Block orthogonal systems for symmetric P-forms

Dieter van Melkebeek* and Adhemar Bultheel†
Department of Computer Science
K.U. Leuven

Abstract

To study polynomials orthogonal with respect to the logarithmic equilibrium measure on the Julia set of a nonlinear polynomial P, Daniel Bessis and his coworkers introduced a symmetric bilinear form, the symmetric P-form. It is a nonhermitian variant of a genuine inner product, the hermitian P-form. The main reason for doing so, was the existence of a three term recurrence relation between the polynomials of successive degree of an orthogonal system.

However, this is only true if the moment matrix of the symmetric P-form is strongly regular. In this paper, we investigate what happens if this condition is not met. We distinguish between the case where an orthogonal system still exists and the one where it does not, and illustrate them by $P(z) = z^2$ and $P(z) = z^2 - 1$ respectively.

1 Introduction

It is known for more than 20 years [11, theorem 2.2 p. 27] that the iterates of a nonlinear complex polynomial P form, up to an additive constant, a subsequence of an orthogonal system, i.e. a sequence of polynomials of successive degree (starting with degree 0) orthogonal to all polynomials of lower degree. The inner product for which this orthogonality relation holds, is defined as

$$(\cdot,\cdot)_h : \mathbb{C}[z] \times \mathbb{C}[z] \to \mathbb{C} : \ (P_1, P_2) \to \int P_1(z)\overline{P_2(z)}\, d\mu_\ell(J(P))(z), \tag{1}$$

where $\mu_\ell(J(P))$ represents the logarithmic equilibrium measure on the Julia set of P. We call $(\cdot,\cdot)_h$ the hermitian P-form.

An obvious question to ask is that for the lacking polynomials of the system. In general however their construction appears to be difficult. The difficulty is related to the fact that there is no simple three term recurrence relation between the successive polynomials of an orthogonal system, which goes hand in hand with the moment matrix being a general (hermitian positive definite) matrix, without any other interesting structure. In the special case where the Julia set of P is a subset of the real line, the moment matrix becomes Hankel-structured, and we do get a three term recurrence relation.

*Research Assistant of the Belgian National Fund for Scientific Research
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Probably inspired by this fact, Daniel Bessis and his coworkers [3, 4, 9] investigated the nonhermitian variant of (1), defined as

$$(\cdot,\cdot)_s : \mathbb{C}[z] \times \mathbb{C}[z] \to \mathbb{C} : \ (P_1, P_2) \to \int P_1(z) P_2(z)\, d\mu_\ell(J(P))(z). \tag{2}$$

We shall call it the symmetric P-form. As its moment matrix is always Hankel-structured, a three term recurrence relation between the successive polynomials of an orthogonal system holds, provided that an orthonormal system exists, which is equivalent to the moment matrix being strongly regular. Moreover, it is equally true that the iterates of P are orthogonal polynomials up to an additive constant. As a consequence, the iterates of P can be calculated using the recurrence relation at the expense of introducing the lacking orthogonal polynomials.

In this paper we investigate the symmetric P-form (2) in the case where the moment matrix is *not* strongly regular. Using a weakening of the concept of an orthogonal system, namely a block orthogonal system, we shall derive a three term recurrence relation, no longer between orthogonal polynomials of successive degree (which may not exist), but between successive so called regularly orthogonal polynomials [7, p. 47]. In the case where the moment matrix is strongly regular, this relation reduces to the ordinary one.

We shall distinguish between two cases. In the first one, an orthogonal system still exists, although there is no orthonormal system. It is illustrated by $P(z) = z^2$. The second case is characterized by the non-existence of an orthogonal system and is illustrated by $P(z) = z^2 - 1$. We examine the structure of block orthogonal systems for both cases, and determine the recurrence relation between the successive regularly orthogonal polynomials. Finally we also derive a connection between these regularly orthogonal polynomials and the iterates of P and see which recurrence relation results between the iterates for the second case, this being impossible for the first one.

