Chapter 1

Introduction and review

1.1 Introduction to ultrashort laser pulses

This book is about ultrafast laser pulses: what they are, linear and nonlinear optical effects which they experience, methods by which they are generated and measured, and how they can be used for measurement of ultrafast physical processes. Let us begin with a definition of the relevant time units.

1 nanosecond (nsec or ns) = 10^{-9} \text{ sec} = 0.000000001 \text{ sec}
1 picosecond (psec or ps) = 10^{-12} \text{ sec} = 0.000000000001 \text{ sec}
1 femtosecond (fsec or fs) = 10^{-15} \text{ sec} = 0.000000000000001 \text{ sec}
1 attosecond = 10^{-18} \text{ sec} = 0.000000000000000001 \text{ sec}

In order to put these very short time units in perspective, it is useful to consider their spatial equivalent. If we could take a snapshot of a one second light pulse, this pulse would stretch over a distance of 186,000 miles (or 300,000 kilometers) - equal to the speed of light multiplied by one second. This is roughly three quarters the distance from the earth to the moon, a distance we will consider very slow! Now skipping over milliseconds and microseconds, we arrive at nanoseconds. One nanosecond has a spatial extent of 30 cm (~ 1 foot). Although still rather slow by the standards of ultrafast optics, the nanosecond is the approximate time scale for high-speed electronic chips and computers. The word “ultrafast” is usually applied to the picosecond time scale and below. A picosecond has an extent of 0.3 mm, roughly the thickness of a business card.

Given that typical garden variety laser beams have beam diameters on the order of a few millimeters, we should perhaps envision pulses a picosecond and shorter not as pencils of light but as pancakes of light! At the time of this writing, most of the research on ultrafast optics technology is occurring at the femtosecond time scale. The shortest pulses reported to date are of the order of 4-5 fsec in duration. In the most typical case the spectrum contains frequencies in the near-infrared and visible, with an ~ 0.8 \mu m center wavelength. The spatial extent of even a 10 fsec laser pulse is only 3 microns, much less than the diameter of a human hair. Since these very shortest pulses are only a few optical cycles
in duration (or a few wavelengths in spatial extent), they are approaching a fundamental pulsewidth limitation of roughly one optical cycle. Research into attosecond pulse generation is also underway. One key theme in attosecond pulse generation schemes is the use of nonlinear optical frequency conversion methods to produce short wavelength radiation, since at shorter wavelengths the duration of a single optical cycle (and hence the attainable pulsewidth limit) is reduced.

Ultrashort pulses have several related characteristics which make them useful for applications. These include the following:

- **High time resolution.** By definition, the pulse duration is in the picosecond or femtosecond range (or below). This provides very high time resolution for excitation and measurement of ultrafast physical processes in solid-state, chemical, and biological materials.

- **High spatial resolution.** The spatial extent of a short light pulse is given by the pulse duration multiplied by the speed of light. As noted above, for very short pulse durations, the spatial pulse length can be on the order of microns. This makes ultrashort pulses useful for some microscopy and imaging applications.

- **High bandwidth.** By the uncertainty principle, the product of the pulsewidth times the optical bandwidth must be of order unity (or larger). As the pulse duration decreases, the bandwidth increases correspondingly. 100-fsec pulses have bandwidths on the order of 10 Terahertz (THz), and the shortest visible laser pulses contain so much of the visible spectrum that they appear white. This high bandwidth feature can be important for optical communications as well as other applications.

- **Potential for high intensity.** For a given pulse energy, the peak power and peak intensity are inversely proportional to the pulse duration. Because the size (hence cost) of high-power lasers usually scales with pulse energy, femtosecond pulse technology can be used to obtain ultrahigh peak intensities at moderate energy levels. Amplified femtosecond pulses have produced peak powers up to the petawatt level (1 petawatt = $10^{15}$ W) and peak intensities exceeding $10^{22}$ W/cm$^2$.

The field of ultrafast optics has traditionally been a highly interdisciplinary one, with a wide range of applications areas. In order to give a flavor for the nature of application areas, we comment on a few of the research applications below:

- **Ultrafast spectroscopy.** Time-resolved spectroscopy is a very successful and probably the most widespread application of picosecond and femtosecond laser technology. The idea is that ultrashort laser pulses can be used to make “stop-action” measurements of ultrafast physical processes, just as high-speed (microsecond) electronic flashes have been used starting several decades ago to make stop-action photographs of bullets.
traveling through apples, milk droplets splashing into milk bowls, etc. [1]. On the femtosecond time scale, macroscopic objects like bullets and milk droplets are motionless, and therefore ultrafast spectroscopy is best applied to study microscopic processes. Examples include investigations of femtosecond interactions of photoexcited electrons and holes with each other and with lattice vibrations in semiconductor crystals, ultrafast laser induced melting, photodissociation and ultrafast solution dynamics of chemical species, and ultrafast internal rearrangements of the large organic molecule bacteriorhodopsin as photons absorbed in the retina initiate the first biochemical steps in the process of vision.

- **Laser-controlled chemistry.** In a research area closely related to ultrafast spectroscopy, researchers are using specially engineered femtosecond laser waveforms to try to influence the course of photoinduced chemical reactions. In addition to observing ultrafast chemical motions as in time-resolved spectroscopy, here the added idea is to control what motions take place. Since the intrinsic time scale for nuclear motions in chemical systems is tens to hundreds of femtoseconds, femtosecond laser pulses are a natural tool in pursuing the challenging goal of laser controlled chemistry.

- **High-speed electrical testing.** Testing is a key issue in the development of high-speed electronic devices and circuits. Electronic test instrumentation based on established technology is usually slower than advanced high-speed research devices. However, since even the very fastest electronic devices only reach into the picosecond range, ultrafast laser technology offers speed to spare. Thus femtosecond optical pulses have been applied to generate subpicosecond electrical pulses and to measure the operation of the highest-speed electronic devices.

- **Laser-plasma interactions.** Lasers with intensities of $10^{13} \text{W/cm}^2$ and above (easily achieved using amplified femtosecond pulses) directed onto solid targets are sufficient to strip electrons from their nuclei, resulting in a laser generated plasma. On the 100 fsec time scale, the resulting free electrons do not have enough time to separate from the ionized nuclei. This provides the opportunity to study solid-density plasmas at temperatures as high as one million degrees.

- **Short wavelength generation.** High intensity ultrashort pulses at visible wavelengths can also be used to generate coherent short wavelength radiation in the vacuum ultraviolet and x-ray ranges through highly nonlinear harmonic generation processes or by pumping x-ray lasers. Coherent short wavelength radiation may be important, for example, for imaging microscopic structures such as DNA.

- **Optical communications.** The low-loss transmission window of optical fibers has a bandwidth comparable to that of a 100 fsec pulse, and therefore, ultrashort pulse technology may play an important role in optical communications. Pulses on the 1-psec time scale have already been used
to demonstrate >100 Gbit/sec data transmission over 200 km of optical fiber and 400 Gbit/sec transmission over 40 km. Ultrashort pulses may also prove important in wavelength-division multiplexing (WDM) systems in which the fiber bandwidth is carved up into different wavelength bands or channels. For WDM applications it is the large bandwidth of the ultrashort pulse (and not the short duration) which is useful, since a single pulse contains enough bandwidth to produce a number of wavelength channels.

- **Medical applications.** Ultrashort pulses as well as broadband incoherent light are finding application in medical imaging. Attractive features include the ability to perform optical imaging within scattering media (i.e., most tissues) and to obtain high resolution depth information. In some cases ultrashort pulses may also reduce collateral tissue damage by reducing heat deposition in laser-assisted surgical procedures.

- **Microscopy.** Substantially improved spatial resolution has been demonstrated in confocal microscopy by relying on two-photon excitation. The ability of ultrashort pulses to provide high intensity without high pulse energy is important for using this technique with sensitive biological samples.

- **Materials processing.** High-power lasers are used for a variety of industrial applications such as cutting and drilling. With continuous-wave or “long” pulse (nanoseconds) lasers, the minimum feature size and the quality of the cut are limited by thermal diffusion of heat to areas neighboring the laser focus. With femtosecond lasers materials processing is possible using lower pulse energies due to the very high peak powers which lead to new physical mechanisms. This reduces the heat deposited into the sample during the laser machining process and leads to a much cleaner cutting or drilling operation.

- **Nuclear fusion.** At very high intensities, the radiation pressure exerted by focused ultrashort laser beams may be sufficient to compress a hydrogen capsule to the densities required for ignition of thermonuclear fusion. This concept represents an entirely new principle for laser-induced fusion, which has been investigated extensively for many years.

### 1.2 Brief review of electromagnetics

Since ultrashort laser pulses are made up of light, and light is a form of electromagnetic radiation, we very briefly review Maxwell’s equations, which describe all forms of electromagnetic radiation, including light. We will use MKS (SI) units here and throughout this book. It is assumed that the reader is already familiar with vector calculus. For a more detailed treatment of electromagnetics, the reader is directed to textbooks on this subject [2] [3].
1.2.1 Maxwell’s equations

Maxwell’s equations are a set of relationships between the electric field $E$ and magnetic field $H$ (boldface symbols denote vectors). Inside a medium we must also consider the charge density $\rho$, current density $J$, polarization density $P$ and magnetization density $M$, and in order to include the effect of the fields on the matter, the electric and magnetic flux densities, $D$ and $B$, are also introduced. Units for these quantities are given in Table 1.1. Maxwell’s equations are then written as follows:

\[ \nabla \cdot D = \rho \] \hspace{1cm} (1.1)

\[ \nabla \cdot B = 0 \] \hspace{1cm} (1.2)

\[ \nabla \times E = \frac{-\partial B}{\partial t} \] \hspace{1cm} (1.3)

\[ \nabla \times H = J + \frac{\partial D}{\partial t} \] \hspace{1cm} (1.4)

The relations defining $D$ and $B$ are

\[ D = \varepsilon_0 E + P \] \hspace{1cm} (1.5)

\[ B = \mu_0 (H + M) \] \hspace{1cm} (1.6)

The constants $\varepsilon_0$ and $\mu_0$ are known as the permittivity and permeability of free space, with numerical values and units given in Table 1.1. Note also that the symbol $\rho$ refers to the free charge density - i.e., any bound charge density associated with the polarization is not included. Likewise, the current density $J$ does not include any currents associated with motion of bound charges (changes in polarization). In free space we would have $\rho = J = P = M = 0$.

For now we specialize to the case of a linear, isotropic, and source-free medium. By source-free we mean that the charge and current densities are zero ($\rho = 0$ and $J = 0$). By linear we mean that the medium response (i.e., the polarization and magnetization) is linear in the applied fields. For the case of the electric field, we write

\[ P = \varepsilon_0 \chi_e E \] \hspace{1cm} (1.7)

where $\chi_e$ is known as the electric susceptibility (dimensionless). Inserting into eq. (1.5), one obtains

\[ D = \varepsilon_0 (1 + \chi_e) E = \varepsilon E \] \hspace{1cm} (1.8)
symbol | name                  | units or numerical value
---|-----------------------|----------------------
E  | electric field        | volt \( m^{-1} \)
D  | electric flux density | coulomb \( m^{-2} \)
H  | magnetic field        | amp \( m^{-1} \)
B  | magnetic flux density | tesla \((\text{volt sec} \ m^{-2})\)
P  | polarization density  | coulomb \( m^{-2} \)
M  | magnetization density | amp \( m^{-1} \)
J  | current density       | amp \( m^{-2} \)
ρ  | charge density        | coulomb \( m^{-3} \)
\(\varepsilon_0\) | permittivity of free space | \(8.85 \times 10^{-12} \ \text{farad} \ m^{-1}\) (or coulomb \(\text{volt}^{-1} \ m^{-1}\))
\(\mu_0\) | permeability of free space | \(4\pi \times 10^{-7} \ \text{Henry} \ m^{-1}\) (or volt \(\text{sec}^{-1} \ m^{-1} \ \text{coulomb}^{-1}\))

Table 1.1: Names and units of symbols in Maxwell’s equations

The proportionality constant \(\varepsilon\) is termed the dielectric constant, with

\[
\varepsilon = (1 + \chi_{e})\varepsilon_0
\]

(1.9)

Other common terms include the relative dielectric constant \((\varepsilon/\varepsilon_0)\) and the index of refraction \(n\), which is commonly used in optics, where

\[
n^2 = \varepsilon/\varepsilon_0
\]

(1.10)

For the case of the magnetic field, we write

\[
M = \chi_m H
\]

(1.11)

where \(\chi_m\) is the magnetic polarizability. Using eq. (1.6), we obtain

\[
\mathbf{B} = \mu_0 (1 + \chi_m)\mathbf{H} = \mu\mathbf{H}
\]

(1.12)

In most cases in ultrafast optics, one is interested in nonmagnetic materials, for which \(M = 0\). In this case of zero magnetization, one has

\[
\mathbf{B} = \mu_0\mathbf{H}
\]

(1.13)

Equations (1.7) and (1.11) are examples of constitutive laws, which specify the response of the material to the fields. The form of these equations as written arises because we have assumed both linear and isotropic media (for nonisotropic media, one would need to replace the assumed scalar susceptibilities with tensors). We note that there are many situations in ultrafast optics where these assumptions are not valid. For example, nonlinear optical effects, which we will discuss in later chapters, require by definition that \(\mathbf{P}\) be a nonlinear function of \(\mathbf{E}\).
1.2.2 The wave equation and plane waves

We now consider electromagnetic wave propagation in linear, isotropic, source-free media. To derive the wave equation, we take the curl of eq. (1.3), and insert eq. (1.4), which using the stated assumptions and a well known vector identity gives the following:

\[
\nabla \times \nabla \times \mathbf{E} = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}
\]

(1.14)

Since \( \nabla \cdot \mathbf{E} = 0 \) also under our conditions, we obtain the wave equation

\[
\nabla^2 \mathbf{E} = \mu \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}
\]

(1.15)

One situation of special interest is the case where the field varies only in one direction, which without loss of generality we take as the z direction. Then the wave equation becomes

\[
\frac{\partial^2 \mathbf{E}}{\partial z^2} = \mu \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}
\]

(1.16)

The general solution takes the form

\[
\mathbf{E} (z, t) = \mathbf{E}_0 \left(t - \frac{z}{v}\right)
\]

(1.17)

where \( \mathbf{E}_0 \) is a vector in the x-y plane (equation (1.1) precludes \( \mathbf{E} \) from having a z-component), and \( v = \frac{1}{\sqrt{\mu \varepsilon}} \). The solution can be verified by plugging back into the wave equation. Equation (1.17) is called a plane wave solution, since the field does not vary in the transverse (x-y) plane. It also represents a travelling wave, since the field propagates in the z direction without changing its form. In the case of a pulsed field, \( \mathbf{E}_0 (t) \) represents the pulse shape. The propagation velocity is given by \( v \). Note that

\[
\frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c \cong 2.998 \times 10^8 \text{m sec}^{-1}
\]

(1.18)

is the velocity of light in free-space. Therefore, for the case most common in optics where \( \mu = \mu_0 \), the velocity of propagation within a medium is given by

\[
v = \frac{c}{n}
\]

(1.19)

where \( n \) is the refractive index as per eq. (1.10). Note also that in deriving equations (1.14-1.17), we have assumed implicitly that the refractive index \( n \) is independent of frequency. When \( n \) does have a frequency dependence, this can change the propagation velocity or cause the pulse to distort during propagation. These effects are discussed in chapter 4.

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1The identity is \( \nabla \times \nabla \times \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \). Note that in Cartesian coordinates \( \nabla^2 \mathbf{A} \) has a very simple form, namely

\[
\nabla^2 \mathbf{A} = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{A}
\]
The case of a sinusoidal solution to the wave equation will be of special importance. Then, eq. (1.17) takes the form

\[ \mathbf{E}(z, t) = \mathbf{E}_0 \cos(\omega t - kz + \phi) \]  

(1.20)

where \( \mathbf{E}_0 \) is now a constant vector, \( \omega \) is the angular frequency, and the propagation constant \( k \) must satisfy the dispersion relation

\[ k = \omega \sqrt{\mu \varepsilon} \]  

(1.21)

or again assuming \( \mu = \mu_0 \)

\[ k = \frac{\omega n}{c} \]  

(1.22)

The wave has a temporal oscillation period equal to \( 2\pi/\omega \) and a spatial period or wavelength in the medium given by \( \lambda = 2\pi/k \). The wavelength in free-space is denoted \( \lambda_0 \) and is given by

\[ \lambda_0 = \frac{2\pi c}{\omega} \]  

(1.23)

Equation (1.20) represents the ideal case of single frequency or monochromatic laser radiation. It can also be written in the equivalent form

\[ \mathbf{E}(z, t) = \text{Re} \left\{ \mathbf{E}_0 e^{j(\omega t - kz)} \right\} \]  

(1.24)

where \( \text{Re}\{\ldots\} \) denotes the real part and the phase \( \phi \) has been incorporated into the complex vector \( \mathbf{E}_0 \). We refer to this form as complex notation. As we will see shortly, ultrashort light pulses are conveniently described as superpositions of sinusoidal solutions of the form (1.20) or (1.24) with different frequencies.

Finally, we note that similar solutions can be written for propagation in directions other than along \( z \), as follows:

\[ \mathbf{E}(\mathbf{r}, t) = \text{Re} \left\{ \mathbf{E}_0 e^{j(\omega t - \mathbf{k} \cdot \mathbf{r})} \right\} \]  

(1.25)

Here \( \mathbf{k} \) is the propagation vector; it points along the direction of propagation and its magnitude \( k = |\mathbf{k}| \) still satisfies the dispersion relation (1.21).

\subsection*{1.2.3 Poynting’s vector and power flow}

We also review the expressions for energy flow with electromagnetic waves. To arrive at the required formulas, we form the dot product of eq. (1.3) with \( \mathbf{H} \) and subtract from this the dot product of eq. (1.4) with \( \mathbf{E} \). Using another vector identity, \(^2\) we find that

\[ \nabla \cdot (\mathbf{E} \times \mathbf{H}) + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{E} \cdot \mathbf{J} = 0 \]  

(1.26)

\(^2\)The identity is \( \nabla \cdot \mathbf{A} \times \mathbf{B} = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \)
We also make use of the divergence theorem

\[ \int \nabla \cdot \mathbf{A} \, dV = \int \mathbf{A} \cdot \mathbf{n} \, dS \]  \hspace{1cm} (1.27)

which states that the surface integral of a vector \( \mathbf{A} \) over a closed surface is equal to the volume integral of \( \nabla \cdot \mathbf{A} \) over the volume bounded by that surface. \( \mathbf{n} \) is the unit vector normal to the surface and pointing outwards. The result is

\[ \int (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{n} \, dS + \int dV \left\{ \frac{\partial}{\partial t} \left( \frac{1}{\varepsilon} |\mathbf{E}|^2 \right) + \frac{\partial}{\partial t} \left( \frac{1}{\mu} |\mathbf{H}|^2 \right) + \mathbf{E} \cdot \mathbf{J} \right\} = 0 \]  \hspace{1cm} (1.28)

Finally, assuming a linear medium and substituting for \( \mathbf{D} \) and \( \mathbf{B} \) using equations (1.8) and (1.12), we obtain

\[ \int (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{n} \, dS + \int dV \left\{ \frac{\partial}{\partial t} \left( \frac{1}{\varepsilon} |\mathbf{E}|^2 \right) + \frac{\partial}{\partial t} \left( \frac{1}{\mu} |\mathbf{H}|^2 \right) + \mathbf{E} \cdot \mathbf{J} \right\} = 0 \]  \hspace{1cm} (1.29)

Equations (1.28-1.29) are representations of Poynting’s theorem, which describes conservation of energy in electromagnetic systems. We can identify specific meanings for each of the terms. Looking at eq. (1.29) for example:

- \( \int (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{n} \, dS \) is the net rate of energy flow out of the closed surface. It has units of power (Watts). \( \mathbf{E} \times \mathbf{H} \) is called the Poynting vector and has units of intensity (W/m\(^2\)). It gives the power density carried by an electromagnetic wave and the direction in which power is carried.

- \( \frac{1}{\varepsilon} |\mathbf{E}|^2 \) and \( \frac{1}{\mu} |\mathbf{H}|^2 \) are the local energy densities (Joule/m\(^3\)) associated with the electric and magnetic fields, respectively. \( \frac{\partial}{\partial t} \int \frac{1}{\varepsilon} |\mathbf{E}|^2 \, dV \) and \( \frac{\partial}{\partial t} \int \frac{1}{\mu} |\mathbf{H}|^2 \, dV \) represent the time rate of change of electric and magnetic field energy stored within the volume, respectively.

- \( \int \mathbf{E} \cdot \mathbf{J} \, dV \) represents power dissipation or generation within the volume (units watts). When \( \mathbf{E} \cdot \mathbf{J} \) is positive, this term represents power dissipation, e.g., due to Ohmic losses. Energy is transferred out of the fields and into the medium, typically as heat. When \( \mathbf{E} \cdot \mathbf{J} \) is negative, this term represents power supplied by the currents and fed into the electromagnetic fields.

Overall, Poynting’s theorem is a power balance equation, showing how changes in stored energy are accounted for by power dissipation and energy flow.

It is worth specializing once more to the case of single frequency, sinusoidal fields, with \( \mathbf{E} \) given by eq. (1.25). The \( \mathbf{H} \) field is obtained using eq. (1.3), with the result

\[ \mathbf{H} = Re \left\{ \sqrt{\frac{\varepsilon}{\mu}} k \times \frac{\mathbf{E}_0}{k} e^{j(\omega t - k \cdot r)} \right\} \]  \hspace{1cm} (1.30)
Thus $\mathbf{H}$ is perpendicular to both $\mathbf{E}$ and $\mathbf{k}$, and its magnitude is equal to $\sqrt{\frac{\varepsilon}{\mu}} |\mathbf{E}|$.

The factor $\sqrt{\frac{\varepsilon}{\mu}}$ is termed the characteristic impedance of the medium, and $\sqrt{\frac{\varepsilon}{\mu}}$ therefore is the admittance.

