra $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{C})$.

Proof
That $\mathcal{A}$ is a semi-ring and that $\sigma(\mathcal{A}) \subseteq \mathcal{B}(\mathbb{C})$ is obvious. The other inclusion $\mathcal{B}(\mathbb{C}) \subseteq \sigma(\mathcal{A})$ can be established by writing any open subset of $\mathbb{C}$ as a countable union of segments $S(r_1, r_2, \theta_1, \theta_2) \in \mathcal{A}$ with rational parameters r_1, r_2, θ_1 and θ_2. □

The second lemma we need in order to prove theorem 5, expresses the fact that when a measure on $\mathbb{C}$ is invariant under a rotation around the origin over an irrational multiple of 2π, it is invariant under any rotation around the origin.

Lemma 2 *Let $\mu \in \mathcal{M}(\mathbb{C})$ and*

$$T_\theta : \mathbb{C} \to \mathbb{C} : z \to e^{i\theta} z \tag{9}$$

for any $\theta \in \mathbb{R}$. If $\mu T_{\theta_o}^{-1} = \mu$ for some $\theta_o \in \mathbb{R} \setminus \pi\mathbb{Q}$ then $\mu T_\theta^{-1} = \mu$ for any $\theta \in \mathbb{R}$.

Proof
Let $\theta \in \mathbb{R}$ be arbitrary. Because $\frac{\theta_o}{2\pi} \in \mathbb{R} \setminus \mathbb{Q}$, for any $k \in \mathbb{N}_o$, there exist $p_k, q_k \in \mathbb{N}_o$ such that $|(p_k\theta_o - \theta) - q_k \cdot 2\pi| < \frac{1}{k}$. So for any $A \subseteq \mathbb{C}$ we have that $(T_{\theta_o}^{-1})^{\circ p_k}(A) \to T_\theta^{-1}(A)$, $k \to \infty$. Hence, the invariance relation $\mu T_{\theta_o}^{-1} = \mu$ and the continuity of μ imply that $\mu T_\theta^{-1}(B) = \mu(B)$ for any $B \in \mathcal{B}(\mathbb{C})$, which establishes the lemma. □

As a corollary, we apply this knowledge to simplify the expression for the measure of segments defined by (7) used in lemma 1.

Corollary 1 *Let $\mu \in \mathcal{M}(\mathbb{C})$ and $\mu T_{\theta_o}^{-1} = \mu$ for some $\theta_o \in \mathbb{R} \setminus \pi\mathbb{Q}$, where T_{θ_o} is defined by (9) for $\theta = \theta_o$. Let $S \triangleq S(r_1, r_2, \theta_1, \theta_2) \in \mathcal{A}$ as defined by (7) and (8) in lemma 1, then*

$$\mu(S) = \frac{\theta_2 - \theta_1}{2\pi} \cdot \mu(\{z \in \mathbb{C} \,|\, r_1 \leqslant |z| < r_2\}).$$

Proof
For rational values of $\frac{\theta_2 - \theta_1}{2\pi}$, the corollary is easily verified using the previous lemma for $\theta = \theta_2 - \theta_1$. The continuity of μ then completes the proof. □

The theorem itself can be stated as follows.

Theorem 5 *Let $R(z) \in \mathbb{C}(z)$ be of degree greater than one, $C \triangleq \cup_{i=1}^{p} R^{\circ i}(D)$ be the union of rotation domains of a cycle of period p, one of which is D, ϕ the homeomorphism satisfying (3) or (4) in case of Siegel disks or Herman rings respectively, $\phi_i \triangleq \phi \circ R^{\circ(p-i)}$ for $i = 1(1)p$, $\mu \in \mathcal{M}(\overline{\mathbb{C}})$ vanishing outside C, λ the uniform probability measure on $[0, 2\pi)$ and $T : \mathbb{R}^+ \times [0, 2\pi) \to \mathbb{C} : (r, \theta) \to re^{i\theta}$.*
μ is R-invariant iff there exists a measure $\nu \in \mathcal{M}(\mathbb{R}^+)$ such that $\mu = \sum_{i=1}^{p} \tilde{\mu}\phi_i$, where $\tilde{\mu} = (\nu \times \lambda)T^{-1}$ and $\nu \times \lambda$ represents the product measure of ν with λ.

Proof
First, we shall prove the implication $\Rightarrow$.

- Let $D_i \triangleq R^{\circ i}(D)$ be any of the rotation domains of the cycle considered ($i = 1(1)p$). Define ν_i as the unique measure on $\mathbb{R}^+$ satisfying

$$\nu_i(A) = \mu\left(\phi_i^{-1}(\{z \in \mathbb{C} \,|\, |z| \in A\})\right)$$

for any set A belonging to the semi-ring of halfopen (left closed, right open) subintervals of $\mathbb{R}^+$, and $\tilde{\mu}_i \triangleq (\nu_i \times \lambda)T^{-1}$. T being a continuous transformation, $\tilde{\mu}_i$ is a well defined measure on $\mathbb{C}$.
We now prove that $\tilde{\mu}_i = \mu\phi_i^{-1}$, by showing that both measures agree on any segment $S \triangleq S(r_1, r_2, \theta_1, \theta_2) \in \mathcal{A}$ as defined by (7) and (8) in lemma 1. Since μ is $R^{\circ p}$-invariant and $R^{\circ p}|_{D_i}$ is homeomorphic via ϕ_i to a rotation around the origin over an irrational multiple of 2π on a disc or an annulus, $\mu\phi_i^{-1}$ is invariant under such a rotation. Corollary 1 of lemma 2 and the definition of ν_i then yield that

$$\begin{aligned}\mu\phi_i^{-1}(S) &= \frac{\theta_2 - \theta_1}{2\pi} \cdot \mu\phi_i^{-1}(\{z \in \mathbb{C} | r_1 \leqslant |z| < r_2\}) \\ &= \lambda([\theta_1, \theta_2)) \cdot \nu_i([r_1, r_2)).\end{aligned}$$

On the other hand, $T^{-1}(S) = [r_1, r_2) \times [\theta_1, \theta_2)$, so

$$\begin{aligned}\tilde{\mu}_i(S) &= \nu_i([r_1, r_2)) \times \lambda([\theta_1, \theta_2)) \\ &= \mu\phi_i^{-1}(S).\end{aligned}$$

So by lemma 1, $\tilde{\mu}_i = \mu\phi_i^{-1}$, and also for any $B \in \mathcal{B}(\overline{\mathbb{C}})$,

$$\begin{aligned}\mu(B \cap D_i) &= \mu\phi_i^{-1}(\phi_i(B \cap D_i)) \\ &= \tilde{\mu}_i\phi_i(B \cap D_i).\end{aligned} \tag{10}$$

- Since $\mu(B) = \sum_{i=1}^p \mu(B \cap D_i)$ holds for any $B \in \mathcal{B}(\overline{\mathbb{C}})$, we have already established by (10) that $\mu = \sum_{i=1}^p \tilde{\mu}_i\phi_i$. Moreover, $\tilde{\mu}_i$ is independent of i, since for any $A \in \mathcal{A}$

$$\begin{aligned}\nu_i(A) &= \mu\left(\phi_i^{-1}(\{z \in \mathbb{C} \,|\, |z| \in A\})\right) \\ &= \mu\left((R^{\circ(p-i)})^{-1}\phi_p^{-1}(\{z \in \mathbb{C} \,|\, |z| \in A\}) \cap D_i\right) \\ &= \mu\left((R^{\circ(p-i)})^{-1}\phi_p^{-1}(\{z \in \mathbb{C} \,|\, |z| \in A\})\right) \\ &= \mu\left(\phi_p^{-1}(\{z \in \mathbb{C} \,|\, |z| \in A\})\right) \\ &= \nu_p(A).\end{aligned}$$

Putting $\tilde{\mu} = \tilde{\mu}_p$, we therefore have established the implication $\Rightarrow$.

The key observation for the proof of the other implication is that $\tilde{\mu}\phi_i R^{-1} = \tilde{\mu}\phi_{i+1}$ for $i = 1(1)p$, where $\phi_{p+1} \triangleq \phi_1$. $\square$

3.4 Conclusion

To conclude this section, we incorporate theorems 2, 3, 4 and 5 into theorem 1, yielding the following result.

Theorem 6 *Let $R(z) \in \mathbb{C}(z)$ be of degree greater than one and $\mu \in \mathcal{M}(\overline{\mathbb{C}})$, then μ is R-invariant iff μ is the sum of*

- *an R-invariant measure supported on $J(R)$,*
- *for every superattractive or attractive periodic cycle of points, equal discrete measures on the points of the cycle, and*
- *for every cycle of rotation domains, an R-invariant measure satisfying theorem 5.*

4 Characterization by a weight function

Let $R(z) \in \mathbb{C}(z)$ be of degree $d > 1$. In the previous section, we have described R-invariant measures by decomposing them into a finite sum of mutually singular R-invariant measures. We were able to characterize individually each component measure, except for one, namely the one supported by $J(R)$. In this section we shall develop a general method for characterizing R-invariant probability measures using a weight function γ for R. A weight function for R is defined using the concept of a complete set of branches of R^{-1}.

Definition 1 (complete set of branches of R^{-1}) *Let $R(z) \in \mathbb{C}(z)$ be of degree $d > 1$, $D_i \subset \overline{\mathbb{C}}$ for $i = 1(1)d$ and $R_i \triangleq R|_{D_i}$ be injective, then $\{R_i^{-1}\}_{i=1}^{d}$ is a complete set of branches of R^{-1} iff for any $z_o, \zeta_o \in \overline{\mathbb{C}}$, the number of indices i such that $R_i(z_o) = \zeta_o$, equals the multiplicity of z_o as a root of the equation $R(z) = \zeta_o$ over $\overline{\mathbb{C}}$.*

We define a weight function for R as follows.