2 Symmetric P-forms

Let P be a complex polynomial. The sequence of the successive iterates of P is denoted by $(P^{\circ k})_{k=0}^{\infty}$ and can be defined recurrently by $P^{\circ 0}(z) = z$ and $P^{\circ(k+1)} = P \circ P^{\circ k}$ for $k \in \mathbb{N}$. Strongly related to the iterates of a nonlinear complex polynomial P is its Julia set $J(P)$. It is the boundary of the set of points of the complex plane that converge to ∞ under the successive iterates of P:

$$J(P) = \partial\{z \in \mathbb{C} \mid \lim_{k\to\infty} P^{\circ k}(z) = \infty\}$$

On this Julia set $J(P)$, one can consider the logarithmic equilibrium measure $\mu_\ell(J(P))$, defined as the unique Borel probability measure μ on $\mathbb{C}$ with support on $J(P)$ minimizing its associated logarithmic potential energy

$$E(\mu) = \iint \ln\left(\frac{1}{|z-\zeta|}\right)\, d\mu(\zeta)\, d\mu(z).$$

It has the property of being balancedly P-invariant [2, theorem 1 p. 510-511], which can be used to prove the following lemma ([1, theorem 2 p. 382] for the monic case and [12, lemma 3.1 p. 64-65] for the general one).

Lemma 1 *Let* $P : \mathbb{C} \to \mathbb{C} :\ z \to \sum_{i=0}^{n} p_i z^i$ *be of degree* $n \geqslant 2$ *and* $\mu = \mu_\ell(J(P))$*, then*

$$\forall\, f \in \mathcal{L}(\mu) :\ \int z^i f(P(z))\, d\mu(z) = \frac{s_i}{n} \int f(z)\, d\mu(z) \text{ for } i = 0(1)n-1,$$

where

$$\begin{aligned} s_0 &= n \\ s_i &= -i\frac{p_{n-i}}{p_n} - \sum_{j=1}^{i-1} \frac{p_{n+j-i}}{p_n} s_j \text{ for } i = 1(1)n-1. \end{aligned}$$

$\mathcal{L}(\mu)$ denotes the set of all functions integrable with respect to μ.

$J(P)$ being a bounded subset of $\mathbb{C}$, $\mu_\ell(J(P))$ can be used to define a symmetric bilinear form for complex polynomials as in (2), the so called symmetric P-form. It is a Hankel-form, i.e. $(zP(z), Q(z))_s = (P(z), zQ(z))_s$ holds for any $P(z), Q(z) \in \mathbb{C}[z]$, and it has the following interesting property.

Theorem 1 *Let* $P \in \mathbb{C}[z]$ *of degree* $n \geqslant 2$*,* $Q \in \mathbb{C}[z]$ *and nonconstant and* $(\cdot,\cdot)_s$ *defined by (2), then*

$$\begin{aligned} &(z^i, Q(z))_s \begin{cases} = 0 & \text{for } i = 0(1)k-1 \\ \neq 0 & \text{if } i = k \end{cases} \\ \Longrightarrow\ &(z^i, Q(P(z)))_s \begin{cases} = 0 & \text{for } i = 0(1)kn-1 \\ \neq 0 & \text{if } i = kn \end{cases} \end{aligned}$$

holds for any $k \in \mathbb{N}$.

Proof
Let $\mu = \mu_\ell(J(P))$. Let l and m be any integers such that $0 \leqslant l < n$ and $0 \leqslant m < k$, and

$$f : \mathbb{C} \to \mathbb{C} :\ z \to z^m Q(z).$$

μ having a bounded support, it follows that $f \in \mathcal{L}^1(\mu)$. So lemma 1 is applicable to f. It asserts that

$$\int z^l f(P(z))\, d\mu(z) = \frac{s_l}{n} \int f(z)\, d\mu(z),$$

that is to say,

$$\int z^l (P(z))^m Q(P(z))\, d\mu(z) = \frac{s_l}{n} \int z^m Q(z)\, d\mu(z)$$

The right-hand side of this equation vanishes because of the premise of the theorem. Since $\{z^l(P(z))^m\}_{l=0(1)n-1, m=0(1)k-1}$ spans the space of complex polynomials of degree less than $n - 1 + (k-1)n + 1 = kn$, the first part of the theorem is established.
An analogue argument choosing $l = 0$ and $m = k$, in conjunction with the first part, proves the second part of the theorem. □

This theorem can be used to prove inductively the orthogonality relation mentioned in the introduction ([1, theorem 3 p. 383] for the monic case and [12, corollary 3.5 p. 77-78] for the general one).