In optics one is usually interested in the time-average power flow. This is calculated in complex notation as follows. First consider scalar functions $f(t)$ and $g(t)$, where

$$f(t) = \Re \left\{ \tilde{f} e^{j\omega t} \right\} \quad \text{and} \quad g(t) = \Re \left\{ \tilde{g} e^{j\omega t} \right\}$$

(1.31)

The time-average of $f(t)g(t)$ is given by

$$\langle fg \rangle = \frac{1}{2} \Re \left\{ \tilde{f} \tilde{g}^* \right\}$$

(1.32)

Here $\langle \ldots \rangle$ denotes the time average, and * means complex conjugate. Similarly if $\mathbf{f}(t), \mathbf{g}(t), \tilde{\mathbf{f}},$ and $\tilde{\mathbf{g}}$ now denote vectors, the time-average of $\mathbf{f} \times \mathbf{g}$ is given by

$$\langle \mathbf{f} \times \mathbf{g} \rangle = \frac{1}{2} \Re \left\{ \tilde{\mathbf{f}} \times \tilde{\mathbf{g}}^* \right\}.$$  

(1.33)

Using these relations, the time-average Poynting vector for the plane-waves of equations (1.25) and (1.30) becomes

$$\langle \mathbf{E} \times \mathbf{H} \rangle = \frac{1}{2} \sqrt{\frac{\varepsilon}{\mu}} |\tilde{\mathbf{E}}_o|^2 \frac{\mathbf{k}}{k}$$

(1.34)

where we have assumed $\varepsilon$ and $\mu$ are real. Power is carried along the direction of $\mathbf{k}$. In the case of a nonmagnetic material, we can write the magnitude of the time-average Poynting vector, commonly called the intensity $I$, in the following useful form:

$$I = |\langle \mathbf{E} \times \mathbf{H} \rangle| = \frac{1}{2} \varepsilon_{\varepsilon_0} c n |\tilde{\mathbf{E}}_o|^2$$

(1.35)

### 1.3 Review of laser essentials

We will shortly discuss in some detail methods by which lasers can be made to produce ultrashort light pulses. First, however, we will give a brief and simple review of lasers in general. More detail can be found in texts on lasers, such as [4] [5].

#### 1.3.1 Steady-state laser operation

Schematic drawings of two simple laser geometries are shown in Fig. 1.1. Both lasers consist of a set of mirrors and a gain medium. The gain medium is an optical amplifier, which coherently amplifies light passing through it. The mirrors may be curved or planar and together make up the laser cavity or resonator. The cavity is aligned so that light reflects back and forth again
and again, passing along the same path every time. If we imagine even a very weak light intensity in the cavity (due to spontaneous emission from the gain medium), then for sufficiently high gain, the intensity increases from one round trip through the laser to the next, eventually resulting in an intense beam. In steady state the gain per round trip must equal the loss. Part of the light passes through the partially transmissive output coupler, and this forms the output laser beam which can be used for experiments.

In the linear or Fabry-Perot laser cavity shown in Fig. 1.1a, light passes along the same path traveling from left to right and right to left. The light makes two passes through the gain medium per round trip. In the ring cavity shown in Fig. 1.1, light passes through the gain medium only once per round trip. In the diagram we have assumed unidirectional operation - i.e., light is traveling around the ring in only one direction. In real ring lasers either unidirectional or bidirectional operation is possible. Both linear and ring geometries have been used in femtosecond laser design.

For our purposes we will usually consider the gain medium as a black box. The physics of laser gain media is usually analyzed using quantum mechanics in courses on laser fundamentals. Here we will usually stick to a classical description, although we will use the indispensable energy level concept, which does come from quantum mechanics. Common laser media used for ultrafast lasers include impurity-doped crystals or glasses such as Ti:sapphire, Nd:YAG or Nd:glass, organic dyes, doped optical fibers, and semiconductor heterojunction diodes. Note that in thermodynamic equilibrium, materials can absorb light but cannot amplify it. To achieve gain, power must be supplied to the medium in order to promote electrons into excited-state energy levels. When the electrons are "pumped" to the excited state at a sufficiently high rate (i.e., when enough power is supplied), a population inversion, in which the population of electrons in an excited energy level exceeds that in a lower level, can be achieved. A population inversion is a necessary condition to achieve optical gain. Power can be supplied to the laser medium in many different ways. Optical pumping, in which absorption of pump photons from a flashlamp or an external laser promotes the electrons to the excited state, is often used in ultrashort pulse lasers. Other pumping methods include current injection in semiconductor diode lasers or electric discharges in gas lasers. The need for a pump source in order to obtain optical gain can be likened to the need to plug in an electronic amplifier.

In order to achieve steady-state laser operation, the electric field must repeat itself after a round trip through the laser cavity. Therefore, let us now consider a single round trip of a monochromatic field through a cavity. To be specific, we will consider the linear cavity of Fig. 1.1a. The optical path length between mirrors is denoted \( l \); the total optical path length is therefore \( 2l \). The electric field amplitude of a monochromatic plane wave propagating in the \( +z \) direction can be written

\[
E(z, t) = Re \left[ E_0 e^{j(\omega t - kz)} \right]
\]

(1.36)

where the field is now taken as a scalar for convenience. \( E_0 \) is the field amplitude.
just to the right of the high reflector at \( z = 0 \), \( \omega \) is the angular frequency, \( k = \frac{\omega n}{c} \) is the propagation constant, and \( n = n' + jn'' \) is the complex refractive index (\( n' \) and \( n'' \) are both real numbers). The field just before the output coupler is written

\[
E = \text{Re} \left[ E_0 e^{\frac{2\pi n t_0}{c}} e^{i\omega t} e^{-j\frac{\omega}{c}(n' l + l_a)} \right]
\]  

(1.37)

In eq. (1.37) \( n' \) and \( n'' \) refer to the gain medium; the regions outside the gain medium are taken as air with \( n = 1 \). \( l_g \) and \( l_a \) are the physical lengths of the gain medium and of air region, respectively. We can now identify \( l = n' l_g + l_a \) as the optical path length. We also see that for \( n'' > 0 \), the field has been amplified.

The field at \( z = 0 \), corresponding to a round trip path through the resonator, is obtained by reflecting off the output coupler with amplitude \( r_2 \), passing back through the cavity, and then reflecting off the high reflector with amplitude \( r_1 \). Although ideally we would have \( r_1 = 100\% \), we keep the variable \( r_1 \) in our expressions to account for any imperfections of the high reflector as well as any other losses in the laser cavity besides those arising from the output coupler. The resulting expression for the field is as follows:

\[
E = \text{Re} \left[ r_1 r_2 e^{\frac{2\pi n t_0}{c}} E_0 e^{i\omega t} e^{-j\frac{2\pi}{l}l} \right]
\]  

(1.38)

In order to satisfy the steady-state requirement, eq. (1.38) must be equal to the initial field, eq. (1.36), evaluated at \( z = 0 \); \( E = \text{Re} \left[ E_0 e^{i\omega t} \right] \). This leads to two conditions: the gain condition and the phase condition. The gain condition,

\[
r_1 r_2 e^{\frac{2\pi n t_0}{l}} = 1,
\]  

(1.39)

states that the round trip gain exactly balances the round trip loss. The phase condition,

\[
\frac{2\omega l}{c} = 2m\pi,
\]  

(1.40)

requires that the the round trip phase shift is equal to an integer \( m \) times \( 2\pi \). This means that the laser is only allowed to oscillate at certain discrete angular frequencies, given by

\[
\omega_m = \frac{m\pi c}{l}.
\]  

(1.41)

These frequencies are known as the longitudinal modes of the cavity. In terms of the frequency \( f = \omega/2\pi \), the mode frequencies are

\[
f_m = \frac{mc}{2l}
\]  

(1.42)

and the mode spacing (taking \( n' \) as frequency independent) is

\[
\Delta f = f_m - f_{m-1} = \frac{c}{2l}
\]  

(1.43)
A frequency domain view of basic laser operation is pictured schematically in Fig. 1.2. A comparison of the gain vs. loss is shown at the top, with the locations of the longitudinal modes shown below. The resonator loss is taken as frequency independent, while the gain is assumed to have a bandpass spectral response. Laser oscillation occurs only for those modes where the gain lies above the loss line. In the situation shown, gain exceeds loss for several longitudinal modes, and multiple output frequencies appear simultaneously. This is called multimode operation.

In order to obtain monochromatic or single-mode laser radiation, it is usually necessary to insert a frequency dependent loss element (a filter) to ensure that gain exceeds loss for only a single longitudinal mode. This situation is sketched in Fig. 1.3. Note however that the steady-state laser gain condition eq. (1.39) requires that the gain exactly equal rather than exceed the loss. For this reason, we now distinguish between small signal gain and saturated gain. The small signal gain refers to the gain available under conditions of zero or at least very weak light intensity and depends only on the properties of the laser medium and the pump level. In order for weak spontaneous emission to build up to produce a significant laser intensity, this small signal gain must indeed exceed the loss. However, as the intracavity intensity increases, the small signal condition is violated. As laser photons are amplified in the laser medium through stimulated emission, at the same time electrons in the excited energy state are stimulated back down to the lower energy state. The intracavity field extracts energy that was stored in the gain medium by the pump and reduces the number of excited state electrons available for amplification. As a result, the gain is reduced. The actual gain which results is known as the saturated gain and depends on the properties of the laser medium, the pump level, and the intracavity laser intensity. Note that saturation should be quite familiar to you in the context of electronic amplifiers, where the full amplifier gain is available only for input signals below a certain voltage level; for higher voltage levels the output may appear to be clipped.

In order to clarify the role of gain saturation in lasers, let us imagine that the pump intensity is slowly increased from zero in a single mode laser cavity. As long as the small signal gain remains below the loss, the laser intensity is zero. When the pump intensity is sufficient to raise the small signal gain to exactly equal the loss, the laser reaches threshold. The laser power is still zero at this point. When the pump is increased above threshold, the laser power increases, and this saturates the gain. For a given pump level, the laser intensity builds up just enough to maintain the saturated gain at exactly the loss level, as pictured in Fig. 1.3. Thus, pump power above the threshold value is converted into stimulated emission. As a result, the laser intensity increases linearly with pump power above threshold.

In our discussion we have implicitly assumed the gain medium is homogeneously broadened. This means that all the excited state electrons have identical gain spectra, so that the overall gain is simply the gain spectrum per electron times the population difference between upper and lower laser states. Gain saturation in a homogeneously broadened medium results from a decrease in this
population difference, and therefore the saturated gain has the same spectral shape as the small signal gain. Inhomogeneously broadened media and inhomogeneously broadened lasers, in which different excited state electrons have different gain spectra, are also possible. This may result, for example, due to the fact that different impurity ions in a glass laser experience different local environments. Because inhomogeneously broadened lasers are not in common use for ultrashort pulse generation, we do not discuss them further.

1.3.2 Gain and gain saturation in four-level atoms

In order to provide further insight, we discuss gain saturation for the so called four level atom. The four level atom approximation, sketched in Fig. 1.4, is commonly used to model important modelocking media such as Ti:sapphire or dye molecules. Here we discuss continuous-wave (CW) saturation; saturation in response to pulses is covered in chapter 2. Note that we do not mean the word “atom” in the term “four level atom” literally. Rather this word refers to whatever entity (molecule, impurity complex, etc.) is active in the laser gain process.

Referring to Fig. 1.4, electrons are promoted from the lowest state, denoted level 1, up to level 2 (for example, via optical pumping). Electrons in level 2 are usually assumed to relax very rapidly to level 3, so that the population of electrons in level 2 remains close to zero. Physically this relaxation usually arises from vibronic rearrangement of the nuclei within the crystal-impurity or molecular system, which typically occurs on a subpicosecond time scale following the electronic transition from 1 → 2. The transition from 3 → 4 is the lasing transition, and levels 3 and 4 are the upper and lower laser levels, respectively. Electrons from level 3 can undergo stimulated emission down to level 4, giving up a photon to the laser field in the process, or they can also relax spontaneously down to level 4 with rate $\tau_{G}^{-1}$, in which case their energy is not available to the laser field. $\tau_{G}$ is the energy storage time of the gain medium, which typically ranges from nanoseconds to tens of microseconds in materials used for femtosecond pulse generation. Electrons in level 4 are again assumed to relax very rapidly back to level 1, so that the lower laser level remains nearly empty (this is the principal advantage of four level atoms).

Mathematically we denote the total density of atoms as $N_{G}$ (units $m^{-3}$), and the number density in each of the four levels as $N_{1}, N_{2}, N_{3},$ and $N_{4}$, respectively. Naturally we have

$$N_{1} + N_{2} + N_{3} + N_{4} = N_{G}$$

(1.44)

The level populations are described by the following rate equations:

$$\dot{N}_{1} = -W(N_{1} - N_{2}) + \frac{N_{4}}{\tau_{14}}$$

(1.45)

$$\dot{N}_{2} = W(N_{1} - N_{2}) - \frac{N_{2}}{\tau_{23}}$$

(1.46)
$\dot{N}_3 = -S (N_3 - N_4) - \frac{N_3}{\tau_{23}} + \frac{N_2}{\tau_{23}}$  

(1.47)

$\dot{N}_4 = S (N_3 - N_4) - \frac{N_4}{\tau_{41}} + \frac{N_3}{\tau_{41}}$  

(1.48)

Here $W$ is the pumping rate per atom (units s$^{-1}$) from level 1 to 2, and its strength is controlled via the external excitation of the laser medium. For example, in the case of optical pumping, $W$ is proportional to the intensity of the pump laser. $S$ is the stimulated emission rate per atom (units s$^{-1}$), which is proportional to intensity and given in physical units by

$S = \frac{\sigma_{34} I (\omega_{34})}{\hbar \omega_{34}}$  

(1.49)

Here $\sigma_{34}$ is the “cross-section” of the laser transition, which characterizes the strength of the laser-matter interaction, $I(\omega_{34})$ is the laser intensity inside the resonator, and $\hbar \omega_{34}$ gives the photon energy of the $3 \rightarrow 4$ transition. In the case of optical pumping, a similar expression applies for $W$. $\tau_{23}$ and $\tau_{41}$ are the $2 \rightarrow 3$ and $4 \rightarrow 1$ relaxation times, respectively. The gain is proportional to the population difference between the laser levels, namely:

$g = \frac{\sigma_{34}}{2} (N_3 - N_4) I_g$  

(1.50)

The factor of $1/2$ arises because we have written the gain coefficient for the field in eq. (1.50); this factor is not present when writing the gain coefficient for the intensity. In steady-state all the time derivatives in the rate equations are set to zero. Furthermore, since the relaxation from levels 2 and 4 is very fast, we can also approximate $N_1 - N_2 = N_1$ and $N_3 - N_4 = N_3$ in equations (1.45-1.48). Solving for the upper laser level population under these conditions yields the following expression for the gain:

$g = \frac{1}{2} \frac{\sigma_{34} W N_G \tau_G I_g}{1 + (W + S) \tau_G}$  

(1.51)

The gain is a function of both $W$ and $S$. In the small signal regime ($S = 0$), the small signal gain $g_o$ is given by

$g_o = \frac{1}{2} \frac{\sigma_{34} W N_G \tau_G I_g}{1 + W \tau_G}$  

(1.52)

$g_o$ starts at zero, then increases linearly with pump rate $W$ at first but saturates for large $W$ when $N_3$ approaches $N_G$. In the large signal regime ($S$ is finite), the gain can be rewritten

$g = \frac{g_o}{1 + \frac{S}{S_{sat}}}$  

(1.53)

where
\[ S_{sat} = W + \frac{1}{\tau g} \]  

(1.54)

is a saturation parameter. Within a laser, the gain first increases with increasing pump until threshold is reached. Above threshold the small signal gain \( g_o \) continues to increase as \( W \) increases, but the actual gain \( g \) is clamped or saturated at the value needed for threshold, which we denote \( g_{th} \). The laser intensity (inside the laser) is found by setting eq. (1.53) to \( g_{th} \), with the result

\[ S = S_{sat} \left( \frac{g_o}{g_{th}} - 1 \right) \]  

(1.55)

Lasing can occur in three level systems as well as the four level systems discussed above. Further discussion can be found in standard laser texts.

### 1.3.3 Gaussian beams and transverse laser modes

Real laser beams have a finite transverse extent: they are not plane waves. In most cases of interest to us, however, laser beams may be considered paraxial. This means they are made up of a superposition of plane waves with propagation vectors close to a single direction (which we take as \( z \)). Equivalently, the variation of the field in the transverse \((x - y)\) direction must be much slower than in the \( z \) direction. Under these conditions the electric field vector still lies mainly in the \( x-y \) plane. In the following we give a brief summary of paraxial laser beams and the related transverse mode structure of lasers. More detail can be found in most texts on lasers, e.g., [4] [5] [6].

For a monochromatic, paraxial wave it is useful to write

\[ E(z,t) = \text{Re} \left\{ \tilde{E}_0 u(x,y,z)e^{i(\omega t - kz)} \right\} \]  

(1.56)

where \( k \) is given by eq. (1.22) and \( E \) refers to a single transverse polarization component. In this form the most rapid, wavelength scale variation of the field is carried by the \( e^{-jkz} \) term; \( u(x,y,z) \) is a slowly varying envelope. Substitution into the wave equation, eq. (1.15) yields

\[ \nabla_T^2 u + \frac{\partial^2 u}{\partial z^2} - 2jk \frac{\partial u}{\partial z} = 0 \]  

(1.57)

where \( \nabla_T^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \). Since \( u \) is assumed to be slowly varying, \( |\partial^2 u/\partial z^2| \ll 2k|\partial u/\partial z| \), and therefore \( \partial^2 u/\partial z^2 \) can be neglected. The resulting paraxial wave equation is

\[ \nabla_T^2 u - 2jk \frac{\partial u}{\partial z} = 0 \]  

(1.58)

The paraxial wave equation has Gaussian beam solutions that provide a good description of laser beams both inside and outside the laser cavity. A particularly useful solution is written as follows:
\[ u_{00}(x,y,z) = \frac{w_0}{w(z)} e^{-(x^2+y^2)/w^2(z)} e^{-jk(x^2+y^2)/2R(z)} e^{i\phi(z)} \]  

(1.59)

where we define

\[ w^2(z) = w_0^2 \left[ 1 + \left( \frac{z}{z_o} \right)^2 \right] \]  

(1.60-a)

\[ \frac{1}{R(z)} = \frac{z}{z^2 + z_o^2} \]  

(1.60-b)

\[ \phi(z) = \tan^{-1} \left( \frac{z}{z_o} \right) \]  

(1.60-c)

\[ z_o = \frac{\pi w_0^2 n}{\lambda} \]  

(1.60-d)

This solution, sketched in Fig. 1.5, describes a beam which comes to a focus at \( z = 0 \) with a radius \( w_0 \) (measured at \( 1/e \) of the on-axis field). The beam radius \( w(z) \) changes only slowly within the region \( |z| < z_o \); far outside of this region the beam spreads with a full angle equal to \( \lambda/\pi w_0 n \). The depth \( b \) over which the beam radius remains less than \( \sqrt{2} w_0 \) is given by \( b = 2z_o \), where \( b \) is termed the depth of focus or the confocal parameter. \( R(z) \) is the radius of curvature of the phase fronts. \( R = \infty \) at the beam waist \( (z = 0) \), corresponding to a planar phase front; for \( |z| \gg z_o \), the phase fronts become spherical with a center of curvature at \( z = 0 \). Equation (1.60-c) indicates a phase shift that accompanies propagation through the focal region. This Gaussian beam solution describes both laser beam propagation in free-space as well as the spatial behavior of the fundamental transverse modes that are allowed inside a laser resonator with flat or spherical mirrors. The particular transverse mode of this type allowed within a given laser resonator is selected by the requirement that \( R(z) \) must match the radii of curvature of the mirrors within the laser, which determines the location of the beam waist and the value of \( w_0 \); for free propagation outside of a laser, any solution of this type is allowed.

It is useful to know the power \( P \) carried by a Gaussian beam, which is obtained by integrating the intensity over its cross-sectional area. The result is

\[ P = \int dA \frac{\varepsilon_o n c}{2} \left| \vec{E}_o \right|^2 |u|^2 = \frac{\varepsilon_o n c}{2} \left| \vec{E}_o \right|^2 \left( \frac{w_0^2}{w^2(z)} \right) \int_0^\infty 2\pi r dr e^{-2r^2/w^2(z)} \]

\[ = \frac{1}{2} \frac{\varepsilon_o n c}{2} \left( \frac{\pi w_0^2 n^2}{2} \right) \]  

(1.61)

The power is equal to the on-axis intensity at the focus, multiplied by the area of a circle of radius \( w_0 / \sqrt{2} \), which is the radius of the intensity at the \( e^{-1} \) point.

Equation (1.59) is the lowest order member of a whole family of Hermite-Gaussian solutions to the paraxial wave equation, given by
\[ u_{mn}(x,y,z) \sim \frac{1}{w(z)} H_m \left( \frac{\sqrt{2}x}{w(z)} \right) H_n \left( \frac{\sqrt{2}y}{w(z)} \right) e^{-\left( a^2 + b^2 \right) / w^2(z)} e^{-j k (a^2 + b^2) / 2 R(z)} e^{i (m+n+1) \phi(z)} \]

(1.62)

The \( H_\nu(\xi) \) with \( \nu \) a nonnegative integer are Hermite polynomials [7], where for example \( H_0(\xi) = 1 \), \( H_1(\xi) = 2\xi \), and \( H_2(\xi) = 4\xi^2 - 2 \). In general, the \( H_\nu \) are polynomials of order \( \nu \), which become wider and exhibit a greater number of zero crossings as \( \nu \) increases. The expressions for \( w(z) \), \( R(z) \), and \( \phi(z) \) are still as given above. Although the fundamental Gaussian beam is by far the most important in most ultrafast optics applications, the higher order (\( m \neq 0 \) or \( n \neq 0 \)) Hermite-Gaussian solutions will be relevant in our study for several reasons. For example, all the \( u_{mn} \) in eq. (1.62) are allowed transverse modes of laser resonators, termed \( TEM_{mn} \) modes, although usually the fundamental \( TEM_{00} \) mode has the lower loss and is therefore favored. We shall see however that higher order modes were significant in the discovery of the important Kerr lens mode locking technique for short pulse generation with solid-state lasers. Additionally, the Hermite-Gaussians form a complete orthogonal basis set, which means that an arbitrary spatial beam profile can be decomposed into a superposition of these functions.