Definition 2 (weight function for R) *Let $R(z) \in \mathbb{C}(z)$ be of degree $d > 1$ and $\gamma : \overline{\mathbb{C}} \to [0,1]$, then γ is a weight function for R iff*

$$\sum_{i=1}^{d} \gamma \circ R_i^{-1} \equiv 1 \tag{11}$$

for any complete set $\{R_i^{-1}\}_{i=1}^{d}$ of branches of R^{-1}.

Because in (11) the sum is taken over all branches of a complete set, it is obvious that we could have stipulated equivalently in definition 2 that (11) holds for *some* complete set of branches of R^{-1}.

The relation between an R-invariant measure μ and a corresponding weight function γ for R is given by

$$\forall\, f \in \mathcal{L}^1(\mu) : \int f\, d\mu = \int \sum_{i=1}^{d} \gamma(R_i^{-1}(z)) f(R_i^{-1}(z))\, d\mu(z) \tag{12}$$

for any complete set $\{R_i^{-1}\}_{i=1}^{d}$ of branches of R^{-1}. A specific complete set $\{R_i^{-1}\}_{i=1}^{d}$ of branches of R^{-1} being fixed, γ can be decomposed into d functions $(\gamma_i)_{i=1}^{d} \triangleq (\gamma \circ R_i^{-1})_{i=1}^{d}$, having the usual meaning of weight functions in the sense that $\gamma_i \geqslant 0$ for $i = 1(1)d$ and $\sum_{i=1}^{d} \gamma_i \equiv 1$ because of (11). Putting $f = \chi_B$ for $B \in \mathcal{B}(\overline{\mathbb{C}})$, it is clear from (12) that these weight functions describe how the R-invariant measure μ is spread out over each of the fixed branches of R^{-1}.

Let Γ denote a class of weight functions for R. As to the relation $\mathcal{R}$ consisting of the ordered pairs $(\mu, \gamma) \in \{\nu \in \mathcal{P}(\overline{\mathbb{C}}) \,|\, \nu R^{-1} = \nu\} \times \Gamma$ for which (12) holds, several fundamental questions can be raised.

1. Can we find for any R-invariant probability measure μ a weight function $\gamma \in \Gamma$ for R satisfying (12), i.e. does the domain of $\mathcal{R}$ equal $\{\nu \in \mathcal{P}(\overline{\mathbb{C}}) \,|\, \nu R^{-1} = \nu\}$?

2. Is this weight function γ unique in Γ, i.e. is $\mathcal{R}$ a function?

3. Is there for any weight function $\gamma \in \Gamma$ for R, an R-invariant probability measure μ satisfying (12), i.e. is $\mathcal{R}$ surjective?

4. Is μ uniquely determined by γ, i.e. is $\mathcal{R}$ injective?

For a full characterization of R-invariant probability measures, it is evident that the answer to all four of these questions should be affirmative for a suitably chosen class Γ of weight functions for R. We shall investigate this matter in this section. First we shall deal with questions 1 and 2, concerning the association of γ with μ, and secondly with questions 3 and 4, about the association of μ with γ. In the next section, we shall consider constructive aspects of this association.
Before doing so, we have to do some preliminary work on complete sets of measurable branches of R^{-1}.

4.1 Complete sets of measurable branches

For a rigorous development, we need complete sets of measurable branches of R^{-1}, defined in an obvious way.

Definition 3 (complete set of measurable branches of R^{-1}) *Let $R(z) \in \mathbb{C}(z)$ be of degree $d > 1$, $D_i \subset \overline{\mathbb{C}}$ for $i = 1(1)d$, $R_i \triangleq R|_{D_i}$ be injective and $\{R_i^{-1}\}_{i=1}^{d}$ be a complete set of branches of R^{-1}, then $\{R_i^{-1}\}_{i=1}^{d}$ is a complete set of measurable branches of R^{-1} iff R_i^{-1} is Borel measurable for each $i = 1(1)d$.*

Actually, we mostly need complete sets $\{R_i^{-1}\}_{i=1}^{d}$ of branches of R^{-1} such that R_i is measurable, but this condition is equivalent to the measurability of R_i^{-1}, because of the following proposition.

Proposition 1 *Let $R(z) \in \mathbb{C}(z)$, $D \subset \overline{\mathbb{C}}$ and $R|_D$ be injective, then $R|_D$ is Borel measurable iff D is Borel iff $R|_D^{-1}$ is Borel measurable .*

Proof
For constant functions and Möbius transformations, the proposition is obvious. Therefore, we only consider rational functions of degree greater than one in the sequel of the proof.

The first equivalence is the easiest to prove. Since for any set $A \subseteq \overline{\mathbb{C}}$,

$$R|_D^{-1}(A) = R^{-1}(A) \cap D \tag{13}$$

holds, the continuity of R yields that $R|_D(B) \in \mathcal{B}(\overline{\mathbb{C}})$ for any $B \in \mathcal{B}(\overline{\mathbb{C}})$, if $D \in \mathcal{B}(\overline{\mathbb{C}})$. Putting $A = \overline{\mathbb{C}}$ in (13) yields the other implication of the first equivalence.

We now prove the second equivalence.

Let C be the set of critical points of R. There exists a finite open cover of $\overline{\mathbb{C}} \setminus C$, say $\{O_i\}_{i=1}^k$, such that $R|_{O_i}$ is injective for any $i = 1(1)k$.

- Suppose $D \notin \mathcal{B}(\overline{\mathbb{C}})$, then for at least one O_m $(1 \leqslant m \leqslant k)$, $O_m \cap D \notin \mathcal{B}(\overline{\mathbb{C}})$ either. Otherwise $D \setminus C = \cup_{i=1}^k (O_i \cap D)$ would be Borel and hence D too, C being a finite set. Because of the continuity of R and the injectiveness of $R|_{O_m}$, $R(O_m \cap D)$ can not be a Borel set either, since otherwise $R^{-1}(R(O_m \cap D)) = O_m \cap D$ would be Borel. But $R(O_m \cap D) = R|_D(O_m) = (R_D^{-1})^{-1}(O_m)$, so $R|_D^{-1}$ is not measurable, which establishes the implication $\Leftarrow$ by contraposition.

- The other implication also holds, since if $D \in \mathcal{B}(\overline{\mathbb{C}})$, $R(B \cap D) \in \mathcal{B}(\overline{\mathbb{C}})$ for any $B \in \mathcal{B}(\overline{\mathbb{C}})$ because of the closedness of $\mathcal{B}(\overline{\mathbb{C}})$ under R. The latter can be proved by showing that $\mathcal{B}_i \triangleq \{B \in \mathcal{B}(\overline{\mathbb{C}}) \mid R(B \cap O_i) \in \mathcal{B}(\overline{\mathbb{C}})\}$ is a σ-algebra containing the open sets (since the image of an open set under a meromorphic function is open) and hence coinciding with $\mathcal{B}(\overline{\mathbb{C}})$ for any $i = 1(1)k$. As for any set $A \subseteq \overline{\mathbb{C}}$, $R(A) = \cup_{i=1}^k R(A \cap O_i) \bigcup R(A \cap C)$ holds, this implies that $R(B) \in \mathcal{B}(\overline{\mathbb{C}})$ if $B \in \mathcal{B}(\overline{\mathbb{C}})$, C being a finite set.

$\square$

The following proposition guarantees the existence of a complete set of measurable branches of R^{-1}.

Proposition 2 *Let $R(z) \in \mathbb{C}(z)$ be of degree $d > 1$, then there exists at least one complete set of measurable branches of R^{-1}.*

Proof

Let J be any closed Jordan curve connecting the points ∞, $R(\infty)$ and the (other) critical values of R. J partitions $\overline{\mathbb{C}} \setminus J$ into two disjoint simply connected domains D_1 and D_2, both having J as their boundary.

Each point $z_o \in D_1$ has precisely d distinct values under R^{-1}. By the inverse function theorem [17, theorem 3.1 p. 86-87], there exists a single valued inverse of R, analytic in z_o, according to each of these values. It is easy to see [18, theorem 8.20 p. 298] that any of these inverse functions can be continued analytically along any curve inside D_1. So, by virtue of the monodromy theorem, according to each of the d values of z_o under R^{-1}, there exists an analytic inverse of R on D_1 transforming z_o into that value. Consequently, there exist d disjoint domains C_{1i} (i=1(1)d), namely the images of D_1 under the d inverse functions of R constructed, that are transformed conformally into D_1 by R. ∂C_{1i} being a closed Jordan curve and R being continuous, $\overline{C_{1i}}$ is transformed homeomorphically into $\overline{D_1} = D_1 \cup J$ [18, corollary of theorem 2.24 p. 70].

Analogously, there exist d domains D_{2j} $(j = 1(1)d)$, each of which is transformed conformally into D_2 by R, that are mutually disjoint and disjoint with each of the regions $\overline{C_{1i}}$. The domains C_{1i} and C_{2j} are the $2d$ components of $\overline{\mathbb{C}} \setminus R^{-1}(J)$.