Corollary 1 *Let $P : \mathbb{C} \to \mathbb{C} : z \to \sum_{i=0}^{n} p_i z^i$ be of degree $n \geqslant 2$ and $(\cdot,\cdot)_s$ be defined by (2), then*

$$(z^i, P^{\circ k}(z) + \frac{p_{n-1}}{np_n})_s = 0 \text{ for } i = 0(1)n^k - 1$$

holds for any $k \in \mathbb{N}$.

We notice that theorem 1 does not only hold for symmetric P-forms, although in the proof given we make use of lemma 1, which is typical for symmetric P-forms. In fact, another argument can be given for theorem 1, that establishes it for a class of symmetric bilinear forms associated to P, containing the symmetric P-form. In this more general context, corollary 1 reads

$$(z^i, P^{\circ k}(z) - \mu_1) = 0 \text{ for } i = 0(1)n^k - 1$$

for any $k \in \mathbb{N}$, where $(\cdot,\cdot)$ represents the symmetric bilinear form and $\mu_1 = (z, 1)$. We refer to [6, p. 505] and [9, p. 318] for the definition of these symmetric bilinear forms (using line integrals in the complex plane), to [6, theorem 13 and 14 p. 521–522] and [9, p. 322] for the argument and to [12, theorem 3.2 p. 76–77] and [10, theorem 6 p. 302–303] for the relation with symmetric bilinear forms defined analogously to the symmetric P-form (using integrals with respect to a measure).

3 Block orthogonal systems

In this section we introduce the notion of a block orthogonal system for a general symmetric bilinear form and subsequently we shall give some specific properties for the Hankel case. In the next section we shall apply this to symmetric P-forms.

3.1 A general symmetric bilinear form

Consider a symmetric bilinear form $(\cdot,\cdot) : \mathbb{C}[z] \times \mathbb{C}[z] \to \mathbb{C}$, and its moment matrix $M = [(z^i, z^j)]_{i,j=0}^{\infty}$. As M is not guaranteed to be strongly regular, an orthogonal system for $(\cdot,\cdot)$ may not exist. Indeed, if the leading submatrix of dimension $\nu \times \nu$ of M, denoted by $M_{\nu\times\nu}$ is singular, an orthogonal polynomial of degree ν does not exist, or it is not determined up to a multiplicative scalar. What we do know for sure, is that there is a polynomial of degree ν orthogonal to all polynomials of degree less than μ, where $M_{\mu\times\mu}$ is the largest regular leading submatrix of $M_{\nu\times\nu}$. This property can be used to define a generalization of an orthogonal system, that reduces to the latter one if M is strongly regular, but that always exists. We call it a block orthogonal system for reasons that will become apparent from the sequel.

Definition 1 (block orthogonal system and its block structure)
Let $(\cdot,\cdot) : \mathbb{C}[z] \times \mathbb{C}[z] \to \mathbb{C}$ be a symmetric bilinear form and $M = [(z^i, z^j)]_{i,j=0}^{\infty}$ its moment matrix. By $(\nu_k)_{k=0}^{B}$ we denote the possibly terminating sequence of block indices of M, defined such that $(M_{\nu_k\times\nu_k})_{k=1}^{B-1}$ are the successive regular leading submatrices of M, $\nu_0 = 0$ and $\nu_B = \infty$ if the sequence is terminating.
A sequence $(P_j(z))_{j=0}^{\infty}$ of complex polynomials is called a block orthogonal system for $(\cdot,\cdot)$ iff $P_j(z)$ is of degree j for any $j \in \mathbb{N}$ and

$$(z^i, P_j(z)) = 0 \text{ for } i = 0(1)\nu_k - 1\,, j = \nu_k(1)\nu_{k+1} - 1\,, k = 1(1)B - 1 \tag{3}$$

The subsequence $(P_j(z))_{j=\nu_k}^{\nu_{k+1}-1}$ *is called the k-th block of the block orthogonal system* $(0 \leqslant k < B)$. *It contains* $\ell_{k+1} \triangleq \nu_{k+1} - \nu_k$ *polynomials of successive degree.*

The relations (3) say that the polynomials of a block have to be orthogonal to those of the previous blocks. It can be interpreted in terms of matrices as follows. Let R be the upper triangular matrix such that

$$(P_j(z))_{j=0}^{\infty} = \left(\sum_{i=0}^{j} R_{ij} z^i\right)_{j=0}^{\infty},$$

then $D \triangleq R^T M R$ is a block diagonal matrix $D = \mathrm{diag}(D_i)_{i=1}^{B}$, where $\dim(D_i) = \ell_i \times \ell_i$. If M is strongly regular, D becomes scalar diagonal.