In order to calculate the propagation of Gaussian beams through linear optical systems, it is convenient to introduce a complex \( q \) parameter, such that

\[ \frac{1}{q(z)} = \frac{1}{R(z)} - \frac{j \lambda}{\pi w^2(z) n} \]

(1.63)

The effect of the optical system can then be characterized by a bilinear transformation of \( q \), namely

\[ q_{\text{out}} = \frac{A q_{\text{in}} + B}{C q_{\text{in}} + D} \]

(1.64)

The coefficients appearing in eq. (1.64) are usually written in matrix form. The matrices for a few common systems are as follows:

\[ \text{A length } d \text{ of a homogeneous medium:} \]

\[ \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \]

(1.65a)

\[ \text{A lens with focal length } f \text{. This matrix also applies to a spherical mirror of radius } R \text{ if we set } f = R/2: \]

\[ \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \]

(1.65b)

\[ \text{A planar interface, perpendicular to the propagation direction, from a medium with index } n_1 \text{ to a medium with index } n_2:} \]
\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & \frac{1}{m^2}
\end{pmatrix}
\]

(1.65-c)

An important property of such matrices is that the effect of cascaded systems is computed via matrix multiplication. If \(M_1, M_2, \ldots, M_N\) represent the matrices for \(N\) cascaded systems, with the beam entering system 1 first and system \(N\) last, then the effect of the cascaded system is given by \(M_N \ldots M_2 M_1\). For example, the matrix for a homogenous slab of thickness \(d\) and index \(n\) and surrounded on either side by free space is

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & n
\end{pmatrix} \begin{pmatrix}
1 & d \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 1/n
\end{pmatrix} = \begin{pmatrix}
1 & d/n \\
0 & 1
\end{pmatrix}
\]

(1.66)

The matrices given above are also useful in the ray optics description of optical systems. Here one writes

\[
\begin{pmatrix}
x_{\text{out}} \\
x'_{\text{out}}
\end{pmatrix} = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \begin{pmatrix}
x_{\text{in}} \\
x'_{\text{in}}
\end{pmatrix}
\]

(1.67)

where \(x_{\text{in}}\) and \(x'_{\text{in}}\) are the position and slope of the incoming ray, respectively, and similarly for the outgoing ray. For a given optical system, the matrices for the ray optics description and the Gaussian beam (\(q\) parameter) description are identical. The ray optics description is sometimes helpful in identifying or interpreting the ABCD matrices. For example, the matrix in eq. (1.66) says that the slab of physical thickness \(d\) has an effective thickness of \(d/n\). From the ray optics description, it is clear that this occurs due to the bending of the rays towards the normal as they enter the medium.

The \(q\) parameter and the ABCD matrices are useful in solving for the transverse mode of laser resonators. Assume we have calculated the matrix for one complete round trip through the resonator, for example starting at a particular mirror \(M\). Then one condition for an acceptable transverse mode is that field pattern must exactly reproduce itself after one round trip, which can be formulated

\[
q = \frac{Ap + B}{Cp + D}
\]

(1.68)

A further condition is that for a stable mode, the beam radius must be finite, which means \(q\) must have a nonzero imaginary part. From the ray optics perspective, this stability condition means that incoming rays remain confined after many passes through the system. Multiplying through by the denominator in eq. (1.68) yields a quadratic polynomial equation for \(q\). The root of this equation with a negative imaginary part gives \(q\) at mirror \(M\), and \(q\) in turn yields the mode size within the laser resonator. Not all resonators have stable solutions. A well known and important example is the simple two mirror cavity (Fig. 1.6), consisting of mirrors with radii of curvature \(R_1\) and \(R_2\) which are separated by distance \(d\). After calculating the round trip ABCD matrix and requiring that
the roots of eq. (1.68) have nonzero imaginary part, one obtains the stability relation

\[
0 < \left(1 - \frac{d}{R_1}\right) \left(1 - \frac{d}{R_2}\right) < 1
\]

(1.69-a)

or simply

\[
0 < g_1 g_2 < 1
\]

(1.69-b)

where \(g_1 = 1 - d/R_1\) and \(g_2 = 1 - d/R_2\) are symbols often used to express the stability condition succinctly. We will make use of this stability condition when we discuss Kerr lens mode locking in chapter 2.

1.4 Introduction to ultrashort pulse generation through mode locking

The single mode laser discussed in the previous section can have a very narrow optical frequency spectrum. This results in the well known monochromaticity property of lasers. However, short pulse generation requires a broad optical bandwidth and hence multiple longitudinal mode operation. Even with the bandwidth limiting filter removed, however, the saturated gain in a homogeneously broadened laser is below the loss level except for those modes near the peak of the gain spectrum, and this also limits the number of oscillating modes. Methods for forcing a great number of modes to oscillate to obtain broad bandwidths and ultrashort pulses will be covered in detail later. In the following we analyze the time domain characteristics of a laser which we assume already has multimode operation.

For a multimode laser we can write the electric field as follows:

\[
e(z, t) = \text{Re} \left[ \sum_m E_m e^{j(\omega_m t - k_m z + \phi_m)} \right]
\]

(1.70-a)

where

\[
\omega_m = \omega_0 + m \Delta \omega = \omega_0 + \frac{2m \pi c}{L}
\]

(1.70-b)

and

\[
k_m = \frac{\omega_m}{c}
\]

(1.70-c)

Henceforth we will try to use small letters [e.g., \(e(z, t)\)] for the time domain representation of the field and large letters [e.g., \(E_m\)] to refer to the frequency domain. Equation (1.70-a) most accurately models a unidirectional ring laser, where the \(z\) coordinate refers to travel around the ring in the direction of laser oscillation, although the same essential results will also hold for a linear (Fabry-Perot) laser. The round trip distance around the ring is denoted \(L\) (for a Fabry-Perot laser we would use \(L = 2l\), where \(l\) is the Fabry-Perot mirror separation). In this formulation we are assuming the laser is oscillating in a single transverse mode, so that the spatial profile may be dropped.
Equation (1.70-a) can be rewritten as follows:

\[ e(z, t) = \text{Re} \left[ e^{j\omega_0(t-z/c)} \sum_m E_m e^{j[m\Delta\omega(t-z/c)+\phi_m]} \right] \]  

(1.71-a)

\[ = \text{Re} \left[ a \left( t - \frac{z}{c} \right) e^{j\omega_0(t-z/c)} \right] \]  

(1.71-b)

where

\[ a \left( t - \frac{z}{c} \right) = \sum_m E_m e^{j[m\Delta\omega(t-z/c)+\phi_m]}. \]  

(1.71-c)

Thus, the electric field is the product of a complex envelope function \( a \left( t - \frac{z}{c} \right) \) with the optical carrier \( e^{j\omega_0(t-z/c)} \).

This terminology is convenient because in many cases the carrier term, which oscillates with a period of just a few femtoseconds (for visible wavelengths), varies much more rapidly than the envelope function. (However, for the shortest pulses available today, comprising just a few optical cycles, this distinction is blurring.) Both carrier and envelope functions travel around the laser cavity at the speed of light. Furthermore, for a specific location in the cavity \( (z \text{ fixed}) \), the envelope function is periodic with period

\[ T = \frac{2\pi}{\Delta\omega} = \frac{2\ell}{c} = \frac{L}{c}. \]  

(1.72)

This corresponds to the time required for light to make one round trip around the resonator.

In order to say anything further about the shape of the envelope function, we need to specify the mode amplitudes \( E_m \) and phases \( \phi_m \). If we assume that there are \( N \) oscillating modes all with equal amplitudes \( E_0 \) and with the phases identically zero, eq. (1.71-a) becomes easy to evaluate. The process (to be discussed later) by which the modes are held with fixed relative phases is known as mode locking, and as we shall see, having all the phases equal is a particularly useful form of mode locking. Equation (1.71-a) now becomes

\[ e(z, t) = \text{Re} \left[ E_0 e^{j\omega_0(t-z/c)} \sum_{m=0}^{(N-1)/2} e^{j[m\Delta\omega(t-z/c)]} \right] \]  

(1.73)

To evaluate this we substitute \( m' = m + \frac{N-1}{2} \) and use the summation formula

\[ \sum_{m=0}^{g-1} a^m = \frac{1 - a^g}{1 - a}. \]  

(1.74)

Upon some further simplification, the result is

\[ e(z, t) = \text{Re} \left[ E_0 e^{j\omega_0(t-z/c)} \frac{\sin \frac{N\Delta\omega}{2}(t-z/c)}{\sin \frac{\Delta\omega}{2}(t-z/c)} \right] \]  

(1.75)
The laser intensity \( I \), averaged over an optical cycle, is proportional to \( |e(z, t)|^2 \). At a specific cavity location, say \( z = 0 \), we have

\[
I(t) \sim |E_0|^2 \frac{\sin^2 \left( \frac{\Delta \omega t}{2} \right)}{\sin^2 \left( \frac{\Delta \lambda t}{2} \right)}.
\]  \hspace{1cm} (1.76)

The resulting intensity profile is sketched in Fig. 1.7a. Some of the key points are as follows:

- The output consists of a periodic series of short pulses, with period \( T = \frac{1}{\Delta \lambda} = \frac{\lambda}{c} \).
- The pulse duration is approximately \( \Delta t = \frac{2\pi}{N \Delta \omega} = \frac{1}{N \Delta \lambda} \). Thus, the pulse duration is equal to the periodicity divided by the number of modes. Equivalently, the pulse width is equal to the inverse of the total laser bandwidth.
- The peak intensity is proportional to \( N^2 |E_0|^2 \). In comparison, the average intensity (averaged over the pulse period), given by the number of modes times the power per mode, is proportional to \( N |E_0|^2 \). Thus, the peak intensity of a modelocked pulse is enhanced by a factor \( N \).

A very different situation arises if we assume the mode phases are random, as in a "garden variety" multimode laser. In this case the intensity profile takes on a random appearance, as shown in Fig. 1.7b. Key points are as follows:

- The intensity fluctuates randomly about the average intensity value \( \sim N|E_0|^2 \). The average intensity is the same as in the modelocked case, although the peak intensity is a factor of \( N \) lower.
- The time scale over which the fluctuations vary (also known as the correlation time) is still \( \Delta t \approx \frac{1}{N \Delta \lambda} \).
- The intensity fluctuations are still periodic with period \( T = \frac{1}{\Delta \lambda} \).

In addition, in the "garden variety" multimode laser, the random phases are likely to vary slowly with time as well. This introduces yet an additional degree of randomness to the multimode laser output. In order to avoid such fluctuations and obtain a well defined output, one can either utilize single-mode lasers with narrow spectra, as in section 1.3, or mode locked lasers with narrow pulses and broad spectra.

1.5 Fourier series and Fourier transforms

We shall have many occasions where we wish to describe pulses in terms of a frequency domain representation. We have already encountered one example in the previous section, where we wrote a mode locked periodic pulse train as a
summation over the longitudinal cavity modes. In this section we briefly review Fourier series and Fourier transforms, which are the mathematical tools used to convert between time and frequency-domain representations. For more detail, see texts such as [8] [9]. The key point is that any time-dependent signal can be written as a superposition or summation of sines and cosines (or complex exponentials) with different frequencies.

If a function \( f(t) \) is periodic with period \( T \), we can write it as a Fourier series:

\[
f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\Delta \omega t}, \tag{1.77}
\]

where \( \Delta \omega = \frac{2\pi}{T} \) is the [angular] frequency. The \( F_n \) are the Fourier amplitudes or Fourier coefficients and can be obtained from the time-domain signal \( f(t) \) as follows \(^3\):

\[
F_n = \frac{1}{T} \int_{0}^{T} f(t)e^{-jn\Delta \omega t} dt \tag{1.78}
\]

In the case of an aperiodic time-domain function, we can consider that the period \( T \) goes to \( \infty \). We then have the frequency spacing \( \Delta \omega \to 0 \), and therefore we replace the discrete variable \( n \Delta \omega \) with a continuous variable \( \omega \). This results in the Fourier transform:

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega \tag{1.79-a}
\]

and

\[
F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt \tag{1.79-b}
\]

\( F(\omega) \) is known as the Fourier transform of \( f(t) \), and \( f(t) \) is obtained by performing the inverse Fourier transform of \( F(\omega) \).

We now review a few useful properties of Fourier transforms, most of which we will state without proof. For all of the following we assume \( f(t) \) and \( F(\omega) \) are a Fourier transform pair.

- **The reality condition**: If \( f(t) \) is a real function, which it will be whenever it represents an actual physical observable, then \( F(-\omega) = F^*(\omega) \).
- **The scaling formula**:

\[
\text{If } h(t) = f(at), \text{ then } H(\omega) = \frac{1}{a} F\left(\frac{\omega}{a}\right) \tag{1.80}
\]

- **The time delay formula**:

\[
\text{If } h(t) = f(t - \tau), \text{ then } H(\omega) = F(\omega) e^{-j\omega \tau}. \tag{1.81}
\]

\(^3\)Instead of integrating from 0 to \( T \), one can get the same results by integrating over any other interval of duration \( T \) e.g., \( -T/2 \) to \( T/2 \).
• The frequency offset formula:

\[ H(\omega) = F(\omega - \omega_0). \] (1.82)

• The convolution formula:

The convolution of two functions \( f(t) \) and \( g(t) \) is denoted \( f(t) * g(t) \) and is defined by

\[ f(t) * g(t) = \int dt' f(t')g(t-t') \]

The convolution formula states that

\[ H(\omega) = F(\omega)G(\omega). \] (1.83)

• Parseval’s theorem:

We know that the intensity \( I(t) \) of a pulse whose electric field profile is \( f(t) \) is proportional to \( |f(t)|^2 = f(t)\ast f(t) \). Recall that the units of intensity are \( \text{watts/m}^2 \). The pulse energy is given by the time-integrated intensity integrated over the cross-sectional area. Parseval’s theorem says that

\[ \int f(t)f^*(t)dt = \frac{1}{2\pi} \int F(\omega)F^*(\omega)d\omega. \] (1.84)

Therefore, the time-integrated intensity is equal to the frequency-integrated intensity, except for a multiplicative factor. For this reason the quantity \( |F(\omega)|^2 = F(\omega)F^*(\omega) \) is called the power spectral density.

The use of these formulas will be illustrated as they are needed. We do provide a few examples of the Fourier transform:

1. \( f(t) = \delta(t) \), where \( \delta(t) \) is the delta function or unit impulse function. Recall that \( \delta(t) = 0 \) for \( t \neq 0 \), \( \delta(t) = \infty \) for \( t = 0 \), and \( \int_{-\infty}^{\infty} dt \delta(t) = 1 \). By substitution into eq. (1.79-b), we find \( F(\omega) = 1 \).

2. If \( f(t) = e^{j\omega_0t} \) then \( F(\omega) = 2\pi\delta(\omega - \omega_0) \). This is verified by plugging into eq. (1.79-a).

3. \( f(t) \) is a Gaussian, \( f(t) = e^{-t^2/\sigma^2} \). Then

\[ F(\omega) = \int dt e^{-t^2/\sigma^2} e^{-j\omega t} = \int dt e^{-[(t+\omega\sigma^2/2)/\sigma^2]t^2/2} e^{-\omega^2\sigma^2/4} \]

Using the substitution \( u = \frac{t+\omega\sigma^2/2}{\sigma} \) and \( \int_{-\infty}^{\infty} du e^{-u^2} = \sqrt{\pi} \), we obtain

\[ F(\omega) = t_p \sqrt{\pi} e^{-\omega^2\sigma^2/4} \]

The procedure we have used above, called completing the square, is very useful for evaluating Fourier transforms of Gaussian functions.
Finally, we will frequently write a pulse in terms of a slowly varying envelope function times an optical carrier term:

\[ e(t) = \text{Re} \left[ a(t) \ e^{j\omega_0 t} \right] = \frac{1}{2} \left[ a(t) \ e^{j\omega_0 t} + a^*(t) \ e^{-j\omega_0 t} \right] \] (1.85)

The spectrum \( E(\omega) \) is then given by

\[ E(\omega) = \frac{1}{2} \left[ A(\omega - \omega_0) + A^*(-\omega - \omega_0) \right] \] (1.86-a)

where

\[ A(\omega) = \int_{-\infty}^{\infty} a(t)e^{-j\omega t} \, dt \] (1.86-b)

is the Fourier transform of the envelope function \( a(t) \). Equation (1.86-a) is sketched in Fig. 1.8. Separating the field into envelope and carrier terms is most useful when (as in Fig. 1.8) the bandwidth of \( A(\omega) \) is much less than the carrier frequency \( \omega_0 \). In the time domain, this means that the envelope is much longer than the optical cycle.

The pulse \( e(t) \) can be obtained directly by performing the inverse Fourier transform of the double-sided spectrum described by eq. (1.86-a) or from the single-sided spectrum using the formula

\[ e(t) = \text{Re} \left[ \frac{1}{2\pi} \int A(\omega_0 - \omega) \ e^{j\omega t} \, d\omega \right] \] (1.87)
Figure 1.1: (a) Schematic of a linear cavity laser. (b) Schematic of a ring laser. HR: high reflector. OC: output coupler.

Figure 1.2: Laser gain and cavity loss spectra, longitudinal mode locations, and laser output for multimode laser operation.
Figure 1.3: Gain and loss spectra, longitudinal mode locations, and laser output for single mode laser operation.

Figure 1.4: Energy level structure for a four level atom.
Figure 1.5: Sketch of a Gaussian beam solution of the paraxial wave equation, showing evolution of beam radius and representative phase front.

Figure 1.6: Simple two mirror optical resonator.

Figure 1.7: (A) Mode-locked laser output with constant mode phase. (B) Laser output with randomly phased modes.
Figure 1.8: Double-sided spectrum corresponding to eq. (1.86-a)
Chapter 2

Principles of mode locking: I

2.1 Processes involved in mode locking

The single-mode lasers reviewed in the previous chapter were relatively simple and could be adequately modeled including only the laser gain, the cavity loss, and the effects of feedback. However, such lasers do not generate short pulses! In order to obtain a mode locked laser generating ultrashort pulses, one must incorporate either active or nonlinear pulse forming elements (modulators) into the laser cavity. Furthermore, one must also consider new processes affecting the short pulse as it propagates within the laser cavity. The following nonexhaustive list describes some of the processes that can be important in mode locked lasers:

- **gain.** Gain is required in mode locked lasers, just as it is in single mode lasers. As we have seen, gain saturation, in which the gain decreases for increasing laser power, is important in any laser. In single-mode lasers, the gain is sensitive only to the average laser power. In some (not all) mode locked lasers, however, the gain saturates dynamically on the time scale of the pulse and can be an important mode locking process.

- **linear loss.** Linear loss, which is also present in any laser, refers to that portion of the loss which is independent of laser power.

- **bandwidth limitations.** Passive filtering effects can limit the bandwidth which can be transmitted through the laser cavity. Filtering can arise either from frequency dependent loss elements or from the finite bandwidth of the laser gain medium. Bandwidth limiting effects are often undesirable in that they limit how short a pulse can be obtained.

- **dispersion.** This refers to passive, frequency-dependent phase variations encountered by the pulse as it travels through the laser cavity. Dispersion
usually leads to pulse broadening and can be an extremely important limitation for femtosecond pulse generation. Consequently, dispersion compensation devices are commonly included within femtosecond laser cavities.

- **active modulation.** This refers to an externally driven optical modulator which modulates either the amplitude or the phase of the circulating pulse. For mode locking the modulation frequency is usually chosen to coincide with the cavity round trip time. The modulator may be either an acousto-optic or electro-optic loss or phase modulator, or in some cases the gain element itself is modulated.

- **saturable loss (self-amplitude modulation).** The cavity loss may be a function of pulse intensity or pulse energy. The loss changes dynamically in response to the pulse, which is itself modulated by the dynamically changing loss. This nonlinear process leads to a self-induced modulation with a period which is automatically equal to the cavity round trip time. Saturable loss elements, also known as saturable absorbers, are classified either as fast or slow, depending on how rapidly they recover with respect to the pulse duration.

- **self-phase modulation.** The phase can also vary nonlinearly with the time-dependent pulse intensity. This leads to a nonlinear self-phase modulation process, which again occurs at the pulse round trip time. In mode locked lasers the nonlinear phase shift usually has a very rapid response time and thus depends only on the instantaneous laser intensity, but in some types of lasers slower nonlinear phase processes have also been important. Self-phase modulation can interact in a very interesting and useful way with dispersion, leading to soliton pulses which are stable against perturbations. Such solitons are important not only in femtosecond solid-state lasers, but also in optical fiber communications.

Modulation plays a key role in initiating and maintaining mode locked laser operation. Active mode locking refers to the case where the modulator is externally driven. Passive mode locking refers to situations in which the pulse forms its own modulation through nonlinearities; both amplitude and phase nonlinearities can be important. The shortest pulses over the last twenty five years or so have been generated using passively mode locked lasers. In this chapter we will cover the more basic concepts of active and passive mode locking. More advanced mode locking concepts will be covered in chapter 7, after we have had a chance to treat prerequisite information on dispersion and ultrafast nonlinear optics in the necessary depth.

One can imagine many different combinations of the pulse shaping processes listed above, corresponding to many different flavors of mode locked lasers. In our treatment we shall concentrate on a few of the main possibilities that are of historical or current importance. In doing so, we will not only learn about basic mode locking principles, but we will also learn more about processes affecting ultrafast optical signals and about lasers in general.
2.2 Active mode locking

We begin by discussing active mode locking, one of the first mode locking techniques demonstrated and conceptually one of the simplest. As sketched in Fig. 2.1a, mode locking is achieved with the aid of an externally driven intracavity loss modulator. The modulator is driven at a frequency \( \omega_m = 2\pi/T \) corresponding to the longitudinal mode spacing (where \( T \) is the round-trip transit time). In the time domain this means that the modulator acts like a periodic shutter which opens once per pulse round trip time (Fig. 2.1b). We can envision that the initial laser radiation is shortened again and again every time it passes through the modulator. Thus, in order to minimize the intracavity losses, one would expect laser emission to occur in the form of short pulses synchronized with positions of maximum shutter opening. Also in steady state, we will find that the per pass pulse shortening by the modulator must be exactly balanced by the (unwanted) per pass pulse broadening mechanisms. In the following we will first analyze active mode locking from this time domain point of view. We will subsequently present an alternate analysis from a frequency domain perspective.

2.2.1 Active mode locking: time-domain treatment

In our analysis we will consider propagation of an ultrashort pulse around an actively mode locked laser cavity [10] [5]. Our model for the laser cavity is shown in Fig. 2.2. The principle elements in our model are the laser cavity itself, the gain medium, and the modulator. In order to arrive at a stable mode locked solution, we will require that the pulse reproduce itself after one round trip through the laser.

We start off by assuming a Gaussian laser pulse of the form

\[
    e(t) = \text{Re} \left[ a(t) e^{j\omega_0 t} \right] = \text{Re} \left[ E_0 e^{-\Gamma t^2} e^{j\omega_0 t} \right]
\]

where \( a(t) = E_0 e^{-\Gamma t^2} \) is the envelope function and \( \Gamma \) is a pulse width parameter. From our results in section 1.5, the spectrum can be written

\[
    E(\omega) = \frac{1}{2} \left[ A(\omega - \omega_0) + A^*(-\omega - \omega_0) \right]
\]

with

\[
    A(\omega) = \left( \frac{\pi}{\Gamma} \right)^{1/2} e^{-\omega^2/4\Gamma}
\]

We now consider each laser element in turn.