So, we have that any of the regions $C_i \triangleq \overline{C_{1i}} \cup C_{2i}$ for $i = 1(1)d$ is transformed one-to-one and onto $\overline{\mathbb{C}}$ by R. The regions C_i are mutually disjoint except for the critical points of R. By the inverse function theorem for the multiple valued case [17, theorem

3.4 p. 89-90 and corollary 1 p. 90], such a point z_o is a boundary point of a number of domains of the family $\{C_{1i}\}_{i=1}^d$ equal to the multiplicity of z_o as a critical point plus one, i.e. equal to the multiplicity of z_o as a root of the equation $R(z) = R(z_o)$ over $\overline{\mathbb{C}}$. Hence, $\{R_i^{-1}\}_{i=1}^d$ with $R_i \triangleq R|_{C_i}$ is a complete set of branches of R^{-1}. Moreover, each of the branches has the union of a closed and an open set as range, i.e. a Borel set, hence is measurable by proposition 1. □

4.2 Association of γ with μ

Now we are able to prove the following theorem, that associates a measurable weight function γ for R with each R-invariant probability measure μ.

Theorem 7 *Let $R(z) \in \mathbb{C}(z)$ be of degree $d > 1$ and $\mu \in \mathcal{P}(\overline{\mathbb{C}})$. μ is R-invariant iff there exists a Borel measurable weight function $\gamma : \overline{\mathbb{C}} \to [0,1]$ for R such that*

$$\mu(B) = \int \sum_{i=1}^{d} \gamma(R_i^{-1}(z))\chi_B(R_i^{-1}(z))\, d\mu(z) \tag{14}$$

for any $B \in \mathcal{B}(\overline{\mathbb{C}})$ and any complete set $\{R_i^{-1}\}_{i=1}^d$ of branches of R^{-1}.

Before proving this theorem, we notice that we could have stipulated equivalently that (14) holds for any $B \in \mathcal{B}(\overline{\mathbb{C}})$ and *some* (instead of *any*) complete set of branches of R^{-1}. Indeed, the integrand in the right-hand side of (14), being a function (namely $\gamma \cdot \chi_B$) evaluated in the image of a point under a certain branch of R^{-1} and summed over a complete set of branches of R^{-1}, clearly is independent of the choice of the complete set of branches of R^{-1}. Therefore, if (14) is satisfied for some complete set of branches of R^{-1}, it is for any.

Proof of the implication $\Rightarrow$ of theorem 7
Let $\{R_i^{-1}\}_{i=1}^d$ be a complete set of measurable branches of R^{-1}, which exists by virtue of proposition 2. Let $B_i \triangleq R_i^{-1}(\overline{\mathbb{C}})$ for $i = 1(1)d$. They are Borel sets because of proposition 1.
Let i be any natural number satisfying $1 \leqslant i \leqslant d$. μR_i^{-1} is a well defined measure, because R_i is measurable according to proposition 1. Moreover μR_i^{-1} is absolutely continuous w.r.t. μ. Indeed, let B be any Borel subset of $\overline{\mathbb{C}}$ satisfying $\mu(B) = 0$, then $\mu(R^{-1}(B)) = 0$ (R-invariance of μ), and since $R^{-1}(B) = \bigcup_{j=1}^n R_j^{-1}(B)$, $\mu R_i^{-1}(B) = 0$. Consequently, the theorem of Radon-Nikodym guarantees the existence of a measurable function $h_i : \overline{\mathbb{C}} \to \mathbb{R}^+$, such that

$$\mu R_i^{-1}(B) = \int_B h_i \, d\mu \tag{15}$$

holds for any $B \in \mathcal{B}(\overline{\mathbb{C}})$.
Let for any $z \in \overline{\mathbb{C}}$, $\mathcal{P}_z$ be the partition of $\{1, 2, \ldots, d\}$ consisting of the equivalence classes of the indices of elements of $\{R_j^{-1}\}_{j=1}^d$ taking the same value at z. Let $P_z(j)$ be the class containing j for any $j = 1(1)d$. For noncritical values z of R, $P_z(j) = \{j\}$ always holds. For critical values z of R, this equality only holds if $R_j^{-1}(z)$ is a simple root of $R(\zeta) = z$ over $\overline{\mathbb{C}}$. We define

$$\tilde{\gamma}_i : \overline{\mathbb{C}} \to \mathbb{R}^+ : z \to \frac{1}{|P_z(i)|^2} \sum_{j \in P_z(i)} h_j(z).$$

Since $\tilde{\gamma}_i$ only differs from the measurable function h_i in a finite number of points, namely some critical values of R, it is still measurable.

We now prove that

$$\mu(R^{-1}(B)) = \int_B \sum_{i=1}^{n} \tilde{\gamma}_i \, d\mu \tag{16}$$

holds for any $B \in \mathcal{B}(\overline{\mathbb{C}})$. Let C be the set of critical points of R. Let $B \in \mathcal{B}(\overline{\mathbb{C}})$. We establish (16) separately for the cases $B \cap R(C) = \emptyset$ and $B = \{w\} \subseteq R(C)$, the general one being obvious from these two.

1. case $B \cap R(C) = \emptyset$
   Then $R^{-1}(B) = \bigcup_{i=1}^{n} R_i^{-1}(B)$ is a disjoint union of Borel sets, so

$$\begin{aligned}\mu(R^{-1}(B)) &= \sum_{i=1}^{n} \mu R_i^{-1}(B) \\ &\overset{(15)}{=} \sum_{i=1}^{n} \int_B h_i \, d\mu \\ &= \sum_{i=1}^{n} \int_B \tilde{\gamma}_i \, d\mu.\end{aligned}$$

2. case $B = \{w\} \subseteq R(C)$
   Using the notation introduced above, it is obvious that for any $P \in \mathcal{P}_w$, $\tilde{\gamma}_i(w)$ has the same value for any $i \in P$ and

$$\sum_{i\in P} \tilde{\gamma}_i(w) = \frac{1}{|P|} \sum_{i \in P} h_i(w). \tag{17}$$

   Let K be a choice function on $\mathcal{P}_w$, then $R^{-1}(B) = \bigcup_{P\in\mathcal{P}_w} R_{K(P)}^{-1}(B)$, and this is a disjoint union of Borel sets. So

$$\begin{aligned}\mu(R^{-1}(B)) &= \sum_{P\in\mathcal{P}_w} \mu(R_{K(P)}^{-1}(B)) \\ &= \sum_{P\in\mathcal{P}_w} \frac{1}{|P|} \sum_{i\in P} \mu(R_i^{-1}(B)) \\ &\overset{(15)}{=} \sum_{P\in\mathcal{P}_w} \frac{1}{|P|} \sum_{i\in P} h_i(w)\mu(B) \\ &\overset{(17)}{=} \sum_{P\in\mathcal{P}_w} \sum_{i\in P} \tilde{\gamma}_i(w)\mu(B) \\ &= \sum_{i=1}^{d} \tilde{\gamma}_i(w)\mu(B) \\ &= \int_B \sum_{i=1}^{d} \tilde{\gamma}_i \, d\mu.\end{aligned}$$

So (16) is established.
Moreover, μ being R-invariant,

$$\int_B \left(\sum_{i=1}^{n} \tilde{\gamma_i} - 1\right) d\mu = \mu(R^{-1}(B)) - \mu(B) = 0$$

holds for any $B \in \mathcal{B}(\overline{\mathbb{C}})$. Hence $\sum_{i=1}^{d} \tilde{\gamma_i} = 1$ a.e. (μ), so by redefining each of the functions $\tilde{\gamma_i}$ to equal $\frac{1}{d}$ on a fixed set of measure zero, we arrive at a new set of nonnegative measurable functions γ_i satisfying $\sum_{i=1}^{n} \gamma_i \equiv 1$ and (16). At last, we can define the function $\gamma : \overline{\mathbb{C}} \to [0,1]$ by claiming that $\gamma|_{B_i} = \gamma_i \circ R_i$ for every $i = 1(1)d$. These claims are consistent, since we have constructed $\{\tilde{\gamma_i}\}_{i=1}^{d}$ and $\{\gamma_i\}_{i=1}^{d}$ in such a way that whenever $z_o \in B_i \cap B_j$, γ_i and γ_j agree in $R_i(z_o) = R_j(z_o)$, and they determine γ completely. As $\sum_{i=1}^{d} \gamma \circ R_i^{-1} = \sum_{i=1}^{d} \gamma_i \equiv 1$, γ clearly is a weight function for R. Moreover γ is measurable, since $\gamma^{-1}(A) = \cup_{i=1}^{d} \gamma|_{B_i}^{-1}(A)$ for any $A \subseteq \mathbb{R}^+$ and each of the γ_i's and R_i's is measurable.

We now prove that (14) is satisfied for any $B \in \mathcal{B}(\overline{\mathbb{C}})$. We distinguish between the cases $B \cap C = \varnothing$ and $B = \{c\} \subseteq C$, the general one being obvious from these two.