3.2 A Hankel-form

If $(\cdot,\cdot)$ is a Hankel-form, or equivalently, if M is a Hankel-matrix, then the following theorem can be used to determine the structure of the blocks of any block orthogonal system for $(\cdot,\cdot)$.

Theorem 2 *Let* $(\cdot,\cdot) : \mathbb{C}[z] \times \mathbb{C}[z] \to \mathbb{C}$ *be a bilinear form with Hankel-structured moment matrix* M. *If* ν_k *is a block index of* M *and* $P(z) \in \mathbb{C}[z]$ *is an orthogonal polynomial of degree* ν_k, *then* $\nu_{k+1} = \nu + 1$, *where*

$$\nu = \begin{cases} \min\{i \in \mathbb{N} \mid (z^i, P(z)) \neq 0\} & \textit{if the set considered is not empty} \\ \infty & \textit{otherwise.} \end{cases}$$

Proof
Let R be any upper triangular matrix of dimension $(\nu+1) \times (\nu+1)$ such that $R_{\nu_k \times \nu_k}$ is regular and

$$\left(\sum_{i=0}^{j} R_{ij} z^i\right)_{j=\nu_k}^{\nu} = (z^{j-\nu_k} P(z))_{j=\nu_k}^{\nu}.$$

Because of the Hankel structure of M, $z^{j-\nu_k} P(z)$ is orthogonal to all polynomials of degree less than ν_k for $j = \nu_k(1)\nu$, so that

$$D \triangleq R^T M R = \mathrm{diag}(R_{\nu_k \times \nu_k}^T M_{\nu_k \times \nu_k} R_{\nu_k \times \nu_k}, D^\star).$$

Since $M_{\nu_k \times \nu_k}$ and $R_{\nu_k \times \nu_k}$ are regular, this implies that $M_{j \times j}$ is regular iff $D^\star_{(j-\nu_k)\times(j-\nu_k)}$ is regular, for $j = \nu_k + 1(1)\nu + 1$. Moreover $D^\star$ is a regular lower triangular Hankel matrix if ν is finite, and $D^\star = 0$ if ν is infinite. It follows that $\nu_{k+1} = \nu + 1$ in both cases. □

On account of the regular right lower triangular structure of the matrix $D^\star$ used in the previous proof, the next corollary can be established [7, property 1.14 p. 41-43].

Corollary 2
An orthogonal polynomial of degree i *exists iff there is a block index* ν_k *satisfying* $\nu_k \leqslant i < \nu_k + \lceil \ell_{k+1}/2 \rceil$.
An orthonormal polynomial of degree i *exists iff there is a block index* ν_k *satisfying* $i = \nu_k + \frac{\ell_{k+1}-1}{2}$.

Based on the relation between orthogonal polynomials and Padé approximants to the generating function $g(z) \triangleq \sum_{i=1}^{\infty} \frac{\mu_i}{z^i}$ of the moments $\mu_i \triangleq (z^i, 1)$ [8, paragraph 1], the first part of corollary 2 can be rephrased as stating that the Padé approximant to $g(z)$ of order $\left[\frac{i-1}{i}\right]$ around ∞ exists iff there is a block index ν_k satisfying $\nu_k \leqslant i < \nu_k + \lceil \ell_{k+1}/2 \rceil$.

The specific block structure used in the proof of theorem 2, can also be used to derive a three term recurrence relation between the first polynomials of successive blocks of the system [7, theorem 1.5 p. 54-67]. These polynomials are called regularly orthogonal, because they are orthogonal polynomials uniquely determined up to a multiplicative scalar: the coefficient matrices determining their coefficients as a function of the highest order coefficient are regular.