The cavity has two effects: it introduces loss and a round-trip time delay. We can represent these effects by the multiplicative factor

\[
    e^{-\ell e^{-j\omega L/c}}
\]

Here \( e^{-\ell} \) is the round-trip attenuation of the field accounting for everything except the loss modulator. The term \( e^{-j\omega L/c} \) corresponds to a delay by the
round-trip propagation time $L/c$, where $L = l_a + n'_g l_g + n'_m l_m$ is the round-trip optical path length, $l_a, l_g,$ and $l_m$ are the physical lengths of air regions, gain medium, and modulator, respectively, and $n'_g$ and $n'_m$ are the real parts of the refractive index of the gain medium and the modulator. Note that $n'_g$ does not include the part of the gain medium refractive index due to the laser transition - see below.

The purpose of the gain medium is of course to amplify the field. We make the following assumptions about the gain medium:

- The gain medium is homogeneously broadened.

- The relaxation time of the gain medium is much longer than the pulse repetition period. Consequently, the gain saturates only in response to the average laser power. The gain can be considered constant during passage of a single laser pulse.

- The gain medium has a finite optical bandwidth. The finite gain bandwidth acts to limit how short a pulse can be attained.

Specifically, we express the action of the gain medium by the multiplicative factor

$$
\exp \left[ \frac{g}{1 + 2j \left( \frac{\omega - \omega_0}{\omega_G} \right)} \right]
$$

Here $g$ refers to the saturated gain, and the particular gain lineshape function

$$
G(\omega) = \frac{g}{1 + 2j \left( \frac{\omega - \omega_0}{\omega_G} \right)}
$$

is known as a Lorentzian and corresponds to our assumption of a homogeneously broadened atomic-like gain medium. Introducing the notation $\tilde{\omega} = \omega - \omega_0$, we can rewrite eq. (2.3) as

$$
G(\omega) = \frac{g (1 - 2j \tilde{\omega} / \omega_G)}{1 + (4 \tilde{\omega}^2 / \omega_G^2)}
$$

$G(\omega)$ can be related to the complex refractive index $n = n' + jn''$ by identifying $G(\omega) = -j \omega n G / c$ (see eq. (1.37)). Consequently we have

$$
n' = \left( \frac{c}{\omega_G} \right) \frac{2g (\tilde{\omega} / \omega_G)}{1 + (4 \tilde{\omega}^2 / \omega_G^2)} \quad \text{(2.5-a)}
$$

and

$$
n'' = \left( \frac{c}{\omega_G} \right) \frac{g}{1 + (4 \tilde{\omega}^2 / \omega_G^2)} \quad \text{(2.5-b)}
$$

$n''$ is associated with gain, and $n'$ is associated with real refractive index (i.e., phase) effects. Figure 2.3 shows a plot of $n'$ and $n''$. The gain term ($n''$) has a symmetric peak centered at $\omega_0$ with a full-width at half-maximum (FWHM) of
\( \omega_c \). \( n' \) is an antisymmetric function with a zero at \( \omega_o \) and with \( n' < 0 \) below resonance \( (\omega < \omega_o) \).

Before proceeding, we note that the value of the peak gain \( g \) is proportional to the population difference \( (N_u - N_l) \) between upper and lower laser energy levels. For gain to occur, the population must be inverted \( (N_u - N_l > 0) \), and \( n' \) and \( n'' \) are as shown in Fig. 2.3. If the pump is turned off, then the population is not inverted \( (N_u - N_l < 0) \), and the signs of \( n' \) and \( n'' \) are opposite to those shown in the figure. Therefore, for unpumped media, \( n'' < 0 \), signifying absorption, and \( n' > 0 \) for \( \omega < \omega_o \).

In actively modelocked lasers, the pulse usually has a bandwidth narrow compared to the gain bandwidth \( \omega_c \). Therefore, we can approximate eqn. (2.3) using a second order Taylor series expansion. Using

\[
\frac{1}{1 + \Delta} \approx 1 - \Delta + \Delta^2
\]

which is valid for small \( \Delta \), we obtain

\[
G(\omega) \approx g \left[ 1 - 2j\omega/\omega_c - 4\omega^2/\omega_c^2 \right] \tag{2.7}
\]

Now by using eqn. (2.2) and eqn. (2.7), we can obtain an expression for the spectrum as affected by the cavity and the gain medium:

\[
A(\hat{\omega}) = E_o \left( \frac{\pi}{\Gamma} \right)^{1/2} e^{-j\omegaL/c}g \cdot e^{-j\hat{\omega} \left( \frac{1}{\pi \omega_c^2} \right)} e^{-\hat{\omega}^2/4\Gamma^2} \tag{2.8-a}
\]

where

\[
\frac{1}{\Gamma^2} = \frac{1}{\Gamma} + \frac{16g^2}{\omega_c^2} \tag{2.8-b}
\]

If the fractional change in \( \Gamma \) (or equivalently in the pulse width) per pass through the gain medium is small, we can rewrite eq. (2.8-b) as

\[
\Gamma' \approx \Gamma - \frac{16g^2}{\omega_c^2} \tag{2.9}
\]

At this point we Fourier transform back into the time domain. Assuming that \( \omega_c L/c = 2\pi n \) (i.e., \( \omega_o \) is equal to one of the cavity mode frequencies), we obtain the following for the envelope function \( a(t) \):

\[
a(t) = E_o e^{g-\Gamma} \left( \frac{\Gamma'}{\Gamma} \right)^{1/2} e^{-\Gamma' \left[ 1 - \left( \frac{j}{\pi \omega_c^2} \right) \right]^2} \tag{2.10}
\]

We see that \( \Gamma' \) is decreased by passage through the gain medium. This means the gain spectral profile reduces the bandwidth of the circulating pulse, resulting in an increase in pulse duration. Additionally, the pulse is delayed by a time

\[
T = \frac{L}{c} + \frac{2g}{\omega_c} = \frac{L}{c} \left( 1 + \frac{g}{\pi \omega_c} \right) \tag{2.11}
\]
where $\Delta \omega = \frac{2 \pi n}{L}$ is the ”cold cavity” (i.e., the cavity without including the effect of the laser transition) longitudinal mode spacing. The round trip delay time is increased over that of the cold cavity ($L/c$) by an amount $Lg \Delta \omega/c \omega_G$. This extra delay arises due to the spectral variation of $n'$ due to the laser transition. When $n'$ is a function of frequency, the pulse propagation velocity (also known as group velocity $v_g$) differs from the propagation velocity of the carrier (also known as the phase velocity $v_p = c/n'$). We shall discuss group and phase velocities further in a later chapter.

We will now account for the loss modulator by multiplying eq. (2.10) by its transmission function $\eta(t)$, where

$$\eta(t) = \exp[-\Delta_m(1 - \cos \omega_m t)]$$ (2.12)

Thus, $\eta(t)$ varies periodically with period $2\pi/\omega_m$. For best modelocking the modulator frequency is tuned to

$$\omega_m = \frac{2\pi}{T}$$ (2.13)

where $T$ is the actual round trip time given by eqn. (2.11). The peak transmission is unity at $t = 0, T, 2T, ...$, and the minimum transmission is $e^{-2\Delta_m}$. $\Delta_m$ is called the modulation depth. If the pulse is much shorter than the modulation period (as indeed we desire for a modelocked pulse!) and is centered near the times of peak transmission, we can approximate $\cos \omega_m t \approx 1 - \frac{1}{2} \omega_m^2 t^2$ near $t = 0$ and by $\cos \omega_m t \approx 1 - \frac{1}{2} \omega_m^2 (t - T)^2$ near $t = T$, etc. Then the transmission for the pulse in eqn. (2.10) becomes

$$\eta(t) = e^{-\frac{4}{9} \Delta_m \omega_m^2 (t-T)^2}$$ (2.14)

Multiplying (2.10) by (2.14) and using (2.11), we obtain an expression for the pulse after one round trip through the laser:

$$a(t) = E_e e^{\gamma t} \left( \frac{\Gamma'}{\Gamma} \right)^{1/2} e^{-t''(t-T)^2}$$ (2.15-a)

where

$$\Gamma'' = \Gamma' + \frac{1}{2} \Delta_m \omega_m^2 \approx \Gamma - \frac{16 \gamma \Gamma^2}{\omega_G^2} + \frac{1}{2} \Delta_m \omega_m^2$$ (2.15-b)

From eqn. (2.15-b) the modulator decreases the pulse width.

A steady-state mode locked solution requires that the pulse exactly reproduces itself after one round trip. Therefore, we require that eqn. (2.15-a) be identical to our starting point, eqn. (2.1), except for the time delay. As a result, we have

$$\Gamma'' = \Gamma$$ (2.16)

and

$$\left( \frac{\Gamma'}{\Gamma} \right)^{1/2} e^{\gamma t - t} = 1$$ (2.17)
Equation (2.16) says that the pulse width must reproduce itself after a round trip. Substituting in eqn. (2.15-b), we obtain

\[
\frac{1}{2} \Delta m \omega_m^2 - \frac{16 g T^2}{\omega_G^2} = 0
\]  

(2.18)

and

\[
\Gamma = \frac{1}{4} \sqrt{\frac{g}{\Delta m}} \left( \frac{\Delta m}{\omega_m} \right)^{1/2} \omega_m \omega_G
\]  

(2.19)

Equation (2.18) leads to the physical interpretation that the per pass pulse shortening due to the modulator exactly balances the per pass broadening due to the gain medium’s finite bandwidth.

Equation (2.19) is the desired solution for the pulse width parameter \( \Gamma \). It is customary to express the pulse width in terms of the full-width at half-maximum (FWHM) of the intensity, which we will denote \( \Delta t \).

\[
\Delta t = 2 \left( \ln 2 \right)^{1/2} = 2^{3/4} \sqrt{\ln 2} \left( \frac{g}{\Delta m} \right)^{1/4} \left( f_m f_G \right)^{-1/2} \approx 0.45 \left( \frac{g}{\Delta m} \right)^{1/4} \left( f_m f_G \right)^{-1/2}
\]  

(2.20)

Here \( f_m = \omega_m / 2\pi \) and \( f_G = \omega_G / 2\pi \). In order to obtain the shortest pulses, one prefers a large modulation index, a large modulation frequency, and a large gain bandwidth. The pulse width depends only weakly on these parameters, however, and usually relatively long pulses in the range 10 - 100 psec are generated.

Equation (2.17) is the gain condition for the actively mode-locked laser. The peak saturated gain \( (g) \) is slightly greater than the loss in order to balance the losses in the modulator and the fact that the overall gain is lower than \( g \) due to the frequency spread within the mode-locked pulse and the finite gain bandwidth. The average power \( P \) circulating in the laser can be obtained from the gain saturation equation,

\[
g = \frac{g_0}{1 + P/P_{sat}}
\]  

(2.21)

where \( g_0 \) is the small-signal gain and \( P_{sat} \) is the saturation power of the gain medium. In evaluating \( P \), \( g \) can be found exactly from eq. (2.17) or can be approximated by \( g \approx \ell \).

It should be noted that in our analysis, we have assumed that the modulation is exactly synchronous with the cavity transit time. Under this condition, the mode locked pulse is positioned exactly at the peak transmission point of the applied modulation, which was also assumed. In practice, the modulation frequency must match the mode spacing very precisely, with a tolerance typically in the range of a few hundred Hz to a few KHz. For a typical cavity length corresponding to a repetition rate on the order of 100 MHz, this means the frequency must be set correctly within one part in 10^7 or better, and the cavity length must remain stable to better than a few tens of \( \mu \)m. When the modulation frequency is detuned (within this narrow range), the mode locked pulse will be offset from the time of peak transmission. The nonsymmetrical
variation of the modulator transmission during the duration of the modelocked pulse will then act to push the pulse slightly back toward the transmission peak, which provides a timing restoration mechanism. Stable modelocked operation can be maintained only when the amount of timing restoration per laser transit is enough to fully compensate the difference between the modulation period and the laser transit time.

2.2.2 Active mode locking: frequency domain treatment

We now consider an alternate frequency domain theory of active mode locking. As before, we keep track of the effects induced by each laser element and require that the net charge per round trip be zero. Here we do not assume a Gaussian pulse at the outset. Instead, by assuming that the change per element per round trip is small, we arrive at a differential equation for the mode locked pulse. We shall follow a similar course in analyzing other types of mode locking in later sections. A principal merit of the treatment in this section is to illustrate the locking of cavity modes leading to the term ”mode locking.” Our discussion most closely follows [11]. Other related treatments include [12] [13].

We start with the effects of the cavity. We assume that the longitudinal modes of the ”cold cavity” are at frequencies \( \Omega_q = \omega_o + q\Delta\omega \), where \( \Delta\omega \) is the cold cavity mode spacing. For a low loss ("high Q") resonator, the time variation of the field resonating with the \( q \)th cavity mode \( E_q \) can be written

\[
\frac{dE_q}{dt} + j\Omega_q E_q + \frac{E_q}{\tau_o} = \text{source terms}
\]  

(2.22)

The identity of the source terms in our mode locking problem will become clear soon. As before, we assume an intracavity loss modulator driven at frequency \( \omega_m \). For steady-state mode locked operation, the laser output should be periodic with period \( 2\pi/\omega_m \). Hence the laser emission can be written as a Fourier series with harmonics \( \omega_n = \omega_o + n\omega_m \), where \( n \) is an integer. Here we have assumed the center frequency is still at \( \omega_o \), which will be true when the central cavity mode exactly coincides with the peak of the gain spectrum. As long as \( \omega_m \) nearly matches the mode-spacing \( \Delta\omega \), each of the driven frequencies \( \omega_n = \omega_o + n\omega_m \) interacts strongly only with the corresponding cavity resonance at \( \Omega_n = \omega_o + n\Delta\omega \). The equation for the \( n \)th harmonic, written \( \tilde{E}_n = \tilde{E}_n e^{j\omega_nt} \), is

\[
\frac{d\tilde{E}_n}{dt} + (jn\omega_m\delta)\tilde{E}_n + \frac{\tilde{E}_n}{\tau_o} = \text{source terms}
\]  

(2.23-a)

where \( \delta = 1 - \frac{\Delta\omega}{\omega_m} \)  

(2.23-b)

Note also that we identify the loss per round trip, which was taken as \( e^{-t} \) in the previous section, with the factor \( e^{-L/c\tau_o} \) arising from eqn. (2.22), where \( L/c \) is the round trip propagation time of the cold cavity. Thus we can equate

\[
\frac{1}{\tau_o} = \left( \frac{c}{L} \right) \ell
\]  

(2.24)
Equivalently, the loss coefficient \((\tau^{-1})\) in the rate equations for the harmonic amplitudes is obtained by taking the natural logarithm of the single pass loss and then dividing by the round trip time.

In a similar way, the gain is incorporated into eq. (2.23-a) by adding the term \(G(\omega)\tilde{E}_n/(L/c)\) to the right hand side of the equation. Here \(G(\omega)\) is approximated by eqn. (2.7), as in the previous section.

Now we consider the modulator. Using eq. (2.12) for the modulator transmission function \(\eta(t)\), the source term representing the effect of the modulator can be written

\[
\frac{\ln[\eta(t)]}{(L/c)} E_n = \frac{c}{L} \left\{ \frac{\Delta m}{2} \tilde{E}_n \left( e^{j \omega_m t} + e^{-j \omega_m t} \right) \right\}
\]

This expression is inserted into the right hand side of eq. (2.23-a). We see that as a result of the modulation, each harmonic acquires sidebands that act as source terms for the fields resonating with the adjacent cavity modes. Thus, the modulator couples the cavity modes together, which results in mode locking.

Putting all these terms together, the equation for the \(n\)th harmonic is written:

\[
\left( \frac{L}{c} \right) \frac{d\tilde{E}_n}{dt} = (-\ell + g) \tilde{E}_n - j n \omega_m \left( \frac{L\delta}{c} + \frac{2g}{\omega_c} \right) \tilde{E}_n - \frac{4g \omega_m^2}{\omega_c^2} \tilde{E}_n + \frac{\Delta m}{2} \left[ \tilde{E}_{n+1} - 2\tilde{E}_n + \tilde{E}_{n-1} \right]
\]  

(2.26)

For steady-state mode locking the changes affecting any particular harmonic \(\tilde{E}_n\) should cancel out after one round trip. Therefore, in eqn. (2.26) we set \(d\tilde{E}_n/dt = 0\). We also assume \(L\delta/c = -2g/\omega_c\); this corresponds to a modulation frequency matched to the actual round trip as modified by the laser transition, just as in eqn. (2.11). Finally, we replace the discrete harmonic spectrum \(\tilde{E}_n\) by a continuous spectrum \(A(\omega)\), where \(\omega = n \omega_m\). The last term in eqn. (2.26) representing the mode coupling is replaced by a second derivative

\[
\frac{\Delta m}{2} \left[ \tilde{E}_{n+1} - 2\tilde{E}_n + \tilde{E}_{n-1} \right] \to \frac{\Delta m \omega_m^2}{2} \frac{d^2 A(\omega)}{d\omega^2}
\]

(2.27)

These approximations are valid when the spectrum is broad compared to the mode spacing and varies little from one mode to the next. This corresponds to the condition that the mode locked pulses are short compared to the period and hence well separated - a condition that is almost always valid. Equation (2.26) now becomes

\[
-\frac{\Delta m \omega_m^2}{2} \frac{d^2 A(\omega)}{d\omega^2} + \frac{4g \omega^2}{\omega_c^2} A(\omega) = (g - \ell)A(\omega)
\]

(2.28)

Equation (2.28) is identical to the time-independent Schrödinger equation for a particle in a harmonic well, which is usually written

\[
-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2} \frac{1}{m} \omega^2 \psi^2 = E \psi
\]

(2.29)
This has the well-known solution for the ground state (see any quantum mechanics text):

$$\psi \sim H_\nu(u) e^{-u^2/2} \quad (2.30-a)$$

where

$$u = \frac{(Cm)^{1/4}}{\hbar^{1/2}} x \quad \text{and} \quad E = (\nu + \frac{1}{2}) \hbar \sqrt{\frac{C}{m}} \quad (2.30-b)$$

The $H_\nu(u)$ are the Hermite polynomials, which were introduced briefly in section 1.3.3. The ground state of the quantum mechanical problem corresponds to $\nu = 0$, for which the corresponding Hermite polynomial is $H_0(u) = 1$. By identifying $\Delta m \omega_m^2 \leftrightarrow \hbar^2 / m$ and $8g / \omega_m^2 \leftrightarrow C$, we obtain solutions for the mode locked spectrum. For the ground state solution ($\nu = 0$), the result is a Gaussian, identical to eqn. (2.2), with $\Gamma$ given by eqn. (2.19). Our choice of the pure Gaussian solution is justified since this is the only stable solution to the mode locking equation, as determined by a stability analysis published in [11]. From a simple physical point of view, this is consistent with the observation that the $\nu = 0$ solution requires the lowest gain, as can be obtained through inspection of eq. (2.30-b) with the identification $g - \ell \leftrightarrow E$. This result is also expected since the higher order Hermite solutions have a broader temporal duration and therefore experience a greater loss from the modulator.

The key lesson of this section is that from a frequency domain perspective, the effect of the modulator is to generate sidebands which couple adjacent cavity modes together. This coupling stabilizes the relative phases of the modes and leads to mode locking. This process is summarized by Fig. 2.4, which shows the small signal gain spectrum as well as the saturated gain of the mode locked and free running (i.e., continuous wave) laser. As we have seen, for continuous-wave operation of a homogeneously broadened laser, saturated gain just equals loss at line center; ideally, all modes other than the central mode are below threshold. In the case of active mode locking, the side band creation process redistributes energy from modes near line center to modes in the wings of the gain spectrum. As a result the saturated gain must exceed the loss at line center. Furthermore, even modes where the gain is below the loss line can oscillate if sufficient power is injected by the modulator through side band generation. From this picture the role of mode locking in inducing a large number of modes to oscillate should be clear.

Finally, it is worth noting that the differential equation for the spectrum, eqn. (2.28), can be Fourier transformed into a differential equation in the time domain. We use the Fourier transform identities

$$\frac{d}{dt} \leftrightarrow j\omega \quad , \quad \frac{d^2}{dt^2} \leftrightarrow -\omega^2 \quad (2.31)$$

$$\frac{d}{d\omega} \leftrightarrow -j \tau \quad , \quad \frac{d^2}{d\omega^2} \leftrightarrow -\tau^2$$
The interpretation of eqn. (2.31) is that we can replace the factor \(-\omega^2\) appearing in the frequency domain equation with the operator \(\frac{d^2}{dt^2}\) in the time domain equation, etc. The result of the Fourier transform is

\[
\frac{-4g}{\omega_G^2} \frac{d^2 a(t)}{d t^2} + \frac{\Delta_m \omega_m^2}{2} a(t) = (g - \ell) a(t) \tag{2.32}
\]

The solution is obtained once again by comparing to the Schrödinger equation for a particle in a harmonic well. The result is identical to eqn. (2.1) with \(\Gamma\) given by eqn. (2.19). Of course, this solution for \(a(t)\) is just the Fourier transform of the solution for \(A(\omega)\) from eqn. (2.28).

### 2.2.3 Variations of active mode locking

Our treatment of active mode locking via synchronous loss modulation covers the most common form of active mode locking. A very similar variant of this technique is synchronous gain modulation, which is important in semiconductor diode lasers, since the gain can conveniently be modulated by modulating the current drive to the laser. Our active mode locking analysis applies to this case also with only minor modifications. In the following we briefly enumerate several other variations of active mode locking, although no detailed analysis will be included.

- Active mode locking can be achieved using synchronous phase modulation [10] [14]. The analysis given above still applies, provided that the modulation depth \(\Delta_m\) takes on an imaginary value. In the circulating pulse picture, a time derivative of the phase gives the pulse a frequency shift, which reduces its gain due to the finite gain bandwidth. For light sitting exactly at the phase extremum, the frequency shift goes to zero, and therefore the gain is higher. This provides the pulse shortening mechanism needed for successful mode locking.