1. case $B \cap C = \varnothing$
   Then $B = \bigcup_{i=1}^{d} (B \cap B_i)$ is a disjoint union of Borel sets, so $\mu(B) = \sum_{i=1}^{d} \mu(B \cap B_i)$. Let i be any natural number satisfying $1 \leqslant i \leqslant d$. Since $B \cap B_i = R_i^{-1}(R(B \cap B_i))$ and $R(B \cap B_i) \in \mathcal{B}(\overline{\mathbb{C}})$ because $\mathcal{B}(\overline{\mathbb{C}})$ is closed under R as argumented in the proof of proposition 1,
   $$\mu(B \cap B_i) = \int h_i(z) \chi_{R(B\cap B_i)}(z)\, d\mu(z),$$
   holds because of (15). $B \cap B_i$ being disjoint with C, $h_i \chi_{R(B\cap B_i)} = \gamma_i \chi_{R(B\cap B_i)} = (\gamma \circ R_i^{-1}) \chi_{R(B\cap B_i)}$ a.e. (μ), so
   $$\begin{aligned} \mu(B \cap B_i) &= \int \gamma(R_i^{-1}(z)) \chi_{R(B\cap B_i)}(z)\, d\mu(z) \\ &= \int \gamma(R_i^{-1}(z)) \chi_B(R_i^{-1}(z))\, d\mu(z). \end{aligned}$$
   Consequently,
   $$\mu(B) = \int \sum_{i=1}^{d} \gamma(R_i^{-1}(z)) \chi_B(R_i^{-1}(z))\, d\mu(z).$$

2. case $B = \{c\} \subseteq C$
   Using the notation introduced above, let P be the class of $\mathcal{P}_{R(c)}$ the elements of which transform $R(c)$ into c. We know that
   $$\begin{aligned} \mu(B) &= \frac{1}{|P|} \sum_{i \in P} \mu R_i^{-1}(\{w\}) \\ &\overset{(15)}{=} \frac{1}{|P|} \sum_{i \in P} \int_{\{w\}} h_i(z)\, d\mu(z) \\ &= \frac{1}{|P|} \sum_{i \in P} h_i(w) \mu(\{w\}) \end{aligned}$$

$$
\begin{aligned}
&\stackrel{(17)}{=} \sum_{i\in P} \gamma_i(w)\mu(\{w\}) \\
&= \int \sum_{i=1}^{d} \gamma(R_i^{-1}(z))\chi_B(R_i^{-1}(z))\, d\mu(z).
\end{aligned}
$$

This completes the proof of the implication $\Rightarrow$ of theorem 7. $\square$

Proof of the implication $\Leftarrow$ of theorem 7
Let $\{R_i^{-1}\}_{i=1}^d$ be a complete set of measurable branches of R^{-1}, which exists by virtue of proposition 2. Let C be the set of critical points of R. Let B be any Borel subset of $\overline{\mathbb{C}}$.

1. case $B \cap R(C) = \varnothing$
   Then $R^{-1}(B) = \bigcup_{i=1}^d R_i^{-1}(B)$ is a disjoint union, more precisely of Borel sets because R_i^{-1} is measurable for $i = 1(1)d$, so
   $$\mu(R^{-1}(B)) = \sum_{i=1}^d \mu(R_i^{-1}(B)).$$
   Because $B \cap P(C) = \varnothing$,
   $$\chi_{R_i^{-1}(B)} \circ R_j^{-1} = \delta_{ij}\chi_B$$
   holds for any natural numbers i, j satisfying $1 \leqslant i, j \leqslant d$, so by (14)
   $$
   \begin{aligned}
   \mu(R_i^{-1}(B)) &= \int \sum_{j=1}^{d} \gamma(R_j^{-1}(z))\chi_{R_i^{-1}(B)}(R_j^{-1}(z))\, d\mu(z) \\
   &= \int (\gamma \circ R_i^{-1})\chi_B\, d\mu.
   \end{aligned}
   $$
   Consequently,
   $$\mu(R^{-1}(B)) = \sum_{i=1}^d \int_B (\gamma \circ R_i^{-1})\, d\mu = \int_B \left(\sum_{i=1}^d \gamma \circ R_i^{-1}\right) d\mu = \mu(B).$$

2. case $B = \{w\} \subseteq P(C)$
   Using the notation introduced in the proof of the implication $\Rightarrow$ of theorem 7, let K be a choice function on $\mathcal{P}_w$. Then $R^{-1}(B) = \cup_{P\in\mathcal{P}_w} R_{K(P)}^{-1}(B)$ is a disjoint union of Borel sets, so
   $$\mu(R^{-1}(B)) = \sum_{P\in\mathcal{P}_w} \mu(R_{K(P)}^{-1}(B)).$$
   Since for any natural numbers i, j satisfying $1 \leqslant i, j \leqslant d$
   $$\chi_{R_j^{-1}(B)} \circ R_i^{-1} = \chi_{P_w(j)}(i)\chi_B$$
   holds, so by (14)
   $$\mu(R_{K(P)}^{-1}(B)) = \sum_{i\in P} \gamma(R_i^{-1}(w))\mu(B)$$

for any $P \in \mathcal{P}_w$,

$$\begin{aligned}
\mu(R^{-1}(B)) &= \sum_{P \in \mathcal{P}_w} \sum_{i \in P} \gamma(R_i^{-1}(w))\mu(B) \\
&= \sum_{i=1}^{d} \gamma(R_i^{-1}(w))\mu(B) \\
&= \mu(B).
\end{aligned}$$

3. The general case $B \in \mathcal{B}(\overline{\mathbb{C}})$ is obvious from the previous two.

□

Once (14) is established for any $B \in \mathcal{B}(\overline{\mathbb{C}})$, it is commonplace in measure theory to generalize it to (12), (14) being equivalent to (12) for $f = \chi_B$. This technique is formalized in the following proposition.

Proposition 3 *Let $R(z) \in \mathbb{C}(z)$ be of degree $d > 1$, $\{R_i^{-1}\}_{i=1}^d$ be some complete set of branches of R^{-1}, γ be a Borel measurable weight function for R and $\mu, \nu \in \mathcal{P}(\overline{\mathbb{C}})$. If*

$$\nu(B) = \int \sum_{i=1}^{d} \gamma(R_i^{-1}(z))\chi_B(R_i^{-1}(z))\, d\mu(z) \tag{18}$$

holds for any $B \in \mathcal{B}(\overline{\mathbb{C}})$, then

$$\int f\, d\nu = \int \sum_{i=1}^{d} \gamma(R_i^{-1}(z)) f(R_i^{-1}(z))\, d\mu(z) \tag{19}$$

holds for any $f \in \mathcal{L}^1(\mu)$.

Proof
Extend (19) from integrable characteristic functions, as given by (18), to integrable step functions by linearity and then to integrable functions using the dominated convergence theorem. □

An application of this proposition to theorem (7) yields the following corollary.

Corollary 2 *Let $R(z) \in \mathbb{C}(z)$ be of degree $d > 1$ and $\mu \in \mathcal{P}(\overline{\mathbb{C}})$. μ is R-invariant iff there exists a Borel measurable weight function $\gamma : \overline{\mathbb{C}} \to [0, 1]$ for R such that*

$$\int f\, d\mu = \int \sum_{i=1}^{d} \gamma(R_i^{-1}(z)) f(R_i^{-1}(z))\, d\mu(z)$$

holds for any $f \in \mathcal{L}^1(\mu)$ and any complete set $\{R_i^{-1}\}_{i=1}^d$ of branches of R^{-1}.

As to the uniqueness of the weight function γ associated with μ by (14), it is obvious that it never holds w.r.t. the class of all measurable weight functions for R, since γ only occurs in the integrand of (14). More precisely, we can prove the next proposition, claiming γ to be unique if representing an equivalence class of $\mathcal{L}^1(\mu)$ modulo μ.

Proposition 4 *Let $R(z) \in \mathbb{C}(z)$ be of degree $d > 1$, $\mu \in \mathcal{P}(\overline{\mathbb{C}})$ be R-invariant and γ_1, γ_2 be Borel measurable weight functions for R. If*

$$\forall\, B \in \mathcal{B}(\overline{\mathbb{C}}) : \; \mu(B) = \int \sum_{i=1}^{d} \gamma(R_i^{-1}(z))\chi_B(R_i^{-1}(z))\, d\mu(z) \tag{20}$$

is satisfied for any complete set $\{R_i^{-1}\}_{i=1}^{d}$ of branches of R^{-1} and for $\gamma = \gamma_1$ as well as for $\gamma = \gamma_2$, then $\gamma_1 = \gamma_2$ a.e. (μ).

Proof
Because of the symmetric role of γ_1 and γ_2 and because of the σ-additivity of μ, is suffices to prove that $A_\epsilon \triangleq \{c \in \overline{\mathbb{C}} \,|\, \gamma_2 \geqslant \gamma_1 + \epsilon\}$ satisfies $\mu(A_\epsilon) = 0$ for any $\epsilon \in \mathbb{R}_0^+$.
Let $\{R_i^{-1}\}_{i=1}^{d}$ be a complete set of measurable branches of R^{-1}, existing by proposition 2, so $B_i \triangleq R_i^{-1}(\overline{\mathbb{C}}) \in \mathcal{B}(\overline{\mathbb{C}})$ for $i = 1(1)d$ by proposition 1. A_ϵ being a Borel subset of $\overline{\mathbb{C}}$, we know that

$$\begin{aligned}
\mu(A_\epsilon) &= \int \sum_{i=1}^{d} \gamma_2(R_i^{-1}(z))\chi_{A_\epsilon}(R_i^{-1}(z))\, d\mu(z) \\
&\geqslant \int \sum_{i=1}^{d} \gamma_1(R_i^{-1}(z))\chi_{A_\epsilon}(R_i^{-1}(z))\, d\mu(z) + \epsilon \sum_{i=1}^{d} \int \chi_{R(A_\epsilon \cap B_i)}(z)\, d\mu(z) \\
&= \mu(A_\epsilon) + \epsilon \sum_{i=1}^{d} \mu\left(R(A_\epsilon \cap B_i)\right) \\
&\geqslant \mu(A_\epsilon) + \epsilon \sum_{i=1}^{d} \mu(A_\epsilon \cap B_i) \\
&= (1+\epsilon)\mu(A_\epsilon),
\end{aligned}$$

so $\mu(A_\epsilon)$ has to vanish. □

If we know γ_1 and γ_2 to be continuous (on $\mathrm{supp}(\mu)$), proposition 4 implies that $\gamma_1 = \gamma_2$ on $\mathrm{supp}(\mu)$. So, if we would be able to show that in theorem 7, γ can be taken continuous, it is uniquely determined by (20) on $\mathrm{supp}(\mu)$ in the class of continuous weight functions for R. We shall return to this point in the conclusion of this section.