Theorem 3 *Let $(P_i(z))_{i=0}^{\infty}$ be a block orthogonal system for the Hankel-form $(\cdot,\cdot) : \mathbb{C}[z] \times \mathbb{C}[z] \to \mathbb{C}$. For any integer k satisfying $0 \leqslant k < B-1$, there exist a polynomial $c_{k+1}(z) \in \mathbb{C}[z]$ of degree ℓ_{k+1} and a constant b_{k+1}, such that*

$$P_{\nu_{k+1}}(z) = c_{k+1}(z) P_{\nu_k}(z) + b_{k+1} P_{\nu_{k-1}}(z), \tag{4}$$

where $P_{\nu_{-1}}(z) \equiv 0$.

Proof

Let $(Q_i(z))_{i=0}^{\infty} = ((z^j P_{\nu_k}(z))_{j=0}^{\ell_{k+1}-1})_{k=0}^{B-1}$. From the proof of theorem 2, we know that $(Q_i(z))_{i=0}^{\infty}$ is also a block orthogonal system for $(\cdot,\cdot)$.

Let k be any integer satisfying $0 \leqslant k < B-1$.

Theorem 2 itself implies that

$$\begin{array}{rcll} (Q_i(z), z^{\ell_{k+1}} P_{\nu_k}(z)) & = & 0 & \text{for } \; i = 0(1)\nu_k - 2 \\ (Q_i(z), P_{\nu_{k-1}}(z)) & = & 0 & \text{for } \; i = 0(1)\nu_k - 2 \\ (Q_i(z), P_{\nu_{k-1}}(z)) & \neq & 0 & \;\;\text{if } \; i = \nu_k - 1. \end{array}$$

Therefore, we can choose

$$b_{k+1} = -\frac{(z^{\nu_{k+1}-1}, P_{\nu_k}(z))}{(z^{\nu_k - 1}, P_{\nu_{k-1}}(z))},$$

such that

$$(Q_i(z), z^{\ell_{k+1}} P_{\nu_k}(z) + b_{k+1} P_{\nu_{k-1}}(z)) = 0 \text{ for } i = 0(1)\nu_k - 1.$$

Because $(Q_i(z))_{i=0}^{\infty}$ is a block orthogonal system for $(\cdot,\cdot)$,

$$(Q_i(z), \sum_{j=\nu_k}^{\nu_{k+1}-1} \gamma_j Q_j(z)) = 0 \text{ for } i = 0(1)\nu_k - 1$$

and hence also

$$(Q_i(z), z^{\ell_{k+1}} P_{\nu_k}(z) + b_{k+1} P_{\nu_{k-1}}(z) + \sum_{j=\nu_k}^{\nu_{k+1}-1} \gamma_j Q_j(z)) = 0 \text{ for } i = 0(1)\nu_k - 1 \tag{5}$$

holds for any choice of $c \triangleq [\gamma_j]_{j=\nu_k}^{\nu_{k+1}-1} \in \mathbb{C}^{\ell_{k+1} \times 1}$.

Because of the same reason, $[(Q_i(z), Q_j(z))]_{i,j=\nu_k}^{\nu_{k+1}-1}$ is a regular matrix, which implies that there is a (unique) c that makes (5) also hold for $i = \nu_k(1)\nu_{k+1} - 1$. This means that the second argument of $(\cdot,\cdot)$ in (5) is an orthogonal polynomial of degree ν_{k+1}. As it is determined up to a multiplicative scalar, the theorem follows because of the particular choice $Q_j(z) = z^{j-\nu_k} P_{\nu_k}(z)$ for $j = \nu_k(1)\nu_{k+1} - 1$. $\square$

There is a connection between (4) and the following three term relation, which is a generalization of [6, theorem 3 p. 512]. For any $k, l, m \in \mathbb{N}$ satisfying $k < l < m$ and an orthogonal system $(P_i)_{i=0}^{\infty}$ for a Hankel-form, there exist $\alpha_{k,l,m}(z), \beta_{k,l,m}(z), \gamma_{k,l,m}(z) \in \mathbb{C}[z]$ of degree $l-k-1$, $m-k-1$ and $m-l-1$ respectively, such that

$$\alpha_{k,l,m}(z)P_m(z) = \beta_{k,l,m}(z)P_l(z) + \gamma_{k,l,m}(z)P_k(z). \tag{6}$$