- In regenerative mode locking the modulation signal is derived by using a photodiode to detect the beat note between the initially unlocked longitudinal modes of the laser [15]. The beat note is amplified and then used to drive an intracavity modulator. The modulation frequency is therefore automatically adjusted to match the cavity mode spacing, and provided that the beat note is properly phase shifted so that the applied modulation will enhance the initial mode beating fluctuation, mode locking results. This relaxes the requirements on careful adjustment of the cavity length in active mode locking. In some sense regenerative mode locking is a cross between active and passive mode locking; the modulation signal originates due to the laser signal itself as in passive mode locking but is applied via an electrically driven modulator as in active mode locking. Regenerative mode locking has also been used as a starting mechanism for solid-state lasers where passive mode locking dominates the steady-state mode locked dynamics.
• In harmonic mode locking the modulator is driven at a harmonic of the cavity mode spacing. The more rapid variation of the applied modulation function can lead to shorter pulses - note the modulation frequency dependence in eqn. (2.20). Harmonic mode locking allows generation of pulse trains with the pulses spaced more closely than the cavity round trip time. This is especially useful in fiber lasers for optical communications applications, where the round trip time can be very long (corresponding often to hundreds of nanoseconds), while pulse trains may be desired at multiple GHz rates [16]. It should be noted, however, that there is no guarantee that a pulse will be present in every transmission peak in harmonic mode locking; therefore, if this is important, other stabilizing mechanisms must be incorporated into the laser.

• In another form of mode locking most appropriate for optically pumped systems, the laser is synchronously pumped by the pulses from another mode locked laser [17] [18]. This technique was commonly applied to dye lasers, where relatively long pulses (∼ 100 ps) from an actively mode locked argon ion or Nd:YAG laser were used to pump the much broader bandwidth dye laser. Under proper conditions, this could lead to considerably shorter output pulses (∼ ps) at the wavelength of choice (determined by the choice of dye). In these systems the gain is built up by the arrival of the pumping pulse, and then turned off rapidly due to dynamic saturation induced by the resulting dye laser pulse. Such transient gain saturation is discussed in the next section on passive mode locking. Similar to other actively mode locked systems, cavity length matching between the two lasers is critical.

• In hybrid mode locking active modulation is combined with passive mode locking, most commonly using a slow saturable absorber. The active modulation is commonly in the form of an electrically driven gain modulation, as in the case of a semiconductor diode laser [19], or an optically driven gain modulation, as in the case of synchronously pumped dye lasers. Typically in hybrid mode locking, the active modulation assists in pulse formation and helps to stabilize the mode locking process, while the saturable absorber is responsible for significant reduction of the final pulse duration. Much of the insight we develop in the next section on saturable absorber mode locking is also relevant to the hybrid mode locking case.

2.3 Passive mode locking using saturable absorbers

Although active mode locking is a well established technique, it suffers from two disadvantages:

• An externally driven modulator is required, and the modulation frequency must be precisely matched to the cavity mode spacing. (However, this can be an advantage for applications where synchronization to an external clock is required!)
Pulse shortening due to the modulator becomes ineffective for very short pulses, and this limits the attainable pulse width.

These drawbacks can be overcome by using passive mode locking, in which the externally driven modulator is replaced by a nonlinear optical element whose loss depends on the laser pulse intensity. For mode locking the loss should decrease with increasing intensity. An ultrashort pulse circulating in the laser will modulate the intracavity loss, which in turn will modulate the circulating pulse. As a result the loss modulation is automatically synchronized with the laser pulses. Furthermore, if the response time of the nonlinearity is sufficiently fast, then the optically driven modulation function gets faster as the pulse becomes shorter. Hence, the pulse shortening action can remain effective even for very short pulses.

We will begin our treatment of passive mode locking by discussing saturable absorbers as the nonlinear loss element. By “saturable absorber” we mean an optically absorbing medium whose absorption decreases with intensity or energy. Throughout most of the ‘70s and ‘80s and into the early 1990s, the shortest pulses on record were generated using organic dye lasers and saturable absorber mode locking. Furthermore, the theory of saturable absorber mode locking provides important background for today’s solid-state lasers which are mode locked very successfully using artificial saturable absorbers implemented using nonlinear phase modulation effects. A full understanding of such mode locked solid-state lasers requires consideration not only of nonlinear amplitude modulation, but also of nonlinear phase modulation and dispersion effects, which we discuss subsequently. In the current section we concentrate on nonlinear amplitude modulation.

Our general model of saturable absorber mode locking is shown in Fig. 2.5. As usual, we will consider the changes encountered by a pulse during a single round trip through the laser cavity. For steady state mode locking, the original pulse should be reproduced after a single round trip.

The initial pulse envelope is written \( a(t) \) with Fourier transform \( A(\omega) \). We take the cavity into account by writing

\[
A' (\tilde{\omega}) = e^{-\tau_0} e^{-\tilde{\omega}^2/\omega^2} A (\tilde{\omega})
\]

where \( \tilde{\omega} = \omega - \omega_0 \), \( e^{-\tau_0} \) is the linear, time-independent cavity loss, and \( e^{-\tilde{\omega}^2/\omega^2} \) is a filtering term related to the finite bandwidth of the optical cavity. If there are no other bandwidth limiting elements, then the bandwidth may be determined by the gain bandwidth as previously. In many cases, however, an intracavity etalon or other filter may limit the cavity bandwidth. In general there is also a phase term which is linear in frequency. As we have seen, this linear phase term corresponds to a time delay \( T \), where \( T \) is the total linear cavity round trip time including the effects of the filter function. Although we have omitted writing the linear phase term here, one should remember that one round trip through the cavity leads to a delay \( T \).

To develop an analytic theory [20] [21], one assumes that the change wrought by each element (loss, gain, filter) per pass is small, and that the mode locked
spectrum is narrow compared to the filter bandwidth \( \omega_c \). Consequently eq. (2.33) can be approximated using first-order Taylor expansions as

\[
A'(\tilde{\omega}) \approx \left(1 - \ell_o - \frac{\tilde{\omega}^2}{\omega_c^2}\right) A(\tilde{\omega})
\] (2.34)

We now return to the time domain by Fourier transforming eq. (2.34). The result is

\[
a'(t) = \left(1 - \ell_o - \frac{1}{\omega_c^2} \frac{d^2}{dt^2}\right) a(t).
\] (2.35)

Now we can take into account the nonlinear loss and gain. The loss \( \ell(t) \) and [sometimes] the gain \( g(t) \) are taken to be time-dependent and controlled via the optical nonlinearity. \( \ell(t) \) refers to the nonlinear, time-dependent part of the loss only. Assuming again that the nonlinear loss and the gain per pass are small, we can write

\[
a''(t) = e^{-\ell(t)}e^{g(t)}a'(t) \approx \left(1 - \ell_o - \ell(t) + g(t) + \frac{1}{\omega_c^2} \frac{d^2}{dt^2}\right) a(t)
\] (2.36)

Both nonlinear loss and gain are taken to be frequency independent, since bandwidth limitations are already modeled as cavity effects.

We arrive at an equation for the mode locked pulse by requiring that \( a''(t) \) be equal to the initial pulse, possibly shifted in time by an amount \( \delta T \). Here \( \delta T \) is a time shift (not included in the linear cavity delay \( T \)) arising due to the nonlinear pulse shaping action of \( \ell(t) \) and \( g(t) \) and is usually small compared to the steady-state pulse width. \( \delta T > 0 \) corresponds to a shift forward in time. Mathematically this is expressed as

\[
a''(t) = a(t + \delta T) \approx a(t) + \frac{da}{dt} \delta T
\] (2.37)

Putting everything together, we arrive at

\[
\frac{1}{\omega_c^2} \frac{d^2a(t)}{dt^2} + [g(t) - \ell(t) - \ell_o] a(t) - \delta T \frac{da(t)}{dt} = 0.
\] (2.38)

In order to proceed further, we will need to specify the forms of \( g(t) \) and \( \ell(t) \). However, we do note that the second-derivative term represents pulse broadening due to the bandwidth-limiting filter, which must be compensated by pulse shortening arising from \( g(t) \) and \( \ell(t) \).

Equation (2.38) is one example of Haus’ “master equation” for mode locking [20] [21]. The main assumption of the master equation approach is that the per pass change wrought by each laser element is small, so that these changes can be expressed by the leading terms in the Taylor series expansions. It is also assumed that the order in which the laser elements are placed has no effect. The master equation approach provides analytical insight for a wide range of mode locking systems.
2.3.1 Saturation model

Absorber

The most common saturable absorber media which have been used for mode locking are organic dye solutions and semiconductors. These can be modeled as four levels systems (see also section 1.3), as shown in Fig. 2.6. The $1 \rightarrow 2$ transition is a resonant absorption for the laser radiation, and the absorption strength is proportional to the population densities $N_1 - N_2$ (where $N_j$ is the density in units of $m^{-3}$ of absorbers in level $j$). The total density of absorbers is $N_A$. The $2 \rightarrow 3$ and $4 \rightarrow 1$ relaxations are taken to be very fast. The $3 \rightarrow 4$ relaxation time is finite and is denoted $\tau_A$. We assume that the laser radiation does not interact with the $3 \rightarrow 4$ transition, which is red shifted compared to the $1 \rightarrow 2$ transition. We also assume that the absorption spectrum is homogeneously broadened and can be considered constant within the mode locked bandwidth. This assumption is equivalent to ignoring coherent laser-absorber interactions.

With these assumptions we can describe the absorber using a simple rate equation [20] [21]:

$$\frac{\partial N_1}{\partial t} = \frac{N_3}{\tau_A} - \frac{\sigma_A |a(t)|^2}{\hbar \omega_o A_A} (N_1 - N_2) \quad (2.39-a)$$

with

$$N_1 + N_3 = N_A \quad \text{and} \quad N_2 \approx N_4 \approx 0 \quad (2.39-b)$$

The first term on the right is relaxation out of level 3 and the second term represents stimulated absorption. The pulse is normalized so that $|a(t)|^2$ gives the time-dependent power carried by the pulse. $\sigma_A$ is the $1 \rightarrow 2$ absorption cross-section, $\hbar \omega_o$ is the photon energy, and $A_A$ is the beam cross-sectional area in the absorber. Using eq. (2.39-b) we can rewrite eq. (2.39-a) as

$$\frac{\partial N_1}{\partial t} = \frac{N - N_1}{\tau_A} - \frac{|a(t)|^2 N_1}{P_A \tau_A} \quad (2.40-a)$$

where

$$P_A = \frac{\hbar \omega_o A_A}{\sigma_A \tau_A} \quad (2.40-b)$$

is the absorber saturation power. Assuming small loss per pass, the time-dependent loss term $\ell(t)$ in eq. (2.38) is proportional to the ground-state absorber density $N_1$:

$$\ell(t) = \frac{\sigma_A}{2} N_1(t) l_a \quad (2.41)$$

where $l_a$ is the length of the absorber medium.

We now use eq. (2.40-a) to determine $N_1(t)$ in two important limiting cases. These cases are distinguished by the magnitude of the relaxation time $\tau_A$ compared to the mode locked pulse-width (which we designate $t_p$). For $\tau_A \ll t_p$ we speak of a fast saturable absorber. In the opposite case $t_p \ll \tau_A$, we speak of a slow saturable absorber.
Fast saturable absorber:

When $\tau_A \ll t_p$, we can set $\partial N_1 / \partial t = 0$ in eq. (2.40-a). We then solve easily for $N_1$, with the result:

$$N_1(t) = \frac{N}{1 + \frac{|a(t)|^2}{P_A}} \quad \text{(2.42)}$$

$N_1(t)$ and hence $\ell(t)$ vary instantaneously with the laser power $|a(t)|^2$. The absorption decreases with increasing laser power; hence the peak of a mode locked pulse will see lower loss than the wings of the pulse.

Slow saturable absorber:

When $t_p \ll \tau_A$, we can solve for the absorber dynamics during the mode locked pulse by setting $(N - N_1)/\tau_A \approx 0$. With this proviso eq. (2.40-a) becomes

$$\frac{\partial N_1}{\partial t} = -\frac{|a(t)|^2}{P_A \tau_A} N_1 \quad \text{(2.43)}$$

The solution is found by integration, with the result

$$N_1(t) = N_1^{(i)} e^{-\int^t dt |a(t)|^2 / P_A \tau_A} = N_1^{(i)} e^{-U(t)/U_A} \quad \text{(2.44)}$$

where

$$U(t) = \int^t dt |a(t)|^2 \quad \text{and} \quad U_A = P_A \tau_A \quad \text{(2.45)}$$

Here $N_1^{(i)}$ is the initial absorber population in level 1 just before the laser pulse, $U(t)$ is the pulse energy up to time $t$, and $U_A$ is the absorber saturation energy. $N_1(t)$ and $\ell(t)$ decrease monotonically during the pulse. The degree of saturation depends on cumulative pulse energy. Note that it is the saturation energy that is important in the slow absorber case, whereas for the fast absorber the saturation power is important.

After the end of the laser pulse the absorber population relaxes exponentially back to its equilibrium condition. Thus, after the laser pulse we have

$$N_1(t) = N_A + \left( N_1^{(i)} e^{-U/U_A} - N_A \right) e^{-t/\tau_A} \quad \text{(2.46)}$$

where $U$ is the total laser pulse energy, and the pulse is centered at $t = 0$.

Gain medium

We analyze the gain medium using a four-level model (see Fig. 1.4) similar to that for the absorber. The main differences are that the laser radiation is resonant with the $3 \rightarrow 4$ transition, and a pump $W$ drives the $1 \rightarrow 2$ transition out of the ground state. With similar assumptions as previously the gain medium rate equation is
\[
\frac{\partial N_3}{\partial t} = W (N_G - N_3) - \frac{N_3}{\tau_G} - \frac{|a(t)|^2}{P_G \tau_G} N_3
\]  
(2.47-a)

where
\[
P_G = \frac{\hbar \omega_e A_G}{\sigma_G \tau_G}
\]  
(2.47-b)

Here \(N_G\) is the total population density of ions (etc.) responsible for gain, and \(N_3\) is the population density in level 3, the upper laser level. The gain is given by \(g(t) = \sigma_G N_3(t) l_g / 2\), where \(l_g\) is the length of the gain medium.

Only the case of a slow saturable gain medium \((t_p \ll \tau_G)\) is of practical importance for mode locking dynamics. In this case the gain is written \(^1\)

\[
g(t) = g^{(i)} e^{-U(t) / U_G}
\]  
(2.48-a)

where
\[
U_G = P_G \tau_G
\]  
(2.48-b)

\(g^{(i)}\) is the gain just before the laser pulse. After the pulse the gain recovers exponentially to its small signal value \(g_o\), given by eq. (1.52):

\[
g(t) = \left( g^{(i)} e^{-U(t) / U_G} - g_o \right) e^{-t / \tau_G} + g_o
\]  
(2.49)

Note that dynamic gain saturation is not always significant. For media such as semiconductors or dye molecules where \(\tau_G\) is on the order of the pulse repetition time (typically nanoseconds), dynamic gain saturation per eq. (2.48-a) can play an important role in mode locking. However, when \(\tau_G\) is much larger than the pulse period (e.g., doped solid-state media like Ti:sapphire, Nd:YAG, or Er-doped glass fibers), dynamic (per pulse) gain saturation is very small, even though the gain medium does saturate in response to the average power. This is analogous to a resistor-capacitor low pass filter, which exhibits small response for drive frequencies exceeding \(1/RC\). In the laser gain medium, \(\tau_G\) plays the role of the \(RC\) time.

### 2.3.2 Slow saturable absorber mode locking

In the early 1970s Ippen and Shank demonstrated subpicosecond pulse generation from passively mode locked dye lasers using a nonlinear absorbing dye as the mode locking element [22] [23] [24]. In the early 1980s an improved passively mode locked dye laser in a ring cavity, which was termed the colliding pulse mode locked (CPM) ring dye laser, resulted in the first generation of sub-100-fs pulses [25] [26]. The CPM laser became the dominant tool for femtosecond pulse generation until 1990 when new solid state laser mode locking techniques were invented. A very interesting feature of such passively mode locked dye

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\(^1\)In addition to \(t_p \ll \tau_G\), we also assume \(t_p \ll W^{-1}\). This assumption is justified since according to eq. (1.52), the small signal gain saturates for \(W \tau_G > 1\). Therefore, for most passively mode locked lasers, there is usually no point to set the pump rate \((W)\) too far above \(W^{-1}\).
lasers is that the relaxation times of the amplifying and absorbing dyes were typically on the order of nanoseconds, much longer than the femtosecond pulse widths obtained. The following question then naturally arises: how is it that one can generate pulses a thousand times faster than the response time of the saturable dyes responsible for mode locking? In the following we outline a theory of slow saturable absorber mode locking put forth by Haus in the mid-1970s which explains this puzzle [21]. A related treatment is also given in [27].

The time-dependent net (or total) gain function $g_T(t) = g(t) - \ell(t) - \ell_o$ plays an important role in the theory of slow saturable absorber mode locking. $g_T(t)$ must be positive near the center of the pulse to provide amplification. Before and after the pulse $g_T(t)$ must be negative to suppress the wings of the pulse and provide pulse shortening. For steady-state mode locking, the pulse shortening per pass must balance pulse broadening, and the integrated gain must just balance the loss so that pulse energy is kept constant.

Figure 2.7 shows schematically the pulse shortening mechanism in mode locking with a slow saturable absorber. Before the pulse the loss exceeds the gain. As the pulse arrives, it saturates the absorber so that the loss line drops below the gain. At this point the pulse experiences net amplification. Later in the pulse the gain begins to saturate and as a result the gain drops below the loss. The combined saturation dynamics of $g(t)$ and $\ell(t)$ lead to a net gain window during the pulse, with loss before and after the pulse. Thus, the absorber suppresses the leading edge of the pulse, then saturates to pass the center and trailing edge. The gain amplifies the center of the pulse but then saturates to suppress the trailing edge.

To treat this process analytically, the results of the previous sections are used to describe the saturation dynamics. Our treatment closely follows [21]. The saturable loss $\ell(t)$ is written

$$\ell(t) = \ell^{(i)} e^{-U(t)/U_A} \approx \ell^{(i)} \left[ 1 - \frac{U(t)}{U_A} + \frac{1}{2} \frac{U^2(t)}{U_A^2} \right]$$

where we assume the pulse energy is small enough that a second order Taylor series expansion is adequate to describe the saturation dynamics. Following the pulse the nonlinear loss recovers exponentially with time constant $\tau_A$. The initial saturable loss $\ell^{(i)}$ follows the relation

$$\ell^{(i)} = \ell^{(i)}_{sat} + \left( \ell^{(i)} e^{-U(t)/U_A} - \ell^{(i)}_{sat} \right) e^{-t/T_A}$$

Here $\ell^{(i)}_{sat}$ is the small signal value of the nonlinear loss, $U$ is the total pulse energy, and $T$ is the round trip time. The initial nonlinear loss may be less than the small signal value if the absorber does not fully recover between pulses. In Haus’ theory, however, it is assumed that the saturable absorber recovers essentially fully, and therefore,

$$\ell^{(i)} \approx \ell^{(i)}_{sat}$$

This was a reasonable assumption for most subpicosecond dye lasers.
Similarly the gain is given by
\[ g(t) = g^{(i)}e^{-U(t)/U_G} \approx g^{(i)}\left[1 - \frac{U(t)}{U_G}\right] \] (2.53)

Here a first order Taylor series expansion is used, since according to Fig. 2.7, the gain saturates less deeply than the loss and since a first order expression for the gain is sufficient to describe a net gain window. The initial gain \( g^{(i)} \) is related to the small signal gain \( g_o \) by
\[ g^{(i)} = g_o + \left( g^{(i)}e^{-U/U_G} - g_o \right)e^{-T/\tau_G} \] (2.54)

We can make several important observations right away:

- First, the net gain \( g_T(t) \) must be less than zero both before and after the pulse for stability. Therefore,
\[ g^{(i)} < \ell_o + \ell^{(i)} \approx \ell_o + \ell^{(i)}_{sat} \] (2.55-a)
and
\[ g^{(i)}e^{-U/U_G} < \ell_o + \ell^{(i)}_{sat}e^{-U/U_A} \] (2.55-b)

- In order for the laser to self-start, the small signal gain must exceed the small signal loss.
\[ g_o > \ell_o + \ell^{(i)}_{sat} \] (2.56)

Therefore, by comparing with (2.55-a), we see that \( g^{(i)} < g_o \). This means that the gain must not recover completely between pulses.

- In order to achieve a net gain window, the absorber must saturate before the gain. This means that
\[ \frac{g^{(i)}}{U_G} < \frac{\ell^{(i)}_{sat}}{U_A} \] (2.57)

Provided that the gain and absorber cross-sections (\( \sigma_G \) and \( \sigma_A \)) are comparable, this condition can be achieved by focusing more tightly into the absorber.

Although satisfying all of these conditions simultaneously is possible, careful selection and matching of gain and absorber media are clearly required.

We now proceed to a more analytical treatment. The mode locking equation is written
\[ \frac{1}{\omega_0^2} \frac{d^2 a(t)}{dt^2} - \delta T \frac{da(t)}{dt} + g_T(t)a(t) = 0 \] (2.58)
where
\[ g_T(t) = g(t) - \ell(t) - \ell_0 \approx g^{(i)} \left( 1 - \frac{U(t)}{U_G} \right) - \ell^{(i)} \left( 1 - \frac{U(t)}{U_A} + \frac{1}{2} \frac{U^2(t)}{U_A^2} \right) - \ell_0 \]
(2.59)
and \( U(t) \) is the pulse energy up to time \( t \) as given by eq. (2.45). The net gain is plotted vs. energy \( U(t) \) in Fig. 2.8. With the approximations made, \( g_T \) is a quadratic function of \( U(t) \). This results in a net gain parabola which provides the pulse shortening necessary for mode locking. Note that the net gain parabola cannot be selected arbitrarily. The net gain is a function of the mode locked pulse, which is itself a function of the net gain function. One must solve for the net gain and the pulse shape and intensity self-consistently, unlike the case of active mode locking, where the modulation is driven under external control.

We can most easily solve eq. (2.58) in the wings of the pulse, where \( g_T \) becomes a constant with \( g_T < 0 \). The solution is an exponential of the form
\[ a(t) \sim e^{st}, \]
where
\[ s = \frac{\omega^2 \delta T}{2} \pm \omega \sqrt{\left( \frac{\omega \delta T}{2} \right)^2 - g_T} \]  
(2.60)
The plus sign in eq. (2.60) is taken for the leading edge of the pulse \( t < 0 \), and the minus sign is taken for the trailing edge. This ensures that the pulse decays exponentially. Note that since the values of \( g_T \) before and after the pulse need not be the same, the exponential decay constants also do not have to be the same. The exponential rather than Gaussian decay is one clear difference compared to active mode locking.