We now investigate the association of an R-invariant probability measure with a given weight function for R.

4.3 Association of μ with γ

First, we shall examine the third question raised on p. 11, namely whether there exists an R-invariant probability measure μ satisfying (12) for a given γ belonging to a certain class of weight functions for R.

We seek to construct such an R-invariant probability measure μ as a fixed point of an operator τ on the space of probability measures $\mathcal{P}(\overline{\mathbb{C}})$. Such an operator can be defined based on (20) as in the following proposition, not only on the entire space $\mathcal{P}(\overline{\mathbb{C}})$, but also on subspaces $\mathcal{P}_C(\overline{\mathbb{C}})$ consisting of the probability measures with support on a fixed R^{-1}-closed subset C of $\overline{\mathbb{C}}$.

Proposition 5 *Let $R(z) \in \mathbb{C}(z)$ be of degree $d > 1$, γ be a Borel measurable weight function for R and $\{R_i^{-1}\}_{i=1}^d$ be a complete set of branches of R^{-1}, then*

$$\tau : \mathcal{P}(\overline{\mathbb{C}}) \to \mathcal{P}(\overline{\mathbb{C}}) : \mu \to \tau(\mu) \qquad (21)$$
$$\textit{with } \tau(\mu) : \mathcal{B}(\overline{\mathbb{C}}) \to [0,1] : B \to \int \sum_{i=1}^{d} \gamma(R_i^{-1}(z))\chi_B(R_i^{-1}(z))\, d\mu(z)$$

defines an operator on $\mathcal{P}(\overline{\mathbb{C}})$ as well as on $\mathcal{P}_C(\overline{\mathbb{C}})$, where C is any R^{-1}-closed subset of $\overline{\mathbb{C}}$.

Proof
By the argument given after the wording of theorem 7 and proposition 2, we can suppose without any loss of generality $\{R_i^{-1}\}_{i=1}^d$ to be a complete set of measurable branches of R^{-1}. Therefore, the integrand in (21) is easily seen to be integrable w.r.t. μ for any $B \in \mathcal{B}(\overline{\mathbb{C}})$, so the right-hand side of (21) is well defined.

The proof that $\mathcal{P}(\overline{\mathbb{C}})$ is closed under τ, relies on an easy application of the monotone convergence theorem, and that $\mathcal{P}_C(\overline{\mathbb{C}})$ is closed under τ for any R^{-1}-closed subset C of $\overline{\mathbb{C}}$, on the fact that $\text{supp}(\tau(\mu)) \subseteq R^{-1}(\text{supp}(\mu))$. □

So, the question (re)phrased above, is equivalent to the one about the existence of a fixed point of the operator τ defined by (21). Concerning such matters, one can use the theorem of Schauder-Tychonoff [11, point 5 p. 453], stating that a compact convex subset of a locally convex linear topological space has the fixed point property [11, point 1 p. 453], the latter meaning that each continuous transformation has at least one fixed point. The following proposition states that the measure spaces relevant in our context, satisfy the conditions of the Schauder-Tychonoff theorem.

Proposition 6 *The space consisting of the finite complex Borel measures on $\overline{\mathbb{C}}$ is a locally convex linear topological space w.r.t. the weak* topology, and $\mathcal{P}_C(\overline{\mathbb{C}})$ is a compact convex subspace of it for any compact R^{-1}-closed subset C of $\overline{\mathbb{C}}$.*

Proof
By virtue of the Riesz-representation theorem [11, point 3 p. 265], the space consisting of the finite complex Borel measures on $\overline{\mathbb{C}}$ is a Banach space w.r.t. the norm corresponding to the weak* topology, hence [11, point 12 p. 53 and point 2 p. 59] a linear topological space. That it is locally convex, follows from a straightforward application of the definition [11, point 9 p. 417].
As a consequence of the theorem of Alaoglu [11, point 2 p. 424], the subspace consisting of the measures μ the total variation of which satisfies $|\mu| \leqslant 1$, is compact. $\mathcal{P}_C(\overline{\mathbb{C}})$ being a closed subset of that subspace for any compact R^{-1}-closed subset C of $\overline{\mathbb{C}}$, it is also compact. Moreover, it obviously is convex. □

To apply the fixed point property established by the Schauder-Tychonoff theorem, to the operator τ defined by (21), τ has to be continuous w.r.t. the weak* topology. We can guarantee this, if the weight function γ is continuous.

Theorem 8 *Let $R(z) \in \mathbb{C}(z)$ be of degree $d > 1$, $\{R_i^{-1}\}_{i=1}^d$ be a complete set of branches of R^{-1}, C be a compact and R^{-1}-closed subset of $\overline{\mathbb{C}}$ and γ be a weight function for R that is continuous on C, then the operator τ defined by (21) is continuous w.r.t. the weak* topology on $\mathcal{P}_C(\overline{\mathbb{C}})$.*

Proof

1. We first prove that whenever $f : \overline{\mathbb{C}} \to \mathbb{C}$ is bounded and continuous on C, then

$$F : \overline{\mathbb{C}} \to \mathbb{C} : \; z \to \sum_{i=1}^{d} \gamma(R_i^{-1}(z)) f(R_i^{-1}(z)) \tag{22}$$

is bounded and continuous on C too.

   - We prove that F is continuous at any point $w \in C$.
   Let $(w_j)_{j=1}^{\infty}$ be any sequence of points of C converging to w. Let $z_i \triangleq R_i^{-1}(w)$ and $N(z_i)$ denote a neighbourhood of z_i for $i = 1(1)d$. By the inverse function theorem [17, theorem 3.4 p. 89–90 and corollary 1 p. 90], for any set of sufficiently small neighbourhoods $\{N(z_i)\}_{i=1}^{d}$, $|(R^{-1}(w_m) \cap N(z_i))| = \mathrm{mult}(z_i)$ holds for $i = 1(1)d$ and for sufficiently large $m \in \mathbb{N}$, where $\mathrm{mult}(z_i)$ denotes the multiplicity of z_i as a root of the equation $R(z) = w$ over $\overline{\mathbb{C}}$. $\{R_i^{-1}\}_{i=1}^{d}$ being a complete set of branches of R^{-1}, this implies that there is a sequence of permutations $(\pi_i)_{i=1}^{\infty}$ of $\{1, 2, \ldots, d\}$ such that

$$R_{\pi_m(i)}^{-1}(w_m) \to z_i, \; m \to \infty$$

   for $i = 1(1)d$. Consequently, each of the points involved belonging to C because of the R^{-1}-closedness of C, and f and γ being continuous on C,

$$\gamma(R_{\pi_m(i)}^{-1}(w_m)) f(R_{\pi_m(i)}^{-1}(w_m)) \to \gamma(z_i) f(z_i), \; m \to \infty.$$

   So

$$F(w_m) = \sum_{i=1}^{d} \gamma(R_i^{-1}(w_m)) f(R_i^{-1}(w_m)) =$$
$$\sum_{i=1}^{d} \gamma(R_{\pi_m(i)}^{-1}(w_m)) f(R_{\pi_m(i)}^{-1}(w_m)) \to \sum_{i=1}^{d} \gamma(z_i) f(z_i) = F(w), \; m \to \infty.$$

   - The boundedness of F is obvious from the boundedness of f and the fact that γ is a weight function for R.

2. We now prove that τ is continuous w.r.t. the weak* topology on $\mathcal{P}_C(\overline{\mathbb{C}})$.

   Let $(\mu_i)_{i=1}^{\infty}$ be a sequence of probability measures with support on C converging to $\mu \in \mathcal{P}_C(\overline{\mathbb{C}})$ in the weak* sense. Let f be a bounded complex function on $\overline{\mathbb{C}}$ continuous on C, then F defined by (22) also is, so

$$\int F \, d\mu_i \to \int F \, d\mu, \; i \to \infty.$$

   Applying proposition 3 to the definition (21) of τ, we know that

$$\int f d\tau(\mu) = \int \sum_{i=1}^{d} \gamma(R_i^{-1}(z)) f(R_i^{-1}(z)) \, d\mu(z).$$

Hence, we have proved that

$$\int f\, d\tau(\mu_i) \to \int f\, d\tau(\mu),\ i \to \infty$$

for any bounded complex function f on $\overline{\mathbb{C}}$ continuous on C, which establishes the continuity of τ on $\mathcal{P}_C(\overline{\mathbb{C}})$ w.r.t. the weak* topology.

□

Combining propositions 5 and 6 and theorem 8 with the fixed point theorem of Schauder-Tychonoff and proposition 3, yields the ultimate existence theorem.

Theorem 9 *Let $R(z) \in \mathbb{C}(z)$ be of degree $d > 1$, C be a compact R^{-1}-closed subset of $\overline{\mathbb{C}}$ and γ be a weight function for R that is continuous on C, then there is at least one measure $\mu \in \mathcal{P}_C(\overline{\mathbb{C}})$ satisfying*

$$\forall\, f \in \mathcal{L}^1(\mu) : \int f\, d\mu = \int \sum_{i=1}^{d} \gamma(R_i^{-1}(z)) f(R_i^{-1}(z))\, d\mu(z) \tag{23}$$

for any complete set of branches $\{R_i^{-1}\}_{i=1}^d$ of R^{-1}.

As to the uniqueness of μ (question 4 on p. 11), it certainly does not always hold if we put $C = \overline{\mathbb{C}}$. This is clear from the decomposition theorem 6: any of the component measures, rescaled to a probability measure, satisfies (23) for a fixed weight function γ for R, as well as any convex combination of them. So, if a fixed point of τ has more than one component in view of theorem 6, it is not unique.
Concerning weight functions for R corresponding to component measures associated with $F(R)$, we can prove the following proposition.