Using the techniques of [6, appendix p. 526–527] (6) can be established for some block orthogonal system of any symmetric P-form, even if the moment matrix M is not strongly regular. The polynomials $\alpha_{k,l,m}(z)$, $\beta_{k,l,m}(z)$ and $\gamma_{k,l,m}(z)$ turn out to have polynomial coefficients in the coefficients of P and need not be of strict degree if M is not strongly regular. When applied to successive regularly orthogonal polynomials, (6) reduces to (4). A detailed analysis is beyond the scope of this article.

If M is strongly regular, (4) comes to the ordinary three term recurrence relation between orthogonal polynomials of successive degree.

4 Block orthogonal systems for symmetric P-forms

We now investigate the orthogonality properties for symmetric P-forms whose moment matrix M is not strongly regular, using the concept of a block orthogonal system. We can distinguish between two cases, both of which we shall illustrate by an example. As to the choice of these examples, we can make use of [6, theorem 17 p. 525], which is based on [5, p. 136–137]. Interpreted in our terms, this theorem states that a sufficient condition for M not being strongly regular for a polynomial $P(z) = z^2 + c$, is that P has a superstable cycle, say of order k, or equivalently, that $P^{\circ k}(0) = 0$. Moreover, we know that in that case the block index of the first block containing more than one polynomial, is less than 2^k and odd. Solving $P^{\circ k}(0) = 0$ yields $P(z) = z^2$ for $k = 1$ and $P(z) = z^2 - 1$ (and $P(z) = z^2$) for $k = 2$. These two polynomials give rise to two different types of block orthogonal systems.

1. The first is the one where an orthogonal system still exists, although M is not strongly regular. Corollary 2 implies that in this case all orthogonal polynomials up to a certain degree ν are unique apart from a multiplicative scalar, whereas all others are not. In terms of the proof of theorem 2, we are in the case where $D^\star = 0$ for $\nu_k = \nu$. It is therefore easy to see that any polynomial of degree at least ν of a block orthogonal system, is orthogonal to any other polynomial. It follows that the concepts of orthogonal system and that of block orthogonal system coincide. This situation is illustrated by $P(z) = z^2$.

2. In the second case, there is no orthogonal system, because for at least one degree, there is no orthogonal polynomial of that degree. It is illustrated by $P(z) = z^2 - 1$.

4.1 The case $P(z) = z^2$

The Julia set of $P(z) = z^2$ is the unit circle C_1 and $\mu \triangleq \mu_\ell(J(P))$ is the uniform Borel probability measure on C_1. Hence all moments of μ of positive order vanish, since

$$\mu_i = \int z^i \, d\mu(z) = \oint_{C_1} z^i z^{-1} \, dz = \oint_{C_1} z^{i-1} \, dz = 0$$

for any positive integer i, whereas $\mu_0 = 1$ because μ is a probability measure.
So M contains only one nonzero entry, the left upper one, and therefore has only one regular leading submatrix, the one of dimension 1×1. Consequently, each block orthogonal system consists of two blocks, the first of which contains a single nonvanishing constant polynomial and the other one all nonconstant polynomials of the system. The polynomials of the second block are orthogonal to any other polynomial and can be any polynomial of the correct degree without a constant term. Indeed, any two polynomials at least one of which does not have a constant term, are orthogonal to each other.
The recurrence relation (4) degenerates into one single relation.

4.2 The case $P(z) = z^2 - 1$

In the sequel, we shall need the following results on the moments of $\mu \triangleq \mu_\ell(J(z^2 - 1))$.