A symmetric solution of eq. (2.58) is given by [21]
\[ a(t) = \frac{a_o}{\cosh(t/t_p)} = a_o \sech(t/t_p). \]  
(2.61)
(Asymmetric solutions also exist but will not be discussed here). The time dependent energy \( U(t) \) becomes
\[ U(t) = \int^t |a(t)|^2 \, dt = \frac{U}{2} \left[ 1 + \tanh\left( \frac{t}{t_p} \right) \right] \]  
(2.62)
where \( U = 2a_o^2 t_p \) is the total pulse energy (we assume \( a_o \) is real). At this point there are four unknowns which must be solved for: the pulse width parameter \( t_p \), the pulse amplitude \( a_o \) (or equivalently the pulse energy \( U \)), the time shift \( \delta T \), and the initial gain \( g^{(i)} \). All other parameters in the equation are inputs. Substituting equations (2.61) and (2.62) into eq. (2.58) leads to three distinct sets of terms proportional to \( \sech(t/t_p) \), \( \sech(t/t_p) \tanh(t/t_p) \), and \( \sech^2(t/t_p) \), respectively. Requiring that each distinct set of terms sum to zero independently leads to three characteristic equations relating \( t_p, U, \delta T, \) and \( g^{(i)} \). The fourth
The most successful example of a laser mode locked using a slow saturable absorber was the colliding pulse mode locked (CPM) ring dye laser, sketched in Fig. 2.9 [25]. The gain is provided by a flowing stream of rhodamine 6G dye dissolved in a suitable solvent, pumped by several watts of light from a continuous-wave argon ion laser. The saturable absorber is a flowing stream of an absorbing dye solution known as DODCI. The ring geometry leads to an additional mode locking mechanism that improves the efficiency of the pulse shortening process in slow absorber passive mode locking. A ring resonator can support two pulses at the same time, one traveling clockwise and the other counterclockwise. It is most favorable energetically for these two pulses to meet, or collide, in the absorber stream. The standing wave interference pattern generated when the pulses overlap in the absorber minimizes the energy lost because the absorber saturation is greatest where the optical field is most intense and weakest at the nulls of the interference pattern, where there is no optical energy. In order to best utilize this effect, the nozzle used to produce the absorber stream is squeezed to yield a dye thickness below a few tens of μm (compared to several hundred μm for the gain stream), which matches the absorber thickness to the spatial extent of the pulse collision region. This colliding pulse geometry enhances the saturable absorber mechanism, leading to shorter pulses and increased stability. The arrangement of four Brewster-angle prisms (see chapter 4) is used to adjust the sign and magnitude of group velocity dispersion in the resonator. Optimization of the CPM laser eventually led to pulse durations as short as 27 fs; in this mode of operation, intracavity pulse compression due to dispersion acting in concert with self-phase modulation in the dye appeared to supplement the pulse shortening mechanisms caused by saturation [26]. A master equation analysis of passive mode locking including slow saturation as well as both group velocity dispersion and energy and intensity dependent self-phase modulation in the dye streams was published in [28]. However, we will defer treatment of such phase terms until we discuss solid-state laser mode locking, where these phase effects are much more prominent.

### 2.3.3 Fast saturable absorber mode locking

Fast saturable absorber mode locking using real absorbers did not achieve the wide spread use that was enjoyed by slow saturable absorber mode locked systems. One reason for this may be that it is hard to find real absorbers whose relaxation is fast compared to femtosecond pulses! Fast saturable absorbers have been mainly applied for picosecond pulse generation with relatively narrow-band solid state laser systems, either (a) by using fast absorber dyes for simultaneous passive mode locking and Q-switching [29] or (b) by using semiconductor saturable absorbers for steady state mode locking [30]. The theory presented in this section provides excellent background for section 2.4, which covers today’s leading mode locking technology based on passive mode locking of broadband solid state lasers using artificial fast saturable absorbers implemented using the
optical Kerr effect.

In order to analyze fast saturable absorber mode locking, we proceed from eq. (2.38). For more details, see [20]. Since this technique is usually applied to solid-state lasers with low gain cross-sections and long relaxation times ($\tau_g$ typically on the order of microseconds to milliseconds), dynamic gain saturation during the pulse is very small. Therefore, we replace $g(t)$ with a constant value for the saturated gain, $g_s$, which is a function of the small signal gain $g_0$ and the time-average laser power. The time-dependent loss $\ell(t)$ is given by

$$\ell(t) = \ell^{(i)} \left[ 1 - \frac{|a(t)|^2}{P_A} \right]$$

where $\ell^{(i)} = \ell^{(i)}$ is the small-signal value of the saturable loss, which is expanded to first order in the power $|a(t)|^2$. This is justified if the mode locked power remains sufficiently below the saturation power $P_A$. One difference of the treatment here compared to that in [20] is that we assume that the gain is frequency independent, while [20] assumes that the finite gain bandwidth plays the main role in limiting the bandwidth.

The resulting mode locking equation is as follows:

$$\left( \frac{1}{\omega_c^2} \frac{d^2}{dt^2} - \delta T \frac{d}{dt} + \left( g - \ell_{o} - \ell^{(i)} \right) + \ell^{(i)} \frac{|a(t)|^2}{P_A} \right) a(t) = 0$$

The last term proportional to $|a(t)|^2 a(t)$ is referred to as self-amplitude-modulation (SAM). In the wings of the pulse when $|a(t)|^2$ is very small, this SAM term is nearly zero, and the pulse decays exponentially, as in the previous section. Also, as before, a sech hyperbolic pulse is a solution to the mode locking equation, with

$$a(t) = a_o \text{sech}(t/t_p)$$

Substituting (2.65) into eq. (2.64) and requiring that the coefficients of the resulting $\text{sech}(t/t_p)$, $\text{sech}(t/t_p) \text{tanh}(t/t_p)$, and $\text{sech}^2(t/t_p)$ terms independently sum to zero yields three characteristic equations:

$$\frac{1}{\omega_c^2 t_p^2} + g - \ell_{o} - \ell^{(i)} = 0$$

$$\frac{-2}{\omega_c^2 t_p^2} + \ell^{(i)} a_o^2 = 0$$

$$\delta T = 0$$

Equation (2.66-c) says that there is no time-shift arising from the fast SAM process. In order to satisfy eq. (2.66-a), we see that the net gain before and after the pulse must be less than zero. This is in fact a stability condition, since if the net gain were positive before or after the pulse, perturbations before or
after the pulse would grow in amplitude. Figure 2.10 shows the net gain curve corresponding to these observations.

From eq. (2.66-b) we see that the peak mode locked power is inversely proportional to the pulse width squared. The pulse energy \(2a_o^2 t_p\) is inversely proportional to \(t_p\). To actually determine the pulse width and power, eq. (2.66-b) is substituted in eq. (2.66-a), with the result

\[
\ell_o + \ell^{(i)} - g = \frac{\ell^{(i)} a_o^2}{2P_A},
\]

(2.67)

Finally, by using the gain saturation equation

\[
g = \frac{g_o}{1 + \frac{\langle P \rangle}{P_o}}
\]

(2.68)

we obtain

\[
\ell_o + \ell^{(i)} - \frac{g_o}{1 + \left(\frac{\langle P \rangle}{P_o}\right)^{1/2}} \frac{2a_o^2}{\omega_0 P_o T} = \frac{\ell^{(i)} a_o^2}{2P_A}
\]

(2.69)

where we have used \(\langle P \rangle = 2a_o^2 T_p / T\) for the average laser power. A graphical solution to (2.69) is sketched in Fig. 2.11. The figure assumes that \(g_o > \ell_o + \ell^{(i)}\), which is required for the laser to self-start even in continuous-wave operation. The saturated gain still satisfies \(g < \ell_o + \ell^{(i)}\) as required. Depending on the parameter values, eq. (2.69) has either zero or two solutions. No solutions are found when \(g_o\) is too large (for fixed \(\ell_o, \ell^{(i)}, \) etc.). In this case even though there is no single-pulse mode locked solution, nevertheless solutions with two or more separated pulses coexisting within a single round trip time are still possible. When \(g_o\) is reduced, then two distinct solutions to eq. (2.69) are found, as indicated in the figure. Of the two solutions, the solution closer to the origin corresponds to the lower mode locked power and hence the longer pulse. Haus has shown that only this longer pulse solution is stable.

Finally, we note the strength of the SAM term responsible for pulse shortening and mode locking is proportional to the pulse power \(|a(t)|^2\). As long as the pulse energy remains roughly constant, we can conclude that the strength of the SAM is inversely proportional to pulse width. As pulses get shorter and shorter during start up of mode locked operation, the pulse shortening becomes more and more effective. As a result, fast absorber mode locking can in principle support extremely short pulses. By the same token, when the pulses are still very long, the mode locking process is very weak. This means that initiating mode locking may be difficult. This corresponds to the experimental observation that passively mode locked solid-state lasers are often not self-starting.
2.4 Solid-state laser mode locking using the optical Kerr effect

Since the early 1990s there has been a rapid shift away from passively mode locked dye lasers and towards passively mode locked solid-state laser technology. Today such lasers produce femtosecond pulses at a variety of wavelengths, with pulse as short as about 5-6 fs reported [31] [32], and with average powers up to several watts (compared to at most tens of milliwatts from mode locked dye lasers). This solid-state laser mode locking is usually achieved using artificial saturable absorbers based on the nonlinear refractive index, also known as the optical Kerr effect. In addition to fast amplitude modulation, phase effects are very important in such lasers. In the following, we first briefly introduce the nonlinear refractive index and the pertinent new effects. We then discuss in some depth specific examples of solid-state laser geometries where the Kerr effect is converted into a fast amplitude modulation. At this point we write down the mode locking equation and discuss important features of its solution. Finally, we comment briefly regarding self-starting of solid-state mode locked lasers.

2.4.1 Nonlinear refractive index changes

The nonlinear refractive index is usually written

\[ n = n_0 + n_2 I(t) \]  

(2.70)

where \( I(t) \) is the pulse intensity, given by \( I(t) = |a(t)|^2 / A_{\text{eff}} \) normalized to give the time dependent power associated with the pulse as before, and \( A_{\text{eff}} \) is the effective area of the beam in the Kerr medium. Thus, the pulse induces a change in refractive index proportional to the instantaneous intensity. In most laser materials \( n_2 > 0 \) and therefore the laser pulse momentarily increases the refractive index. This is valid for nonresonant nonlinear index changes associated with intensity dependent distortions of the electron clouds [electronic wave functions] in the material. The response time of the nonresonant optical Kerr effect is not known exactly but is usually estimated to be 1-2 femtoseconds. The optical Kerr effect is allowed for gases, liquids, and solids regardless of their symmetry. The size of the nonlinear refractive index \( (n_2) \) is material dependent, but a typical (perhaps somewhat small) value is that of fused silica, \( n_2 = 3 \times 10^{-16} \text{cm}^2/\text{W} \) [33]. Ultrafast nonlinear effects arising due to the optical Kerr effect will be discussed in detail in a chapter 6.

The nonlinear phase shift observed after propagating in a nonlinear medium of length \( L \) is given by

\[ \Delta \phi(t) = -\frac{\omega}{c} n_2 I(t) L = -\frac{2\pi}{\lambda} n_2 I(t) L \]  

(2.71)

For mode locking, this fast time-dependent phase modulation must somehow be converted into a fast amplitude modulation in order to stabilize the wings of
the mode locked pulse. There are in fact numerous schemes for achieving this self-amplitude-modulation. Some of the more popular schemes include Kerr lens mode locking (KLM) in which the nonlinear index leads to self-focusing in the laser cavity, additive-pulse-mode locking (APM) in which the nonlinear index modifies the interference between two coupled laser cavities, and nonlinear polarization rotation. This list is by no means exhaustive; a number of other schemes have also been demonstrated. In addition to self-amplitude-modulation, which is similar to the fast saturable absorber action discussed in 2.3.3, the nonlinear refractive index also leads to new effects due to self-phase-modulation. The interplay of self-phase-modulation with group velocity dispersion is also important. The influence of these effects on laser operation will be discussed initially in this chapter and then in more detail in chapter 7.

2.4.2 Self-amplitude modulation, self-phase modulation, and group velocity dispersion

Self-amplitude modulation

Self-amplitude modulation (SAM), similar to the fast saturable absorption studied previously, can be modeled by writing the loss terms in the mode locking equation as

\[ \ell_o + \ell(t) \approx \ell - \gamma |a(t)|^2 \]  \hspace{1cm} (2.72)

Here we have lumped both the constant cavity loss \( \ell_o \) as well as the small signal value of the nonlinear loss (this would be \( \ell^{(i)} = \ell^{(n)}_s \) in the fast saturable absorber theory) into the loss constant \( \ell \), and \( \gamma \) (equal to \( \ell^{(i)}/P_A \)) is the SAM coefficient. Since \( |a(t)|^2 \) is normalized to give the time-varying power \( P(t) \) carried by the laser beam, i.e., \( P(t) = |a(t)|^2 \), \( \gamma \) is the derivative of the field loss coefficient with respect to power:

\[ \gamma = \frac{\partial \ell}{\partial P} \]  \hspace{1cm} (2.73)

In terms of the loss coefficient for the power, \( \mathcal{L} \), which for small loss is given by \( \mathcal{L} \approx 2\ell \), one has

\[ \gamma = \frac{1}{2} \frac{\partial \mathcal{L}}{\partial P} \]  \hspace{1cm} (2.74)

Calculating \( \gamma \) requires a calculation specific to the particular laser system being considered and can usually be controlled over some range by adjusting the laser parameters. Here we will use \( \gamma \) to characterize the strength of the SAM without specifying a particular laser system.

Self-phase modulation

The SAM is based on a time-varying nonlinear phase shift which itself modifies the pulse. This is called self-phase modulation (SPM). Assuming as usual that
the SPM per pass is small, we can account for this effect by writing

$$a'(t) = e^{j\Delta \phi(t)} a(t) \approx \left[ 1 + j \Delta \phi(t) \right] a(t) \quad (2.75)$$

where $a(t)$ and $a'(t)$ represent the pulse envelope before and after the nonlinear refractive index medium. From eqn. (2.71) the nonlinear phase shift is proportional to the intensity. We introduce a SPM coefficient $\delta$ where $\Delta \phi(t) \approx -\delta |a(t)|^2$ with $\delta > 0$ for positive nonlinear index media ($n_2 > 0$). We can then write

$$a'(t) \approx \left[ 1 - j \delta |a(t)|^2 \right] a(t) \quad (2.76)$$

The principal effects of SPM can be understood by recognizing that the instantaneous frequency $\omega_{\text{inst}}$ of the pulse is simply the time derivative of the total phase:

$$\omega_{\text{inst}} = \frac{d}{dt} (\omega_0 + \Delta \phi(t)) = \omega_0 + \frac{d\Delta \phi(t)}{dt} \quad (2.77)$$

Thus, the time-varying nonlinear phase shift gives rise to time-varying frequency shifts away from the initial carrier frequency. For a medium with $n_2 > 0$, SPM gives rise to red shifts (lower frequency, longer wavelengths) on the front edge of the pulse and blue shifts (higher frequencies, shorter wavelengths) on the trailing edge. This variation of instantaneous frequency as a function of time is called a chirp - in this case an up-chirp since the frequency increases with time. SPM creates new frequency components which can contribute to the mode locking process. It turns out that the spectral broadening is small compared to the input bandwidth for $\Delta \phi_{\text{max}} \ll \pi$. For $\Delta \phi_{\text{max}} \geq \pi$, the output spectrum is roughly $\Delta \phi_{\text{max}}/\pi$ times wider than the input bandwidth [34] [33]. SPM will be discussed in more detail in chapter 6.

**Group velocity dispersion**

We have also noted that group velocity dispersion (GVD) is important in femtosecond solid state lasers (because the path lengths in material are relatively large and the pulses are very short). Dispersion refers to the dependence of group velocity on frequency. In the case of nonzero GVD, different frequencies have slightly different cavity round trip times, and this can be an important pulse broadening mechanism. Mathematically, we can treat dispersion by considering the frequency dependence of the propagation constant $\beta$ (also called $k$ in chapter 1). Using a second order Taylor Series expansion, we write

$$\beta(\omega) = \beta(\omega_0) + \beta'(\omega_0) (\omega - \omega_0) + \frac{\beta''}{2} (\omega - \omega_0)^2 \quad (2.78)$$

Here $\beta' = \frac{\partial \beta}{\partial \omega}$ and $\beta'' = \frac{\partial^2 \beta}{\partial \omega^2}$, with both derivatives evaluated at $\omega = \omega_0$. We account for the dispersion in a medium of length $L$ by writing

$$A''(\tilde{\omega}) = e^{-j \beta' L} e^{-j \beta L} e^{-j \frac{\beta''}{2} L} A(\tilde{\omega}) \quad (2.79)$$
where $A(\tilde{\omega})$ and $A''(\tilde{\omega})$ are the spectrum before and after the dispersive medium, respectively, and $\tilde{\omega} = \omega - \omega_0$. For our purposes we can ignore the first two factors, since $e^{-j\beta' L}$ represents a constant phase shift and $e^{-j\beta'' \tilde{\omega} L}$ represents a constant delay. We identify $\beta'$ as the inverse of the group velocity. Assuming the dispersion per pass is small, we obtain

$$A''(\tilde{\omega}) \approx \left(1 - j \frac{\beta''}{2} \tilde{\omega}^2 L\right) A(\tilde{\omega}) \quad (2.80)$$

Taking the Fourier transform yields

$$a''(t) \approx \left(1 + jD \frac{d^2}{dt^2}\right) a(t) \quad (2.81-a)$$

where

$$D = \frac{\beta'' L}{2} \quad (2.81-b)$$

The physical meaning of these equations is easily identified. We know that the Fourier transform of $a(t - \tau)$ is $A(\tilde{\omega}) e^{-j\tilde{\omega} \tau}$. In the case of a spectrum with an arbitrary spectral phase variation, written $A(\tilde{\omega}) = |A(\tilde{\omega})| e^{j\phi(\tilde{\omega})}$, we can define a frequency-dependent delay $\tau(\omega)$ given by

$$\tau(\omega) = -\frac{\partial \phi(\omega)}{\partial \omega} \quad (2.82)$$

From eqn. (2.79), we find that

$$\tau(\omega) = \beta' L + \beta'' \tilde{\omega} L \quad (2.83)$$

Thus, for $\beta'' > 0$ (called "normal" dispersion), higher frequencies (shorter wavelengths) travel slower and are displaced towards the trailing edge of the pulse. This leads to an up-chirp. For $\beta'' < 0$ ("anomalous dispersion"), higher frequencies move faster and are displaced to the leading edge of the pulse, leading to a down-chirp.

We will discuss dispersion in more detail in chapter 4.

### 2.4.3 Additive pulse mode locking (APM)

The first widely exploited method for producing a self-amplitude-modulation via the optical Kerr effect used the nonlinear phase shift to modify the interference between two coupled laser cavities. This method is called additive pulse mode locking (APM) or coupled cavity mode locking. A schematic set up is shown in Fig. 2.12. The gain medium sits in the main cavity, which is coupled to an auxiliary cavity through a partially reflective mirror. The lengths of the two cavities are very close to equal. A pulse traveling to the right in the main cavity is partially reflected by the coupling mirror. Part of the pulse is transmitted into the auxiliary cavity. Since the length of the two cavities are nearly equal, the pulses transmitted from the auxiliary cavity back into the main resonator arrive in synchronism and interfere with the main cavity pulses reflected from
the coupling mirror. The overall reflectivity seen by the main cavity depends on
the phase between these two sets of pulses. The highest reflectivity occurs when
the round trip phases of the two cavities are identical (modulo $2\pi$). In order to
produce self-amplitude-modulation, a medium with nonlinear refractive index
is inserted into the auxiliary cavity. If the two cavities are initially out of phase
and the nonlinear phase shift brings them closer to in phase, then the overall
reflectivity seen by the main cavity goes up. This results in a favorable self
amplitude modulation, which gives the highest reflectivity for the peak of the
pulse. Of course, if the initial phase mismatch is such that the nonlinear phase
shift makes the cavities more out of phase, then the reflectivity will decrease
with intensity, and mode locking will be suppressed. For this reason the relative
cavity lengths in APM lasers should be interferometrically stabilized.

A forerunner of APM was the "soliton laser," which also used a nonlinear
coupled cavity principle to shorten the pulses from a synchronously pumped
color-center laser [35]. However, a significant difference compared to the APM
principle is that the fiber in the auxiliary cavity was sufficiently long that both
dispersion and self-phase-modulation (SPM) were important. In fact, these
effects interacted to form solitons, and soliton propagation was thought to be
the main pulse shortening mechanism for this laser. Although this was correct
for the soliton laser as originally implemented, several papers in the late 1980s
showed that neither soliton formation nor substantial dispersion were required
to obtain effective passive mode locking in coupled cavity systems [36][37] [38]
[39] [40]. The understanding that nonlinear coupled cavities provided a general
mechanism for achieving an artificial fast saturable absorber without the need
for soliton formation led within a few years to the application of APM in a
number of different laser systems, e.g., [41] [42] [43] [44] [45] [46].