Proposition 7 *Let $R(z) \in \mathbb{C}(z)$ be of degree $d > 1$, C be either the set of points of a superattractive or attractive p-periodic cycle $(z_i)_{i=1}^p$ of R, or the union of a cycle of rotation domains, $\mu \in \mathcal{P}(\overline{\mathbb{C}})$ be R-invariant and vanishing outside C and $\{R_i^{-1}\}_{i=1}^d$ any complete set of branches of R^{-1}, then*

$$\gamma : \overline{\mathbb{C}} \to [0,1] : z \to \begin{cases} \frac{1}{\left|\left\{k \in \{1,2,\ldots,d\} \,|\, R_k^{-1}(R(z)) = z\right\}\right|} & \text{if } z \in C \\ 0 & \text{if } z \in R^{-1}(C) \setminus C \\ \frac{1}{d} & \text{if } z \in \overline{\mathbb{C}} \setminus R^{-1}(C) \end{cases} \tag{24}$$

is a Borel measurable weight function for R satisfying (23).

Proof
In case $C = \{z_i\}_{i=1}^p$, the proposition is easily verified by using theorem 3.
In case C is the union of a cycle of rotation domains, the keys to the proof are that $R|_C$ is injective, that C contains no critical points of R and that $\gamma|_C \equiv 1$. □

The definition (24) of γ is clearly independent of the choice of $\{R_i^{-1}\}_{i=1}^d$.
We notice that $\gamma|_C \equiv 1$ unless C is the set of points of a superattractive cycle. Then C contains at least one critical point z_o, for which $\gamma(z_o) = \frac{1}{\text{mult}(z_o)} < 1$ holds, where $\text{mult}(z_o)$ denotes the multiplicity of z_o as a root of the equation $R(z) = R(z_o)$ over $\overline{\mathbb{C}}$.

So, in case C is the union of a cycle of rotation domains, $\gamma|_C \equiv 1$. By virtue of proposition 4, this means that $\gamma = 1$ a.e. (μ) for any measurable weight function γ for R satisfying (23), if μ is an R-invariant probability measure vanishing outside C. Since there are many such measures μ because of theorem 5, uniqueness of the association of μ with γ also does not hold in this case.
Generally however, a fixed point of the operator τ defined by (21) has only one component in view of theorem 6, namely the one with support on $J(R)$. This follows from proposition 7 combined with proposition 4: only weight functions for R coinciding almost everywhere with the one defined by (24), allow other component measures. Normally, this condition is not met, unless R has a cycle of critical points of multiplicity d. In that case, the definition of a weight function γ for R already implies that $\gamma|_C \equiv \frac{1}{d}$, satisfying (24) a.e. for C being the set of points of this cycle. Such points are known as the exceptional points of R [9, corollary 4.2 p. 105 and corollary 4.7 p. 107]. As a consequence of Montel's theorem, R can have no more than 2 exceptional points [9, corollary 4.2 p. 105]. Polynomials have $z_o = \infty$ as an exceptional fixed point [9, theorem 4.5 p. 106].
Anyway, if $\gamma < 1$, any fixed point of τ can only be a convex combination of a fixed point with support on $J(R)$ and balanced discrete probability measures supported by the points of superattractive cycles of R. If $\gamma > 0$, which implies $\gamma < 1$, the superattractive cycles can only contain exceptional points, if any. Superattractive cycles (of this kind) being relatively rare, the fixed points of τ generally have their support on $J(R)$.
Moreover, we are actually more interested in applying τ on the subspace $\mathcal{P}_C(\overline{\mathbb{C}})$ with $C = J(R)$ than with $C = \overline{\mathbb{C}}$, because in the previous section, we were able to characterize completely all component measures of theorem 6, except for the one supported by $J(R)$.

However, we can not guarantee τ to have a unique fixed point in $\mathcal{P}_{J(R)}(\overline{\mathbb{C}})$ for a general continuous weight function γ for R. We have vainly tried to prove that this restriction of τ is contractive w.r.t. the Prohorov or Hutchinson metric under quite general conditions. In that case, the fixed point theorem of Banach would guarantee the existence of a unique fixed point.
There is only one rather specific result of this kind we know of, using another approach. We shall discuss it in the next section (see proposition 9).

4.4 Conclusion

In this section, we have characterized R-invariant probability measures by weight functions for R, especially those supported on $J(R)$, we had not been able to characterize in the previous section. We have considered two classes of weight functions γ for R, namely the class Γ_m consisting of the measurable ones and the subclass Γ_c containing the continuous ones.
We can associate a measurable weight function $\gamma \in \Gamma_m$ satisfying (23) for any set $\{R_i^{-1}\}_{i=1}^d$ of branches of R^{-1} with any R-invariant probability measure μ. γ is unique if seen as an equivalence class of $\mathcal{L}^1(\mu)$ modulo μ.
Conversely, corresponding to every *continuous* weight function $\gamma \in \Gamma_c$, there exists an R-invariant probability measure μ (supported on $J(R)$ or any other compact R^{-1}-closed subset of $\overline{\mathbb{C}}$) satisfying (23). Uniqueness is still an open question.
To bridge the gap between the two directions of association, we have looked vainly for a class Γ of weight functions for R satisfying $\Gamma_c \subseteq \Gamma \subseteq \Gamma_m$, so that both associations hold w.r.t. that class. On the one hand, one could try to guarantee the existence of a

continuous weight function within the equivalence class of $\mathcal{L}^1(\mu)$ modulo μ, represented by the function γ mentioned in theorem 7. The weight function γ being constructed in the proof of theorem 7 by using Radon-Nikodym derivatives, this seems to be a hard problem. On the other hand, there appears to be no way around claiming the function F defined by (22) to be continuous, guaranteeing the operator τ in (21) to be continuous.

5 Construction

Let $R(z) \in \mathbb{C}(z)$ be of degree $d > 1$, $\{R_i^{-1}\}_{i=1}^{d}$ be any complete set of branches of R^{-1} and γ be a given measurable weight function for R. In this section we shall discuss the methods we know to construct an R-invariant probability measure μ satisfying (23). The first method constructs the measure as such and was first applied in this setting in [10, chapter 3 p. 138–143] to the balanced case $\gamma \equiv \frac{1}{d}$. The second one, being only applicable to polynomials, constructs its moment generating function. It is used in [19, section 2 p. 316–319] and is elaborated in [20] and [5, sections IV.2 and IV.3 p. 78–84].

5.1 Straightforward construction of the measure

As argumented in the previous section, we are actually looking for a fixed point of the operator τ defined by (21) on the space $\mathcal{P}(\overline{\mathbb{C}})$ or on the subspace $\mathcal{P}_C(\overline{\mathbb{C}})$ corresponding to an R^{-1}-closed subset $C \subset \overline{\mathbb{C}}$. A straightforward method to find such a fixed point, is by iterating τ on some arbitrary starting measure $\mu_0 \in \mathcal{P}_C(\overline{\mathbb{C}})$. If this process converges, it furnishes at least a candidate R-invariant measure $\mu \in \mathcal{P}(\overline{\mathbb{C}})$. If C is compact, $\mu \in \mathcal{P}_C(\overline{\mathbb{C}})$. If γ is continuous, so that τ is continuous by theorem 8, the candidate μ satisfies (23).

Interesting starting measures are Dirac measures, say $\mu_0 = \delta_c$ for some $c \in \overline{\mathbb{C}}$. In that case, all measures $\tau^{\circ i}(\mu_0)$ for $i \in \mathbb{N}_0$ generated by iterating τ, are discrete probability measures supported by at most d^i points, namely the points mapped onto c by $R^{\circ i}$. It is clear that, if the iterative process converges to a limit measure μ, the support of μ can only contain points belonging to

$$L \triangleq \left(\bigcup_{i\in\mathbb{N}}(R^{\circ i})^{-1}(c)\right)' \quad \text{or} \tag{25}$$

$$\limsup_{i\to\infty}(R^{\circ i})^{-1}(c), \tag{26}$$

the prime in (25) denoting the derived set, i.e. the set of limit points.
As to (25), this set is empty if c is an exceptional point of R [9, theorem 4.5 p. 106]. Otherwise, $J(R) \subseteq L$ [9, corollary 4.7 p. 107] and the equality holds iff c does not belong to a rotation domain [10, theorem 8.1 p 112, rotation domains being called singular domains in this reference]. If it does belong to a p-periodic rotation domain D, L also contains $\cup_{i=0}^{p-1} R^{\circ i}(\phi^{-1}(C_\rho))$, where ϕ is the homeomorphism satisfying (3) or (4) in case D is a Siegel disk or a Herman ring respectively and $\rho \triangleq |\phi(c)|$, provided $\rho \neq 0$. In case $\rho = 0$, the cycle of transformed circles degenerates into a cycle of irrationally indifferent p-periodic points of R, that is not contained in (25), but in (26).
Indeed, as to (26), it is obvious that this set can only be nonempty if c is a periodic point, and in that case consists precisely of the points of the periodic cycle containing c.
All these observations agree with theorem 6. They also reflect that not all R-invariant

probability measures vanishing outside a cycle of rotation domains, can be constructed this way, since their support is not limited to transformed circles.

Of course, this construction method suffers from the drawbacks discussed in the previous section. If γ can not be chosen continuous so that τ is continuous, the limit measure μ might not satisfy (23) and if we do not know τ to be contractive, we even can not guarantee the limit to exist.