Lemma 2 *Let $\mu = \mu_\ell(J(z^2-1))$. All moments of μ of odd order vanish, and $\mu_0 = \mu_2 = 1$.*

Proof
That all moments of odd order vanish can be proved inductively using lemma 1, stating that $\int z(P(z))^i \, d\mu(z) = 0$ for $i \in \mathbb{N}$.
$\mu_0 = 1$ because μ is a probability measure, whereas

$$\mu_2 = \int z^2 \, d\mu(z) = \int (P(z) + 1) \, d\mu(z) = \int z \, d\mu(z) + \int 1 \, d\mu(z) = \mu_1 + \mu_0 = 1$$

because of lemma 1. □

It follows that all regularly orthogonal polynomials of a block orthogonal system are even or odd.
As to the structure of the blocks of a block orthogonal system, we know by corollary 1 that the iterates of $P(z)$ are orthogonal polynomials. Moreover, another application of lemma 1 yields that their norm equals μ_2. This moment being nonvanishing according to the previous lemma, corollary 2 implies that to each iterate of $P(z)$, say $P^{\circ k}(z)$, there is a corresponding block containing a polynomial of degree $\deg(P^{\circ k}(z)) = 2^k$ in the middle. The next theorem asserts that there are no other blocks.

Theorem 4 *Let $(P_i(z))_{i=0}^{\infty}$ be a block orthogonal system of monic polynomials for the symmetric P-form with $P(z) = z^2 - 1$. Then*

$$\begin{array}{lll}
\nu_0 = 0, & \ell_1 = 1, & P_{\nu_0}(z) \equiv 1 \\
\nu_1 = 1, & \ell_2 = 1, & P_{\nu_1}(z) = z \\
\nu_{2k} = 2\nu_{2k-1}, & \ell_{2k+1} = \nu_{2k} - 1, & P_{\nu_{2k}}(z) = P_{\nu_{2k-1}}(P(z)) \\
\nu_{2k+1} = 2\nu_{2k} - 1, & \ell_{2k+2} = \nu_{2k+1}, & P_{\nu_{2k+1}}(z) = \frac{P_{\nu_{2k}}(P(z))}{z}
\end{array}$$

for any positive integer k.

Proof
We also prove that $P_{\nu_{2k}}(-1) = 0$ and $P_{\nu_{2k+1}}(0) = 0$ for any positive integer k.
The proof is by induction on the subscript l of the block index ν_l.
The basic steps for $l = 0, 1, 2$ and 3 can be easily verified.
We establish the induction step seperately for l even and odd.

- $l = 2k$ $(1 < k \in \mathbb{N})$
  $\nu_{2k} \triangleq \nu_{2k-1} + \ell_{2k} = \nu_{2k-1} + \nu_{2k-1} = 2\nu_{2k-1}$
  $P_{\nu_{2k-1}}(P(z))$ is a monic polynomial of degree $2\nu_{2k-1} = \nu_{2k}$ and because of theorems 1 and 2
  $$\begin{aligned} (z^i, P_{\nu_{2k-1}}(P(z))) &= 0 \quad \text{for} \quad i = 0(1)2(\nu_{2k-1} + \ell_{2k} - 1) - 1 \\ &\neq 0 \quad \text{if} \quad i = 2(\nu_{2k-1} + \ell_{2k} - 1). \end{aligned}$$
  Since $2(\nu_{2k-1} + \ell_{2k} - 1) = \nu_{2k} + (2\ell_{2k} - 1) - 1$, it follows that $P_{\nu_{2k}}(z) = P_{\nu_{2k-1}}(P(z))$ and that $\ell_{2k+1} = 2\ell_{2k} - 1 = 2\nu_{2k-1} - 1 = \nu_{2k} - 1$.
  $P_{\nu_{2k}}(-1) = P_{\nu_{2k-1}}(P(-1)) = P_{\nu_{2k-1}}(0) = 0$.
- $l = 2k + 1$ $(1 < k \in \mathbb{N})$
  $\nu_{2k+1} \triangleq \nu_{2k} + \ell_{2k+1} = \nu_{2k} + \nu_{2k} - 1 = 2\nu_{2k} - 1$
  $P_{\nu_{2k}}(P(z))$ is a monic polynomial of degree $2\nu_{2k}$ and is divisible by z as $P_{\nu_{2k}}(P(z))|_{z=0} = P_{\nu_{2k}}(-1) = 0$. So $\frac{P_{\nu_{2k}}(P(z))}{z}$ is a monic polynomial of degree $2\nu_{2k} - 1 = \nu_{2k+1}$. Moreover,
  $$\begin{aligned} (z^i, \tfrac{P_{\nu_{2k}}(P(z))}{z}) &= 0 \quad \text{for} \quad i = 0(1)2(\nu_{2k} + \ell_{2k+1} - 1) \\ &\neq 0 \quad \text{if} \quad i = 2(\nu_{2k} + \ell_{2k+1} - 1) + 1. \end{aligned}$$
  For $i = 1(1)2(\nu_{2k} + \ell_{2k+1} - 1) + 1$ this is a consequence of theorems 1 and 2 applied to $(z^{i-1}, P_{\nu_{2k}}(P(z))) = (z^i, \frac{P_{\nu_{2k}}(P(z))}{z})$, whereas for $i = 0$ the result holds because $\frac{P_{\nu_{2k}}(P(z))}{z}$ is odd and all moments of $\mu_\ell(J(z^2-1))$ of odd order vanish. Since $2(\nu_{2k} + \ell_{2k+1} - 1) + 1 = \nu_{2k+1} + (2\ell_{2k+1} + 1) - 1$, it follows that $P_{\nu_{2k+1}}(z) = \frac{P_{\nu_{2k}}(P(z))}{z}$ and that $\ell_{2k+2} = 2\ell_{2k+1} + 1 = 2(\nu_{2k} - 1) + 1 = \nu_{2k+1}$.
  $P_{\nu_{2k+1}}(0) = 0$, because $P_{\nu_{2k+1}}(z)$ is odd.