In one of the earliest clear examples of APM [39], a color center laser with
a broad gain bandwidth around 1.5 $\mu$m was modified to include an auxiliary
cavity. The main cavity was synchronously pumped by 100-ps pulses from
an actively mode locked Nd:YAG laser. With the main cavity round trip time
adjusted to match the pump laser periodicity, the synchronous pumping was
sufficient to generate 23 psec pulses. With the addition of the auxiliary cavity
containing a short (40 cm) fiber as the nonlinear index medium, the laser pro-
duced pulses as short as 127 fsec via APM. Figure 2.13 shows data examining
the APM laser operation as the auxiliary cavity length was slowly ramped to
bring the auxiliary cavity repeatedly in and out of phase with the main cav-
ity [39]. For the laser configuration studied, the power in the main cavity and
the output power were at a maximum and at a minimum, respectively, when
the two cavities were in phase. These quantities vary periodically, repeating
when the relative phases of the two cavities change by $2\pi$. The curve marked
SHG indicates the power in the second harmonic beam generated by focusing
the mode locked pulse onto an appropriate nonlinear crystal. Since the average
SHG power is inversely proportional to the pulse width, the periods when the
SHG is high indicate femtosecond pulse operation. The important finding is
that the laser mode locks only for certain relative cavity phases such that the
nonlinear phase shift reduces the cavity loss.
The operation of an APM system can be understood by analyzing the reflectivity seen by the main cavity, taking the nonlinear auxiliary cavity into account. Following [39] [40] [47], we start with the scattering matrix [6] of the mirror separating the main and external cavities. This matrix relates the complex amplitudes of the waves $a_i$ incident on the mirror and the waves $b_i$ reflected from the mirror as follows:

$$
\begin{pmatrix}
  b_1 \\
  b_2 
\end{pmatrix} = \begin{pmatrix}
  r & \sqrt{1-r^2} \\
  \sqrt{1-r^2} & -r 
\end{pmatrix} \begin{pmatrix}
  a_1 \\
  a_2 
\end{pmatrix}
$$

(2.84)

Here $i = 1$ and $i = 2$ refer to waves in the main cavity (left side of the mirror) and the auxiliary cavity (right side of the mirror), respectively. The mirror has field reflectivity $r$, which may be either positive or negative. \(^2\) $a_2$ and $b_2$ are also related, since $a_2$ results simply from $b_2$ after one transit through the auxiliary cavity. Therefore

$$
a_2 = \eta b_2 e^{-j\phi}
$$

(2.85)

where $\eta$ represents the amplitude transmission factor through the auxiliary cavity and $\phi$ is the phase shift in the auxiliary cavity relative to that of the main cavity. The phase $\phi$ is given by $\phi = \phi_b + \Phi$, where $\phi_b$ is the phase bias of the auxiliary cavity ($\phi_b > 0$ when the auxiliary cavity is longer than the main cavity) and $\Phi$ is the nonlinear phase shift due to the double pass through the fiber, which can be written

$$
\phi = \frac{4\pi n_2 |a_f|^2 L_f}{\lambda A_{eff}}
$$

(2.86)

where $n_2$, $L_f$, and $A_{eff}$ are the Kerr coefficient, length, and effective area of the fiber, respectively, and $a_f$ is the normalized amplitude of the pulse in the fiber. It is worth noting that although this formalism is most obviously applicable to continuous-wave light, it also works for mode locked pulses, due to the assumption that the round trip times in the main and external cavities are equal to within a small phase difference. The composite reflection coefficient $\Gamma = b_1/a_1$, seen by the main cavity can be obtained exactly by combining equations (2.84) and (2.85), which yields

$$
\Gamma = \frac{b_1}{a_1} = \frac{r + \eta e^{-j\phi}}{1 + \eta e^{-j\phi}}
$$

(2.87)

In the usual case where an optical fiber is responsible for the nonlinearity, the transmission through the auxiliary cavity is typically small due to coupling losses.

\(^2\)The definition of the scattering matrix requires that $|a_i|^2$ and $|b_i|^2$ represent intensity, so the $a_i$ and $b_i$ actually represent the field amplitudes in medium $i$ times the square root of the refractive index in that medium. Furthermore, the specific form of the scattering matrix given here applies only for certain choices of the reference planes where the $a_i$ and $b_i$ are measured. Transformations of the scattering matrix as the reference planes are translated with respect to the mirror are discussed in [6]. However, the results given below for the SPM and SPM coefficients are independent of the choice of reference planes.
into the fiber ($\eta \ll 1$). In this limit $\Gamma$ can be expanded to first order in $\eta$, which gives

$$\Gamma \approx r + \eta(1 - r^2)e^{-j\phi}$$ \hspace{1cm} (2.88)

This is equivalent to considering only a single pass through the external cavity. If we further assume a small nonlinear phase shift ($\Phi \ll 1$), this can be rewritten as

$$\Gamma \approx r + \eta(1 - r^2)e^{-j\phi_b}(1 - j\Phi)$$ \hspace{1cm} (2.89)

Positive self-amplitude modulation occurs when the nonlinear phase shift brings the second term in eqn. (2.89) closer to zero phase, since this results in the greatest reflectivity for the main cavity. Since the Kerr coefficient $n_2$ is positive, mode locking requires $\phi_b < 0$, i.e., the auxiliary cavity shorter than the main cavity. This is consistent with the experimental observations of [39] in Fig. 2.13.

Using eqn. (2.89) we can determine the SAM and SPM coefficients. If we identify the field amplitude $a_1$ in the main cavity as the quantity of interest for the mode locking equation and assume that $|a_1| = \eta\sqrt{1 - r^2} |a_1|$, which is appropriate if coupling into the fiber dominates the loss in the external cavity, then the SAM coefficient is

$$\gamma = \frac{-4\pi \eta^3 (1 - r^2)^2 n_2 L_f}{\lambda A_{eff}} \sin \phi_b$$ \hspace{1cm} (2.90)

$\gamma$ is maximized by setting $\phi_b = -\pi/2$. For $\phi_b = 0$ or $\pi$ (linear reflectivity either a maximum or minimum, respectively), the SAM coefficient is zero. The SPM coefficient is given by

$$\delta = \frac{4\pi \eta^3 (1 - r^2)^2 n_2 L_f}{\lambda A_{eff}} \cos \phi_b$$ \hspace{1cm} (2.91)

Except for special settings of $\phi_b$, both the SAM and SPM coefficients are nonzero. Furthermore, setting the phase bias of the auxiliary cavity allows the relative strengths of $\gamma$ and $\delta$ to be varied.

It is worth noting that large nonlinear phase shifts are not unusual in APM systems. As a simple estimate, consider a laser at a 1.5 $\mu$m wavelength with a steady state pulse width of 125 fs, a 100 MHz repetition rate, and 5 mW of average power coupled into the fiber (considering typical average powers of several hundred milliwatts to several watts for solid state lasers, 5 mW in the fiber is rather conservative!). If we further assume a 80 cm round trip distance in the fiber and a fiber effective area of 80 $\mu$m$^2$, then we estimate a nonlinear phase shift of roughly 0.5 rad. When large nonlinear phase shifts $\Phi$ occur, the assumption of a SAM simply proportional to the intensity is no longer valid, since from eqn. (2.88) the composite reflectivity $\Gamma$ varies sinusoidally with phase shift. Since the peak intensity and therefore the nonlinear phase shift increase with decreasing pulse width, the nonmonotonic variation of $\Gamma$ with intensity can lead to a saturation of the pulse shortening process, as can dispersion effects.
(not included above), which also become more important for shorter pulses. Nevertheless, this can generally only occur when the laser is already successful at generating pulses which are very short!

At present lasers mode locked via APM using nonlinear coupled cavities are not very common, due to the need for interferometric stabilization and the development of other successful implementations of fast absorber solid-state laser mode locking. However, several schemes widely used for mode locking of fiber lasers are closely related to APM. Although these schemes do not use nonlinear coupled cavities directly, like APM they obtain self-amplitude modulation through the effect of nonlinear phase shifts on interfering waves. Examples include nonlinear polarization rotation, where the nonlinear phase shifts change the coherent summation of the two polarization modes in a fiber, and the nonlinear optical loop mirror, where the interference takes place between counter-propagating waves in a single loop of fiber. Both these implementations have the advantage of a stable geometry where active interferometric stabilization is not necessary. More detail on such mode locked fiber lasers is given in chapter 7.

2.4.4 Kerr lens mode locking (KLM)

In current practice the most important method for pulse generation from broadband solid-state lasers is usually called Kerr lens mode locking (KLM). KLM lasers based on titanium:sapphire (Ti:S) are particularly widespread. Sub-100 fsec pulse generation from Ti:S lasers was first reported by Spence in 1991 [48]. Initially these lasers were described simply as "self mode locked," since the mode locking mechanism was not immediately identified. It was subsequently determined that a nonlinear lensing (or "Kerr lensing") effect associated with the nonlinear refractive index provides a fast self-amplitude-modulation useful for mode locking [49] [50]. This Kerr lensing effect is important in most of today’s shortest pulse Ti:S lasers emitting pulses in the sub-10 fsec range and has also been applied for mode locking of a number of other solid-state laser materials.

General description

The four-mirror laser cavity used by Spence et al is shown in Fig. 2.14(a). Gain is provided by a 2-cm long Ti:S crystal pumped through a dichroic mirror by an argon ion laser. Ti:sapphire’s gain spectrum is extremely broadband, with lasing possible from below 700 nm to longer than 1 µm (the lasing wavelength was around 880 nm in [48]). In some experiments a single-plate birefringent filter was used to select the center wavelength. In other experiments the birefringent filter was removed, and instead a prism pair (see chapter 4) was used to compensate for the group velocity dispersion of the Ti:sapphire rod. In the case of the prism pairs, wavelength tuning could be accomplished using a vertical slit place in front of mirror M3 where the various optical frequencies are slightly displaced spatially. When the laser was properly aligned for a single-transverse mode (TEM00) and lowest threshold continuous-wave operation, no mode locking
occurred. The remarkable observation was that when this laser was misaligned for a mixture of \( \text{TEM}_{00} \) and a higher-order transverse mode such as \( \text{TEM}_{01} \) (corresponding to a higher threshold), mode locking could be induced by a suitable external perturbation, such as tapping one of the cavity mirrors. Although the mode locking was not self-starting, once mode locking was initiated, it was self-sustaining. The cavity containing the birefringent filter and no prisms produced 2-psec pulses which were highly chirped, with an optical bandwidth greatly exceeding that required to support a 2-psec pulse. The cavity containing the prisms generated pulses as short as 60-fsec, with no appreciable chirp. The dramatic improvement seen when using prisms to compensate dispersion clearly indicates the importance of phase effects in such femtosecond lasers, as we shall discuss presently.

One arrangement for Ti:S lasers generating pulses in the range 10 fs and below is shown in Fig. 2.14(b) [51]. The layout is similar to that of the original KLM laser geometry in Fig. 2.14(a), with the differences that an "X" rather than a "Z" arrangement for the spherical mirrors is usually used and the placement of the mirrors and gain rod are optimized. Another important difference [52] [53] [54] [55] [56] [31] [32] is that it is essential to minimize the intracavity dispersion over a very broad bandwidth, including higher order dispersion terms, such as those where phase is proportional to \((\omega - \omega_0)^3\), which have not been included in our simple treatment so far. This is achieved in part by using shorter, more highly doped gain crystals. Furthermore, the intracavity dispersion compensation elements must be optimized. This can be realized by carefully choosing the prism material, and by replacing the prisms with specially designed chirped, dispersion compensating mirrors, or by using a combination of prisms and chirped mirrors. Chirped mirrors also have the advantage of a very broad reflectivity bandwidth, which is an important issue in sub-10 fs laser design. A detailed treatment of dispersion and dispersion compensation will be the subject of chapter 4.

To explain the nonlinear lensing responsible for the self mode locking observed in the very simple laser cavities of Fig. 2.14, we observe that the mode locked pulse is a function of both time and space. Assuming the time and space coordinates are independent, the field can be written most simply as a circular Gaussian beam as per eq. (1.59). The intensity \( I(r,t) \) is given by

\[
I(r,t) = \frac{2}{\pi w^2} |a(t)|^2 e^{-2r^2/w^2(z)} = \frac{2}{\pi w^2} P(t)e^{-2r^2/w^2(z)} \tag{2.92}
\]

where \( w(z) \) is the beam radius which varies along the longitudinal cavity coordinate \( z \), \( r \) is the normal radial distance in cylindrical coordinates, and the pulse shape \( a(t) \) is normalized such that \( |a(t)|^2 \) gives the instantaneous power \( P(t) \), as usual. It is clear that the intensity and hence the nonlinear index change induced in the Ti:sapphire crystal vary with \( r \). The largest index change occurs at the center of the beam. This radially varying index (hence phase) change is equivalent to a self-induced lens which can modify the beam radius \( w(z) \) of the lasing mode. Furthermore, the strength of the self-induced lensing effect is time-dependent and follows the pulse temporal intensity profile. Under
suitable conditions this lensing effect can reduce the loss or increase the gain with increasing power. When this occurs, the Kerr lens acts as a fast saturable absorber.

There are several ways in which a nonlinear change in mode structure or beam radius can reduce the loss. For example:

- The misaligned laser containing higher-order transverse mode content such as $TEM_{06}$ may experience loss due to beam clipping on the edge of one of the end mirrors. Self-lensing may improve the mode quality or decrease the beam size on the end mirrors, hence reducing loss.

- A similar effect may arise with respect to any other optical element in the cavity. For example, the beam may be clipped by the apex of one of the dispersion compensating prisms. In one study [50] a self mode locked Ti:S laser was aligned so that light leaking past one of the prisms accounted for a 2% loss in continuous-wave operation. After mode locking was initiated, the measured prism loss was reduced ten-fold. Measurements showing a substantial change in beam profile with mode locking were also performed and are reproduced in Fig. 2.15 [50]. Data of this sort give direct evidence of the importance of nonlinear mode profile changes in the mode locking process.

- To accentuate and control the beam clipping effects, an appropriately sized slit may be placed near one of the end mirrors. This can increase the beam clipping effects and the size of the loss modulation resulting when the mode size is modified through self lensing. This approach is termed hard aperture-KLM.

- Self-lensing and self-focusing may affect the overlap of the lasing mode with the spatial profile of the pump laser beam in the Ti:S crystal. If spatial overlap improves with power, the net gain can be increased, creating a situation similar to a fast saturable absorber [57]. This approach is termed soft-aperture KLM.

It is important to note that depending on the cavity configuration, the nonlinear lens can either decrease or increase the net loss. When the loss is decreased, mode locking is possible; when the loss is increased, mode locking is suppressed.

**Properties of KLM resonators and the SAM coefficient**

We have seen that the SAM and SPM coefficients are important for modeling the laser behavior. In the following we outline how these are calculated, given knowledge of the (power dependent) spatial mode structure for a specific laser configuration.

For hard aperture KLM, the fraction $T$ of the power transmitted through an aperture with power transmission function $S(x,y)$ is given by [58]
\[ T = \frac{2}{\pi w^2} \int_0^\infty \int_0^\infty dx dy \ S(x,y)e^{-2(x^2+y^2)/w^2} \quad (2.93) \]

where a circular Gaussian beam is assumed and \( w \) is the beam radius at the aperture. For an aperture with circular symmetry, this can be written

\[ T = \frac{2}{\pi w^2} \int_0^\infty 2\pi r \, dr \ S(r)e^{-2r^2/w^2} \quad (2.94) \]

The loss for the power is given by \( \mathcal{L} = 1 - T \).

In the case of a circular aperture which transmits only within a radius \( w_s \), we get

\[ T = \frac{2\pi}{w^2} \int_0^{w_s} 2\pi r \, dr \ e^{-2r^2/w^2} = 1 - e^{-2w_s^2/w^2} \quad (2.95) \]

and

\[ \mathcal{L} = e^{-2w_s^2/w^2} \quad (2.96) \]

To calculate the linear loss, one uses this formula with \( w \) given by its value at low power. The SAM coefficient is given by

\[ \gamma = \frac{1}{2} \frac{\partial \mathcal{L}}{\partial P} = \frac{2w^2}{w^2}e^{-2w^2/w^2} \left[ \frac{1}{w(0)} \frac{\partial w(0)}{\partial P} \right] = \frac{2w^2}{w^2} \left[ \frac{1}{w(0)} \right] \mathcal{L} \quad (2.97) \]

Considerable effort has gone into optimizing cavity geometries to obtain strong SAM, which means from eq. (2.97) that \( \frac{1}{w(0)} \frac{\partial w(0)}{\partial P} \), the fractional change in spot size with respect to power evaluated at \( P = 0 \), should be large. We also note from eq. (2.97) that \( \gamma \) is proportional to the linear loss introduced by the aperture. In order to obtain large \( \gamma \) without driving the linear loss too high, the beam radius \( w \) should be slightly smaller than the aperture radius \( w_s \).

Similar calculations can be applied for a slit aperture, which is the geometry most commonly employed in real hard-aperture KLM lasers. For example, if the slit has width \( 2w_s \) centered at \( x = 0 \), then we obtain from eq. (2.93)

\[ T = \text{ef} \left( \frac{\sqrt{2}w_s}{w} \right) \quad (2.98) \]

where the error function is defined as \( \text{ef}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt \) [7]. The SAM coefficient \( \gamma \) is given by

\[ \gamma = \frac{\sqrt{2}w_s}{\sqrt{\pi}w} e^{-2w_s^2/w^2} \left[ \frac{1}{w(0)} \frac{\partial w(0)}{\partial P} \right] \quad (2.99) \]

In the limit of low loss \( (w_s \gg w) \), the error function in eq. (2.98) can be simplified by using the asymptotic expression \( \text{ef}(z) \approx 1 - \frac{1}{\sqrt{\pi}z} \exp(-z^2) \) for \( z \to \infty \) [7], which yields
\[ \mathcal{L} = \frac{1}{\sqrt{2\pi w_s}} \frac{w}{w_s} e^{-2w^2/w_s^2} \]  

and

\[ \gamma = \frac{2w_s^2}{w^2} \left[ \frac{1}{w(0)} \frac{\partial w(0)}{\partial P} \right] \mathcal{L} \]  

For small \( \mathcal{L} \) the relation between \( \gamma \) and \( \mathcal{L} \) is exactly the same as for a circular aperture.

Note that the calculations above neglect any diffraction caused by the aperture. Therefore, these formulas somewhat underestimate both the loss and the SAM coefficient.

For soft-aperture KLM, the SAM coefficient depends on the power dependent single pass gain, which can be calculated by performing a three dimensional overlap integral between the laser mode in the crystal and the position dependent saturated gain coefficient. Omitting proportionality constants, this integral has the form

\[ g_{sp} = \int dV \frac{|E_{pump}(x, y, z)|^2 |E_{satur}(x, y, z)|^2}{1 + \frac{2P(z, x, y)}{j_0}} \]  

Here \( g_{sp} \) and \( E_{satur}(x, y, z) \) refer to the single pass gain experienced by a specific time slice within the laser pulse and the corresponding spatial profile corresponding to that time slice. The local small signal gain is proportional to the local pump intensity, but the actual gain is saturated by the average laser power \( \langle P(x, y, z) \rangle \) at that point in the crystal. The factor of 2 arises in a linear cavity because the pulse makes a double pass through the crystal. A detailed discussion of this gain integral can be found, e.g., in [59]. Calculation of \( g_{sp} \) and its variation with instantaneous power is usually performed numerically for KLM lasers - for a discussion see, e.g., [50] [60]. For mode locking to be possible, \( g_{sp} \) must be positive.

The SPM coefficient \( \delta \) reflects the nonlinear phase shift caused by the focused beam in the laser crystal. It can be estimated from the on-axis nonlinear phase shift, which for a double pass through a crystal of length \( L \) is [61]

\[ -\delta |a(t)|^2 = -\frac{4\pi n_2}{\lambda} \int_{-\Phi}^{\Phi} 2P(t) dz \]  

where we have used eq. (2.92) for the on-axis intensity \( I(0, t) \). The beam radius varies in the crystal according to the Gaussian beam law

\[ w^2(z) = w_o^2 \left[ 1 + \frac{z^2}{z_o^2} \right] \]  

where \( z_o = \pi n w_o^2 / \lambda \) and we have assumed that the beam waist occurs in the center of the crystal. In the limit of weak focusing \( \langle L \ll z_o \rangle \), \( w(z) = w_o \) is a constant, and
$$\delta = \frac{8n_2L}{\lambda w_0^2} \quad \text{for} \quad L \ll z_o \quad (2.105)$$

In the opposite tight focusing extreme ($L \gg z_o$), the integral can be extended to infinity, with the result

$$\delta = \frac{8\pi^2 n_2}{\lambda^2} \quad \text{for} \quad L \gg z_o \quad (2.106)$$

Note that calculation of $\delta$ for a given laser configuration only requires knowledge of the linear mode properties (i.e., the beam radius in the crystal).

In order to evaluate $\gamma$, from eq. (2.97) for example, one must know the variation in mode size with power. This requires analysis of the laser resonator. Let us first discuss the stability conditions for the resonator in the low power (linear) regime. A simplified diagram for the analysis of the resonator is shown in Fig. 2.16(a). Comparing to Fig. 2.14, the pair of spherical mirrors is represented by a pair of internal lenses (with lens focal length equal to half the mirror radius of curvature). Usually the two spherical mirrors have identical radii of curvature, typically 10 cm. The distances $s_1$ and $s_2$ from the lenses to the planar end mirrors are typically between 30 and 150 cm. Prism pairs which may be used for dispersion compensation are not included in Fig. 2.16, since they do not affect the stability analysis. From eq. (1.66) the effective distance between the curved mirrors is given by $d^* = d - (n - 1)d_{crystal}/n$, where $d$ is the physical distance and $d_{crystal}$ is the length through the laser crystal with refractive index $n$. The resonator analysis is simplified by recognizing that each lens - planar mirror combination can be replaced by an equivalent curved mirror as shown in Fig. 2.16(b) [62] [63]. The equivalent curved mirrors sit in the planes where the planar mirrors are imaged, i.e.,

$$\frac{1}{s'_i} = \frac{1}{s_i} + \frac{1}{f_i} \quad , \quad \text{for} \quad i = 1, 2 \quad (2.107)$$

and the equivalent radii of curvature $^{3}$ are given by

$$R_i^{(eq)} = -\frac{f_i^2}{s_i - f_i} \quad , \quad \text{for} \quad i = 1, 2 \quad (2.108)$$

If we also introduce a parameter $\Delta$ which characterizes the range of stable operation, defined by

$$d' = f_1 + f_2 + \Delta \quad (2.109)$$

we find that the equivalent distance $t$ between the two equivalent curved mirrors is

$$t = R_1^{(eq)} + R_2^{(eq)} + \Delta \quad (2.110)$$

$^{3}$In most mode locked Ti:S lasers, $s_i > f_i$, and therefore the equivalent radii of curvature are negative.
This result is useful because the simple two mirror resonator of Fig. 2.16(b) has been thoroughly analyzed. For example, if we substitute eq. (2.110) into the stability condition for the two mirror resonator, eq. (1.69-a), we find that the resonator is stable only for values of $\Delta$ satisfying

$$0 < \Delta < -R_2^{(eq)} \quad \text{or} \quad -R_1^{(eq)} < \Delta < -\left( R_1^{(eq)} + R_2^{(eq)} \right) \quad (2.111)$$

where without loss of generality we have taken $s_1 \leq s_2$ and therefore $\left| R_2^{(eq)} \right| \leq \left| R_1^{(eq)} \right|$. Figure 2.17 shows the stability regions and the calculated beam diameters at the planar end mirrors for a cavity with $f_1 = f_2 = 5\, \text{cm}$, $s_1 = 68\, \text{cm}$, and $s_2 = 118\, \text{cm}$ as a function of the mirror spacing parameter [51]. There are two distinct stability zones, each with a calculated width of 2.2 mm, and separated by 1.8 mm. At the edges of the stability zones, the calculated beam diameters vary very rapidly, either diverging or collapsing to zero. It turns out that the largest SAM coefficient in hard aperture KLM also occurs near the edges of a stability zone. Intuitively, this makes sense. Since the mode parameters are very sensitive to the spacing of the curved mirrors, it is also reasonable that they should be very sensitive to a nonlinear lens placed between these same curved mirrors.