If $\gamma \equiv \frac{1}{d}$, we know this iterative process to converge. The limit measure is always the same for any $c \in \overline{\mathbb{C}}$ except for the exceptional points of R, if any: it is the unique balancedly R-invariant probability measure. For polynomials, this property is established in [10, theorem 16.1 p. 141–142] and the limit measure can also be described as the logarithmic equilibrium measure on $J(R)$. In general, it is proven in [16, theorem 2 p. 362–364] and the limit measure can be characterized as the probability measure of maximal entropy w.r.t. R [16, sections 5 and 6 p. 366–384].

5.2 Construction of the moment generating function

Another approach to find an R-invariant probability measure μ satisfying (23), is by constructing its moment generating function g_μ defined as

$$g_\mu : D_{g_\mu} \subseteq \overline{\mathbb{C}} \to \mathbb{C} : \ z \to \int \frac{1}{z-\zeta}\, d\mu(\zeta).$$

Provided the support of μ is a bounded subset of $\mathbb{C}$, the domain D_{g_μ} of g_μ comprises a neighbourhood of ∞ and is indeed the generating function of the moments $\mu_i \triangleq \int \zeta^i \, d\mu(\zeta)$ ($i \in \mathbb{N}$), since

$$g_\mu(z) = \sum_{i=1}^{\infty} \frac{\mu_{i-1}}{z^i} \tag{27}$$

for $z \in \overline{\mathbb{C}}$ with sufficiently large modulus $|z|$.
The point ∞ being a superattractive fixed point for any nonlinear polynomial $P(z) \in \mathbb{C}[z]$, it is clear from theorem 6 that this condition is satisfied for any P-invariant probability measure vanishing at ∞. For a general rational function R of degree greater than one, the moments do not even have to exist [4]. Since in addition, we shall explicitly need that R is polynomial, we only consider polynomials in the sequel.

Let $P(z) \in \mathbb{C}[z]$ be of degree $d > 1$ with highest order coefficient p_d and $\{P_i^{-1}\}_{i=1}^d$ be a complete set of branches of P^{-1}. Let μ be any measure with bounded support on $\mathbb{C}$, so that g_μ exists. Since $\mathrm{supp}(\tau(\mu)) \subseteq P^{-1}(\mathrm{supp}(\mu))$ and ∞ is a superattractive fixed point of P, $\mathrm{supp}(\tau(\mu))$ is bounded also and $g_{\tau(\mu)}$ also exists. Hence, we can look for a relation between g_μ and $g_{\tau(\mu)}$. The combination of the definition (21) of the operator τ with proposition 3 yields that

$$\forall\, f \in \mathcal{L}^1(\mu) : \int f d\tau(\mu) = \int \sum_{i=1}^{d} \gamma(P_i^{-1}(\zeta)) f(P_i^{-1}(\zeta))\, d\mu(\zeta).$$

We can take f to be the Cauchy kernel $\frac{1}{z-\zeta}$ seen as a function of ζ. If $z \notin \mathrm{supp}(\mu)$, this function is guaranteed to be integrable w.r.t. μ, so

$$g_{\tau(\mu)}(z) \quad = \quad \int \sum_{i=1}^{d} \frac{\gamma(P_i^{-1}(\zeta))}{z - P_i^{-1}(\zeta)}\, d\mu(\zeta)$$

$$= \int \frac{W(z,\zeta)}{P(z)-\zeta}\, d\mu(\zeta), \tag{28}$$

where

$$W : \overline{\mathbb{C}} \times \overline{\mathbb{C}} \to \overline{\mathbb{C}} : (z,\zeta) \to p_d \cdot \sum_{i=1}^{d} \left(\gamma(P_i^{-1}(\zeta)) \prod_{i \neq j=1}^{d} (z - P_j^{-1}(\zeta)) \right) \tag{29}$$

is a polynomial of degree $d-1$ in z having the same highest order coefficient p_d as P and other coefficients that generally depend on ζ. The conditions γ has to satisfy, in order the latter not to hold, i.e. in order W to be independent of its second argument, are easily expressed.

Proposition 8 *Let $P(z) \in \mathbb{C}[z]$ be of degree $d > 1$ with highest order coefficient p_d, $\{P_i^{-1}\}_{i=1}^d$ be a complete set of branches of P^{-1}, $C \subseteq \overline{\mathbb{C}}$, $\gamma : P^{-1}(C) \to \overline{\mathbb{C}}$ and W be defined by (29) on $\overline{\mathbb{C}} \times C$, then $W(z,\zeta)$ is independent of $\zeta \in C$ iff there exists a polynomial $W^\star(z) \in \mathbb{C}[z]$ of degree $d-1$, such that*

$$\gamma(z) = \frac{W^\star(z)}{P'(z)} \tag{30}$$

for $z \in P^{-1}(C)$. In that case $W(z,\zeta) \equiv W^\star(z)$ on $\overline{\mathbb{C}} \times C$, and $\sum_{i=1}^d \gamma \circ P_i^{-1} \equiv 1$ on C iff the highest order coefficient of $W^\star$ equals p_d.

Proof
The implication $\Rightarrow$ is obvious, since if $W(z,\zeta)$ is independent of $\zeta \in C$ for $z \in \overline{\mathbb{C}}$,

$$\begin{aligned} W^\star(P_i^{-1}(\zeta)) &= W(P_i^{-1}(\zeta),\zeta) \\ &= p_d \cdot \gamma(P_i^{-1}(\zeta)) \prod_{i \neq j=1}^{d} (P_i^{-1}(\zeta) - P_j^{-1}(\zeta)) \\ &= \gamma(P_i^{-1}(\zeta)) \cdot P'(P_i^{-1}(\zeta)) \end{aligned}$$

holds for any index i and any $\zeta \in C$.
The keys to the other implication are that

$$\begin{aligned} W(z,\zeta) &= (P(z)-\zeta) \cdot \sum_{i=1}^{d} \frac{\gamma(P_i^{-1}(\zeta))}{z - P_i^{-1}(\zeta)} \\ &= (P(z)-\zeta) \cdot \sum_{j=0}^{\infty} \left[\sum_{i=1}^{d} \gamma(P_i^{-1}(\zeta))(P_i^{-1}(\zeta))^j \right] \frac{1}{z^{j+1}} \end{aligned}$$

holds for any $\zeta \in C$ and $z \in \overline{\mathbb{C}}$ sufficiently large in modulus w.r.t. ζ, and that, having chosen γ as stipulated in (30) on $P^{-1}(C)$,

$$\sum_{i=1}^{d} \gamma(P_i^{-1}(\zeta))(P_i^{-1}(\zeta))^j$$

is a polynomial in ζ of degree $\lfloor \frac{j}{d} \rfloor$ for any $j \in \mathbb{N}$ and $\zeta \in C$, as proven in [6, lemma 2.1 p. 159]. The coefficient of z^{d-1} in $W(z,\zeta)$ clearly equals $p_d \cdot \sum_{i=1}^d \gamma(P_i^{-1}(\zeta))$. □

We notice that the function γ in proposition 8 is not always (the restriction to $P^{-1}(C)$ of) a weight function for P, because it is not necessarily nonnegative. We also observe that the choice $W^\star = \frac{1}{d}P'$ leads to the balanced weight function $\gamma \equiv \frac{1}{d}$.

Let C be a compact P^{-1}-invariant subset of $\mathbb{C}$ and γ be a weight function for P so that $W(z,\zeta)$ defined by (29) is independent of ζ for $\zeta \in C$. By virtue of proposition 8, such a weight function γ can be constructed (and can only be constructed) by choosing a polynomial $W^\star(z) \in \mathbb{C}[z]$ of degree $d-1$ having highest order coefficient p_d, and claiming that $\gamma = \frac{W^\star}{P'}$ on C. Obviously this function, being defined on the P^{-1}-invariant and hence (since P is a surjective function) P-invariant subset C, can be extended to a weight function for P on $\overline{\mathbb{C}}$ iff it is a weight function for P on C. Since γ satisfies $\sum_{i=1}^d \gamma \circ P_i^{-1} \equiv 1$ on C, this condition is met iff γ is nonnegative on C. So the construction is possible for and only for polynomials $W^\star$ having the same highest order coefficient as P and satisfying $\frac{W^\star}{P'} \geqslant 0$ on C. Let μ be any probability measure with support on C. Since $W(z,\zeta) = W^\star(z)$ for $z \in \overline{\mathbb{C}}$ and $\zeta \in \mathrm{supp}(\mu)$, we know by (28) that

$$\begin{aligned} g_{\tau(\mu)}(z) &= W^\star(z) \int \frac{1}{P(z)-\zeta}\, d\mu(\zeta) \\ &= W^\star(z) \cdot g_\mu(P(z)) \end{aligned}$$

for $z \in \overline{\mathbb{C}}$ sufficiently large in modulus. So, the moment generating function we are looking for, is a fixed point of the operator

$$\tilde{\tau} : H(\infty) \to H(\infty) : g \to W^\star \cdot (g \circ P), \tag{31}$$

where $H(\infty)$ is the set of functions $g : D_g \subseteq \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ the domain D_g of which comprises a neighbourhood of ∞ and that have an expansion of the form $g(z) = \sum_{i=1}^\infty \frac{c_i}{z^i}$ with a positive radius of convergence and $c_1 = 1$, just as (27). Concerning these fixed points, the following theorem can be established [19, 20, 5]

Theorem 10 *Let $P(z) \in \mathbb{C}[z]$ be of degree $d > 1$ having p_d as highest order coefficient, $W^\star(z) \in \mathbb{C}[z]$ be of degree $d-1$ having the same highest order coefficient p_d and $\tilde{\tau}$ be the operator defined by (31), then $\tilde{\tau}$ has precisely one fixed point $h \in H(\infty)$. The iterative process $(\tilde{\tau}^{\circ i}(g))_{i=0}^\infty$ converges to h for any starting function $g \in H(\infty)$. The fixed point h is analytic in the superattractive domain of P containing ∞.*