$\square$

An explicit expression for the block indices is $\nu_0 = 0$ and $\nu_k = \lceil 2^k/3 \rceil$ for any positive integer k.

Using the previous theorem, we can determine the coefficients occurring in the three term recurrence relation (4) between the regularly orthogonal polynomials of a block orthogonal system consisting of monic polynomials.

Theorem 5 *Let $(P_i(z))_{i=0}^{\infty}$ be a block orthogonal system of monic polynomials for the symmetric P-form with $P(z) = z^2 - 1$. Then*
$$\forall\, k \in \mathbb{N}: \; P_{\nu_{k+1}}(z) = c_{k+1}(z) P_{\nu_k}(z) - P_{\nu_{k-1}}(z),$$
where
$$\begin{aligned} & P_{\nu_{-1}}(z) \equiv 0 \\ & c_1(z) = c_2(z) = z \\ & c_{k+1}(z) = \frac{\prod_{i=2}^{\lfloor (k+1)/2 \rfloor} P^{\circ(k+1-2i)}(z)}{\prod_{i=2}^{\lfloor k/2 \rfloor} P^{\circ(k-2i)}(z)} P^{\circ(k-2)}(z) \; \text{for } 1 < k \in \mathbb{N} \end{aligned}$$

The proof is by induction on k

In the same way, the following relation between the regularly orthogonal polynomials and the iterates of $P(z)$ occurring in the same block, can be established.

Theorem 6 *Let $(P_i(z))_{i=0}^{\infty}$ be a block orthogonal system of monic polynomials for the symmetric P-form with $P(z) = z^2 - 1$. Then*

$$\forall\, k \in \mathbb{N} : \; P^{\circ k}(z) = \prod_{i=1}^{\lfloor k/2 \rfloor} P^{\circ(k-2i)}(z) \cdot P_{\nu_{k+1}}(z)$$

The recurrence relation between the iterates of $P(z)$ that results from the two previous theorems, reads

$$P^{\circ(k+1)}(z) = \left[P^{\circ(k-1)}(z)\right]^2 P^{\circ k}(z) - \left[P^{\circ(k-1)}(z)\right]^2,$$

for any $k \in \mathbb{N} \setminus \{0, 1\}$, which is no longer linear and quite obvious, since

$$\begin{aligned}
P^{\circ(k+1)}(z) &= P(P^{\circ k}(z)) \\
&= \left[P^{\circ k}(z)\right]^2 - 1 \\
&= \left[P^{\circ k}(z) + 1\right] \left[P^{\circ k}(z) - 1\right] \\
&= \left[P^{\circ(k-1)}(z)\right]^2 \left[P^{\circ k}(z) - 1\right] \\
&= \left[P^{\circ(k-1)}(z)\right]^2 P^{\circ k}(z) - \left[P^{\circ(k-1)}(z)\right]^2.
\end{aligned}$$

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