Proof

See for instance [5, theorem 3 p. 81–82] and [5, theorem 4 p. 82–83]. □

Moreover, in our setting, the unique limit function h of theorem 10 is the moment generating function of at least one measure $\tilde{\mu} \in \mathcal{P}(\overline{\mathbb{C}})$ with support on C. Indeed, let $\mu_0 \in \mathcal{P}_C(\overline{\mathbb{C}})$ be arbitrary. Because of proposition 5 and the compactness of $\mathcal{P}_C(\overline{\mathbb{C}})$ (see proposition 6), we know that $\{\tau^{\circ i}(\mu_0)\}_{i=0}^\infty$ has at least one limit measure $\tilde{\mu} \in \mathcal{P}_C(\overline{\mathbb{C}})$. So, there exists an increasing row $(k_i)_{i=1}^\infty$ of natural numbers satisfying $\tau^{\circ k_i}(\mu_0) \xrightarrow{\star} \tilde{\mu}$, $i \to \infty$. All the measures involved being supported on C, so that their moment generating function exists in a neighbourhood $N(\infty)$ of ∞, applying the definition of weak* convergence to the Cauchy kernel yields that

$$g_{\tau^{\circ k_i}(\mu_0)}(z) = \int \frac{1}{z-\zeta}\, d(\tau^{\circ k_i}(\mu_0))(\zeta) \to \int \frac{1}{z-\zeta}\, d\tilde{\mu}(\zeta) = g_{\tilde{\mu}}(z),\ i \to \infty$$

for $z \in N(\infty)$. On the other hand, it is obvious that

$$g_{\tau^{\circ k_i}(\mu_0)} = \tilde{\tau}^{\circ k_i}(g_{\mu_0}),$$

and by theorem 10, we know that

$$\tilde{\tau}^{\circ i}(g_{\mu_0}) \to h,\ i \to \infty,$$

so h equals the moment generating function $g_{\tilde{\mu}}$.

This result can also be obtained by applying the existence theorem 9. Indeed, γ being a weight function for P satisfying (30) on C, it obviously is continuous on C. So, C being a compact P^{-1}-invariant and a fortiori P^{-1}-closed subset of $\mathbb{C}$, theorem 9 guarantees the existence of a fixed point μ of τ in $\mathcal{P}_C(\overline{\mathbb{C}})$, yielding the unique fixed point $h = g_\mu$ of $\tilde{\tau}$. Actually, this result is a bit stronger than the former one, because the measure $\tilde{\mu}$ constructed above, although satisfying $\tilde{\tau}(g_{\tilde{\mu}}) = g_{\tilde{\mu}}$, is not known to be a fixed point of τ. Indeed, in view of the expansion (27) it is clear, that the equality of $g_{\tilde{\mu}}$ and $\tilde{\tau}(g_{\tilde{\mu}}) = g_{\tau(\tilde{\mu})}$ is equivalent to the equality of all the corresponding moments of $\tilde{\mu}$ and $\tau(\tilde{\mu})$, i.e. with the claim

$$\int \zeta^i \, d\tilde{\mu}(\zeta) = \int \zeta^i \, d\tau(\tilde{\mu})(\zeta)$$

for any $i \in \mathbb{N}$. On the other hand, the equality of $\tilde{\mu}$ and $\tau(\tilde{\mu})$ is equivalent to

$$\int f \, d\tilde{\mu} = \int f \, d\tau(\tilde{\mu})$$

for any complex function f continuous on C (and hence, C being compact, bounded on C). Generally, the set of polynomials is not dense in the set of continuous complex functions on a compact subset C of $\mathbb{C}$ w.r.t. the topology induced by the supremum norm, because a sequence of polynomials, just as any other sequence of analytic functions, can only converge in a neighbourhood to an analytic function w.r.t. this topology.

However, if C has no interior points and the complement of C is connected, it is true that the set of polynomials is dense in the set of continuous complex functions on C as a consequence of the theorem of Mergelyan [23, p. 425]. So in that case, any limit measure $\tilde{\mu}$ of the iterative process $(\tau^{\circ i}(\mu_0))_{i=0}^{\infty}$ is a fixed point μ of the operator τ in $\mathcal{P}_C(\overline{\mathbb{C}})$ for any starting measure $\mu_0 \in \mathcal{P}_C(\overline{\mathbb{C}})$. Moreover, this fixed point μ is unique, because its moments are the coefficients in the expansion of the form (27) for the unique fixed point h of $\tilde{\tau}$, so for any complex function f continuous on C, $\int f \, d\mu$ is independent of the fixed point μ of τ chosen, hence all fixed points coincide.
This is the situation we referred to at the end of our discussion about the uniqueness of the association of μ with γ in the previous section. It is formalized in the following proposition [20, 5].

Proposition 9 *Let $P(z) \in \mathbb{C}[z]$ be of degree $d > 1$, $W^\star(z) \in \mathbb{C}[z]$ be of degree $d-1$ and have the same highest order coefficient as P, γ be a weight function for P and C be a compact P^{-1}-closed subset of $\mathbb{C}$ having no interior points and such that the complement of C is connected and $\frac{W^\star}{P'} \geqslant 0$ on C.*
If γ satisfies $\gamma = \frac{W^\star}{P'}$ on C, then there is a unique $\mu \in \mathcal{P}_C(\overline{\mathbb{C}})$ satisfying

$$\forall f \in \mathcal{L}^1(\mu) : \int f \, d\mu = \int \sum_{i=1}^{d} \gamma(P_i^{-1}(z)) f(P_i^{-1}(z)) \, d\mu(z) \tag{32}$$

for any complete set $\{P_i^{-1}\}_{i=1}^d$ of branches of P^{-1}. Moreover, the iterative process $(\tau^{\circ i}(\mu_0))_{i=0}^{\infty}$ converges to it for any $\mu_0 \in \mathcal{P}_C(\overline{\mathbb{C}})$, where τ is the operator on $\mathcal{P}_C(\overline{\mathbb{C}})$ defined by (21).

Proof

A straightforward generalization of [20, theorem 6 p. 302–303]. □

The Julia set $J(P)$ being a P^{-1}-closed compact subset of $\mathbb{C}$ having no interior points [9, corollary 4.6 p. 106], we can apply proposition 9 to $C = J(P)$, provided $F(P)$ is connected. This condition is met iff all points of $\overline{\mathbb{C}} \setminus J(P)$ converge to ∞ under the iterates of P [9, p. 114], or equivalently, iff the complement of the superattractive domain containing ∞ equals $J(P)$. This occurs if $J(P)$ is a subset of the real line and this situation is of special interest in theoretical physics [21, 3, 4, 20, 19, 6, 5].

If C does not satisfy the extra conditions of having no interior points or having a connected complement, we only know the limit function h in theorem 10 to be the moment generating function g_μ of some $\mu \in \mathcal{P}_C(\overline{\mathbb{C}})$, but we do not know whether it satisfies (32), nor whether it is unique. In analogy to the use of Dirac measures as starting points for the straightforward construction of μ by iterating τ, we can use here the moment generating function of a Dirac measure, say δ_c with $c \in \mathbb{C}$, as a starting point for an iterative process based on $\tilde{\tau}$, namely $g_{\delta_c}(z) = \frac{1}{z-c}$. In that case, the i-th element of the sequence produced by iterating $\tilde{\tau}$ is a rational function having $P^{\circ i}(z) - c$ as denominator, namely the moment generating function of the discrete measure obtained by iterating i times τ on δ_c. So, we get a sequence of rational functions of order $\left[\frac{d^i-1}{d^i}\right]$ converging to the unique fixed point h of $\tilde{\tau}$. If $c = h''(\infty)/2$, or equivalently, if $c = \frac{w^\star_{d-2} - p_{d-1}}{p_d}$ where $w^\star_{d-2}$ and p_{d-1} are the one but highest order coefficients of $W^\star$ and P respectively, they are the Padé-approximants of h around ∞ of the given order [5, corollary of theorem 5 p. 84]. Based on what we have said about (25) and (26) in the previous subsection, it is obvious that the poles of these rational approximants do not have any accumulation points in the superattractive domain containing ∞, which agrees with the analyticity of h in that domain as established by theorem 10.

The major drawbacks of this method to find the moment generating function of a measure μ satisfying (32), is its limited applicability, because of the fact that W defined by (29) has to be independent of its second argument. In addition, if we want to be sure that the candidate solution we find, really is the moment generating function of a measure μ satisfying (32), the conditions that have to be met become much stricter yet.

Once again, there are no problems if we apply the method to the balanced case $\gamma \equiv \frac{1}{d}$.

5.3 Conclusion

This section addressed the construction problem of an R-invariant probability measure μ associated with a given measurable weight function γ for a rational function $R(z) \in \mathbb{C}(z)$ of degree $d > 1$.

The obvious method of iterating the operator τ, having μ as a fixed point, on some starting measure, was shown to be inadequate in the general setting because no convergence guarantee can be given.

Convergence of this iterative process to a unique R-invariant measure associated with γ can be guaranteed in case R is polynomial, γ is chosen within a certain affine subspace of

dimension d and some conditions on the preset support are met. Although the premises are rather strict, some interesting R-invariant probability measures on the Julia set $J(R)$ can be generated this way. If only the conditions on the preset support are not fulfilled, the operator τ acting on measures can still be translated into the operator $\tilde{\tau}$ acting on the corresponding moment generating functions, and for this operator $\tilde{\tau}$ convergence to a unique fixed point still holds.
The drawbacks of this construction process are intrinsically linked to the difficulties discussed in the previous section.

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