To calculate the power dependent mode profile, one must model the propagation in the laser crystal as influenced by the nonlinear lens. Such nonlinear propagation has been studied since many years in nonlinear optics under the name "self focusing" [64] [65]. In KLM lasers self focusing has been handled both by numerical and analytical methods. In the numerical approaches, e.g., [50], the crystal is typically divided into many thin slices, and in each slice the effect of linear diffraction and of the nonlinear lens are calculated essentially independently. For thin enough slices, this accurately models the nonlinear propagation through the entire crystal. The remainder of the laser resonator can be modeled via conventional diffraction theory. This calculation is performed iteratively for many passes through the laser until a self-consistent mode profile is obtained.

In the analytical methods self focusing is treated under the so-called "aberrationless approximation," in which the nonlinear lens is assumed to have a parabolic index profile [66] [67] [68]. This approximation is useful because a Gaussian beam propagating through a medium with a parabolic index profile remains Gaussian. Therefore, the entire laser resonator can be treated using the formulas for Gaussian beam propagation or the ABCD matrices, which were introduced in chapter 1. This simplifies the calculation and allows analytic results to be derived. The index profile is taken as

$$n(r) = n_0 + \Delta n(0)e^{-2r^2/w^2(z)} = \Delta n(0) \left( 1 - \frac{2r^2}{a_z w^2(z)} \right) \quad (2.112-a)$$

where $\Delta n(0)$ is the on-axis index change given by
$$\Delta n(0) = n_2 I(t) = \frac{2n_2 |a(t)|^2}{\pi w^2(z)}$$ (2.112-b)

Here \( w^2(z) \) is the Gaussian beam radius (which varies as the beam propagates through the crystal). Note that the index change is not taken as simply proportional to the second order expansion of the intensity, which would overestimate the effect of the nonlinear lens on the wings of the beam profile. Instead an adjustable parameter \( a_{SF} \) is introduced in order to improve the accuracy of the aberrationless approximation [68] [69]. Adjusting \( a_{SF} \) to minimize the mean square error in \( \Delta n(r) \) weighted by the spatial intensity profile yields \( a_{SF} = 4 \) [70]. Furthermore, the aberrationless approximation has been compared with experimental studies of the nonlinear transmission through an aperture placed in the far field of a thick self-focusing medium; good agreement with data was achieved when \( a_{SF} \) was adjusted between 3.77 and 6.4, depending upon the details of specific experiments [68]. Therefore, using eq. (2.112-a) with an appropriate adjustment factor \( a_{SF} \) is expected to yield useful results.

The nonlinear propagation within the laser rod has been modeled analytically by several authors, e.g., [69] [63] [70]. The behavior is found to depend on the ratio \( P/a_{SF} P_{crit} \), where \( P_{crit} \) is the critical power for self-focusing, given by

$$P_{crit} = \frac{a_{SF} \lambda^2}{8 \pi m \rho_2}$$ (2.113)

The significance of the critical power in three dimensional self-focusing theory is that in a long enough medium, a collimated or initially focused beam will always eventually spread due to diffraction for \( P < P_{crit} \), while for \( P > P_{crit} \), the beam will always self-focus catastrophically to a very small spot (where for example damage may ensue). The distance at which the beam self focuses (assuming \( P > P_{crit} \)) is shorter for higher \( P/P_{crit} \). It is also informative to note that the nonlinear phase shift which would be predicted using eq. (2.106) for the case of Gaussian beam in a long medium is \( (a_{SF}) \pi P/P_{crit} \) (this equation of course neglects the nonlinear beam reshaping and focusing). Thus, we have the intuitive result that large self-focusing effects are expected when the integrated nonlinear phase shifts approach or exceed \( \pi \). The parameters in a KLM laser are chosen of course to remain below the range for catastrophic self-focusing within the crystal; nevertheless, the ratio \( P/P_{crit} \) is still the important parameter.

As long as the laser crystal can be described analytically, the rest of the resonator can be analyzed using the conventional ABCD matrices. Although detailed coverage of these calculations is beyond the scope of our treatment, there are several general results that emerge from the analysis [71] that we enumerate here. These results are expressed in terms of a coefficient

$$\delta_w = \frac{1}{w(0)} \frac{\partial w(0)}{\partial (P/P_{crit})}$$ (2.114)

which characterizes the variation in mode size with respect to normalized power:
• For resonators with planar end mirrors, as is usually the case for KLM lasers, the maximum attainable value of $\delta_w$ occurs at one of the end mirrors and at positions internal to the resonator corresponding to image or Fourier transform planes of that end mirror. This means that placing an aperture near an end mirror is optimum for achieving large $\gamma$ (assuming of course that $\delta_w$ is negative at that end mirror).

• The maximum first order variation of the normalized spot size with respect to power is given by

$$|\delta_w| \leq \frac{1}{4\sqrt{g_1 g_2 (1 - g_1 g_2)}} \quad (2.115)$$

where $g_1$ and $g_2$ are the resonator stability parameters from eq. (1.69-b). Recall that the resonator is stable only for $0 < g_1 g_2 < 1$. Therefore, the largest $\gamma$ coefficient scan be attained close to the stability limits.

• To reach the limit given by eq. (2.115) above, the resonator including the position of the gain medium must be designed optimally. Convenient formulas expressing $\delta_w$, in terms of the placement of the resonator elements are given in [71]. The most critical parameters are the distance between the spherical mirrors, which determines the stability of the resonator, and the longitudinal position of the laser rod with respect to the spherical mirrors, which largely determines the nonlinear variation of the far-field spot size for a given resonator stability. Calculated values of $\delta_w$ are plotted in Fig. 2.18 as a function of spherical mirror spacing and the position of the laser rod [72]. Two different cases corresponding to two different degrees of resonator symmetry are shown. The contour plots, which include only negative values of $\delta_w$, demonstrate that (1) the largest negative values of $\delta$ are achieved near stability limits, and (2) negative $\delta_w$ is obtained only when the laser rod is positioned asymmetrically with respect to the midpoint between the two spherical mirrors. Figure 2.18 also indicates (filled circles) the resonator configurations for which self-sustaining hard aperture KLM was observed in corresponding experiments. The close correspondence between these data and the $\delta_w$ contours provides strong evidence of the need for a sufficiently large SAM coefficient for successful KLM.

The dependence of $\delta_w$ on the position of the laser rod can be understood by noting that a nonlinear self-focusing medium (i.e., $n_2 > 0$) placed prior to the focus of a converging beam tends to enhance the focusing, leading to greater divergence and reduced transmission through an aperture in the far-field; conversely a self-focusing medium placed after the focus tends to make the beam more collimated, increasing transmission through a far-field aperture. Consequently, the nonlinear transmission through the aperture changes sign when the nonlinear medium is scanned through the focus and is small or zero
when the nonlinear medium is placed symmetrically with respect to the focus [68] [73].

At this point we are ready to discuss the size of the nonlinear effects in KLM lasers. As an example, we consider the power levels in the original self mode locked laser of [48], which are fairly typical for KLM Ti:S lasers. From the 450 mW output power emerging from a 3.5% output coupler, we calculate an average intracavity power is 12.8 W. To estimate the size of the SAM effect, we use eq. (2.99). We take \( \delta_m = 1 \), which according to Fig. 2.18 is a relatively large value, and we assume \( w/w_o = 0.8 \), which gives a linear loss of 1.4% for the power. This yields \( \gamma = 0.044 P_{\text{crit}} \). For Ti:S the critical power is usually taken as 2.6 MW [69], which yields \( \gamma \approx 0.02MW^{-1} \). Let us assume that before the laser is mode locked, the initial power fluctuations due to random mode beating between multiple unlocked longitudinal laser modes are of the order of the average continuous-wave power, i.e., 12.8 W. Then we have \( \gamma P \approx 2 \times 10^{-7} \) for the self-amplitude modulation term, which is quite weak. For this reason many KLM lasers are not self-starting. On the other hand, once the laser is mode locked, the nonlinear pulse shaping effects can be quite strong. For mode locked operation the peak pulse power is enhanced by the ratio of the pulse round trip time to the pulse width, so for 60 fsec pulses with a 12 nsec round trip time, the peak power is \( \sim 2.6MW \), which is roughly \( P_{\text{crit}} \) in this particular laser. We estimate a SAM term \( \gamma P \approx 0.05 \). Since the analysis we are using consists of a first order expansion in \( P \), this estimate for \( \gamma P \) may become inaccurate at the high peak powers encountered in mode locked operation; nevertheless, the basic conclusion that SAM is a large effect does not change! Since \( P \approx P_{\text{crit}} \) in this laser, from our previous discussion we conclude that the nonlinear phase shift is itself of the order of \( \pi \), and therefore SPM can also be a strong effect. Thus, even in the 60 fs regime, both the nonlinear amplitude modulation and the nonlinear phase modulation are becoming large. For even shorter pulses (e.g., 6 fsec), the peak powers which drive the nonlinearities can be an order of magnitude higher.

In the following section we describe the solutions to the master equation governing mode locking of such solid-state lasers. Although the master equation is derived with the assumption of small changes per element per round trip, nevertheless, the solutions provide important insight into mode locked laser operation, especially for pulses a few tens of femtoseconds and above. Additional phenomena, associated with violation of the assumption of small changes per round trip, are discussed in chapter 7.

Before concluding this section, we briefly discuss the role of astigmatism in KLM lasers, which we have neglected so far. By astigmatism we mean that the Gaussian beam radius, and the position of the beam waist, may be different for the \( x \) and \( y \) transverse directions. KLM lasers as depicted in Fig. 2.14 contain two sources of astigmatism: the spherical mirrors used away from normal incidence and the Brewster angle cut laser rod [74]. In astigmatic systems one distinguishes the tangential plane, which lies in the plane of Fig. 2.14, from the sagittal plane, which lies perpendicular to the figure. Thus, the beam folding at the spherical mirror and the change of direction due to refraction at
the surface of the Brewster cut rod both lie within the tangential plane. In astigmatic resonators the stability zones calculated for tangential and sagittal planes can be displaced and may not overlap at all. Fortunately, astigmatically compensated cavities can be realized in which the angles of incidence on the spherical mirrors are chosen so that the mirror astigmatism compensates for that of the laser rod; such astigmatically compensated cavities, which were analyzed in [74], are generally used for KLM lasers. Note that although with astigmatic compensation the stability regions are the same for the tangential and sagittal planes, the positions of the beam waists and the beam radii in the laser rod may still be different for the two planes. As a result, astigmatism may still influence KLM even with an astigmatically compensated laser cavity.

Many of the analyses of KLM resonators account for astigmatism by calculating the mode properties and $\delta_\omega$ independently for tangential and sagittal planes. That is, the calculation is performed for a cylindrically symmetric beam; but different values for the focal lengths, etc., are used to predict the different behavior for the two planes. The results indicate that for most KLM resonators, the astigmatism enhances (decreases) $\delta_\omega$ in the tangential (sagittal) plane [63][72]. This is the reason that a vertical slit is generally used for hard aperture KLM. In reality, however, the beam radii in the two directions are not strictly independent, since a change in beam radius along one direction, $x$ for example, will change the intensity and therefore the nonlinearity seen by the entire beam [75]. An analytical treatment of astigmatic Gaussian beams in KLM resonators, which does account for the $x-y$ coupling, is given in [76]. Using this treatment the $\delta_\omega$ contour plots, which were shown in Fig. 2.18 for the tangential plane using a cylindrical beam treatment, were recalculated with the coupling included. The results showed that in most cases the overall trends observed from the $\delta_\omega$ contour plots in Fig. 2.18 remain valid.

### 2.4.5 Mode locking solutions

We now write down the mode locking equation and discuss its solution, following the treatment of [77]. The main bandwidth limiting effect is assumed to arise from the gain itself, so that the $\frac{1}{\omega^2} \frac{d^2}{dt^2}$ term in eq. (2.38) is replaced with $\frac{g}{\omega^2} \frac{d^2}{dt^2}$.

The changes in the pulse in one round trip through the laser, caused by gain, loss, SAM, SPM, and dispersion, are given by

$$a_{n+1}(t) = \left\{ 1 + g \left( 1 + \frac{1}{\omega^2} \frac{d^2}{dt^2} \right) - \left( \ell - \gamma |a(t)|^2 \right) - j\delta |a(t)|^2 + jD \frac{d^2}{dt^2} \right\} a_n(t)$$

(2.116)

where $a_n(t)$ and $a_{n+1}(t)$ represent the pulse after $n$ and $n + 1$ passes through the laser, respectively, and $g$ as usual is the saturated gain. For steady state mode locking we require that the pulse reproduce itself to within a phase shift $\psi$:

$$a_{n+1}(t) = a_n(t) e^{j\psi}$$

(2.117)
Note that previously we implicitly assumed that any such phase shift could be
ignored in the equations for active mode locking and passive mode locking using
saturable absorbers. This is exactly correct when the gain line shape and any
bandwidth limiting filters are exactly centered on one of the longitudinal cavity
modes. However, the assumption is also approximately correct whenever the
cavity modes are closely spaced relative to the gain or filter bandwidth, which
is usually satisfied in practice. Here, however, we must include \( \psi \) due to the
possibility of a contribution from self-phase-modulation. If we assume that the
phase shift per pass is small, then \( \exp(j\psi) \approx 1 + j\psi \), and we can combine
equations (2.116) and (2.117) to obtain

\[
\left\{ \left( \frac{g}{\omega_G} + jD \right) \frac{d^2}{dt^2} + (g - \ell - j\psi) + (\gamma - j\delta) |a(t)|^2 \right\} a(t) = 0 \tag{2.118}
\]

A number of new phenomena can occur in mode locked lasers described by
this equation. These include the following:

- SPM can create new frequency components which can broaden the spec-
  trum and sometimes lead to shorter output pulses.
- Due to the action of SPM, GVD, or both, the mode locked pulses may be
  chirped. In the case of chirped pulses, a chirp compensator located outside
  of the laser may be used to compress the output pulse duration. Pulses
discussed in previous sections (active mode locking, saturable absorber
passive mode locking) were always chirp-free.
- SPM and GVD can interact strongly. The character of the interaction
depends on the sign of the dispersion, which can be tuned to be positive,
negative, or approximately zero using prism pairs or other devices.

The solution of eqn. (2.118) is a chirped sechant hyperbolic [77], written as
follows:

\[
a(t) = a_o \sech(t/t_p) \left[ 1 + j\beta \right] = a_o \sech(t/t_p) e^{j\beta \ln(\sech(t/t_p))} \tag{2.119}
\]

For convenience we take \( a_o \) to be real. The new feature is the time-dependent
phase term, which leads to a time-dependent frequency shift \( \Delta \omega(t) \), given by

\[
\Delta \omega(t) = \frac{\partial \phi(t)}{\partial t} = \frac{-\beta}{t_p} \tanh(t/t_p) \tag{2.120}
\]

The frequency shift is equal to \( \beta / t_p \) before the pulse, zero at the center of
the pulse, and \( -\beta / t_p \) after the pulse. \( \beta < 0 (\beta > 0) \) corresponds to an up-
chirp (down-chirp). Since the bandwidth of an unchirped sechant hyperbolic
would be on the order of \( t_p^{-1} \), the chirp corresponds to large additional spectral
broadening when \( |\beta| \gg 1 \) and small spectral broadening when \( |\beta| \ll 1 \). Roughly,
the spectral width is increased by a factor \( \sqrt{1 + \beta^2} \) compared to the bandwidth
in the unchirped (\( \beta = 0 \)) case.
Inserting eq. (2.19) into the mode locking equation (2.18) and requiring that the coefficients of sech$^{1+j\beta}(t/t_p)$ and sech$^{3+j\beta}(t/t_p)$ independently sum to zero yields two complex characteristic equations:

$$g - \ell - j\psi + \frac{(1 + j\beta)^2}{t_p^2} \left( \frac{g}{\omega_g^2} + jD \right) = 0$$

(2.121-a)

$$\left( \frac{g}{\omega_g^2} + jD \right) \left( \frac{2 + 3j\beta - \beta^2}{t_p^2} \right) = (\gamma - j\delta)a_o^2$$

(2.121-b)

These equations (equivalent to four real equations) are supplemented by the gain saturation equation (2.68) which gives $g$ in terms of the small signal gain and average laser power. This system of nonlinear equations can be solved simultaneously to yield the five unknowns, namely, $a_o$, $t_p$, $\beta$, $g$, and $\psi$.

This calculation can be simplified by noting that the saturated gain is not usually too far below the loss line, and therefore to first order one can approximate $g \approx \ell$. Under this approximation the average power and the pulse energy $U = 2\omega_g^2 t_p$ can then be obtained directly from eq. (2.68). It is then convenient to introduce a normalized (dimensionless) dispersion parameter

$$D_n = \left( \frac{\omega_g^2}{g} \right) D$$

(2.122)

which describes the relative strength of dispersive pulse spreading compared to pulse spreading due to bandwidth filtering. It also useful to define a normalized (not dimensionless!) pulse width

$$t_{p,n} = \left( \frac{U \omega_g^2}{2g} \right) t_p$$

(2.123)

Equation (2.121-b) can be rewritten in terms of these normalized variables as follows:

$$(1 + jD_n) \left( 2 + 3j\beta - \beta^2 \right) = (\gamma - j\delta)t_{p,n}$$

(2.124)

This single complex equation can be solved to yield the chirp parameter $\beta$ and the normalized pulse width parameter $t_{p,n}$.

Results taken from [77] are shown in Fig. (2.19), in which $\beta$, $t_{p,n}$, a normalized mode locked bandwidth $(1/t_{p,n})\sqrt{1 + \beta^2}$, and a stability parameter (discussed below) are all plotted as a function of $D_n$. This figure assumes a fixed SAM coefficient ($\gamma = 1$) with varying degrees of SPM ($\delta = 0, 1, 2, 4$). Due to the approximation $g \approx \ell$ and the use of the normalized variables, the small signal gain does not explicitly appear, and therefore a single set of curves is able to represent solutions for various $g_o$. The value of $g_o$ and $U$ do enter when eq. (2.123) is used to convert the normalized pulse width back to real pulse width units.

When there is no SPM ($\delta = 0$), the curves are symmetric. The shortest pulses and highest bandwidth occur at $D = 0$, for which the pulses are chirp-free. (With no dispersion or SPM, the behavior is the same as for fast absorber mode locking, considered previously.) For $D \neq 0$ the bandwidth decreases while
the pulses grow longer and pick up a chirp. The sign of the chirp changes when the sign of the dispersion changes.

When SPM is turned on ($\delta > 0$), the behavior depends strongly on the sign of the dispersion. Even at zero dispersion, the pulses are up-chirped, with shorter pulses and higher bandwidth compared to $\delta = 0$. These effects are attributed to the new frequency components generated through SPM.

For $D > 0$ ("normal dispersion"), SPM and GVD acting individually both lead to an up-chirp. The down-shifted frequency components which SPM creates on the leading edge of the pulse are shifted to even earlier times by GVD, while the up-shifted frequency components created on the trailing edge of the pulse are shifted to even later times. Thus, under the combined influence of SPM and GVD, pulses spread faster and experience a greater chirping than they would under the influence of GVD alone. In Fig. 2.19 this manifests itself as a rapid increase in the pulse width and in the magnitude of the chirp parameter, either for increasing $D_n$ with $\delta$ fixed or for increasing $\delta$ with $D_n$ fixed. For fixed dispersion the bandwidth also increases with increasing SPM, so chirp compensation outside the laser can lead to compressed pulses shorter than could be obtained without SPM but with the same intracavity dispersion. Finally, it is interesting to note that when a highly chirped pulse passes through a bandwidth limiting filter (e.g., due to the finite gain bandwidth), it can actually be narrowed in time as well as in frequency. This is in contrast to chirp-free pulses considered earlier, where filters were always associated with pulse broadening effects.

For $D < 0$ ("anomalous dispersion"), GVD leads to a down-chirp while SPM creates an up-chirp. If the strengths of these two effects are matched, they compensate each other, and no net chirping or pulse spreading occurs. In non-linear fiber optics such compensation results in pulses called solitons, which can propagate without temporal or spectral distortion [78] [79] [33]. Solitons have been applied in optical communication systems for distortionless propagation of short pulses over extremely long distances (e.g., pulses a few tens of picoseconds long over 10,000 km of fiber - corresponding to trans-Pacific distances!) [80].

The operation of mode locked lasers with $D_n < 0$ can be heavily influenced by soliton-like effects, and such effects can lead to new pulse compression mechanisms which contribute strongly to pulse shortening in the mode locked laser. These effects manifest themselves in Fig. 2.19 in several ways. First, for fixed $\delta$ the shortest pulses and the largest bandwidth actually occur for $D_n$ slightly less than zero. Compared to the positive dispersion case, the pulse width increases rather slowly as the negative dispersion is increased beyond the optimum point, especially for large SPM. For a given $D$, the pulses become shorter with increasing $\delta$. At the same time the pulses remain remarkably chirp free over a substantial range of negative dispersion.

Mode locked lasers operating in a soliton-like regime exhibit definite relations between pulse intensity, pulse width, and dispersion (since the SPM effect, whose strength is proportional to peak intensity, must balance dispersive effects, whose strength turns out to depend on pulse width squared divided by the dispersion parameter $D$). This can be seen by inserting $\beta = 0$, which we have seen is approximately true within much of negative dispersion range, into eq. (2.124).
The imaginary part of this equation yields

\[ t_{p,n} = \frac{2|D_n|}{\delta} \]  \hspace{1cm} (2.125)

In terms of the original rather than the normalized variables this gives

\[ \frac{\delta U t_p}{4|D|} = \frac{\delta a_o^2 \epsilon_p^2}{2|D|} = 1 \]  \hspace{1cm} (2.126)

As we shall discuss in further detail in chapter 6, this is exactly the relationship which applies for ideal soliton propagation.

Figure 2.19(c) shows the stability parameter plotted vs. \( D_n \). The stability parameter is taken as \( \ell - g \), which gives the net loss before and after the pulse. As in the case of saturable absorber mode locking, loss must exceed gain before and after the pulse to suppress amplification of perturbations on the pulse's leading or trailing edge. Therefore, \( \ell - g > 0 \) for stability (see Fig. 2.10). Since this stability parameter cannot be evaluated using the approximation \( g \approx \ell \), it is obtained from the full solution of equations (2.121-a) and (2.121-b). The key point to be learned from Fig. 2.19(c) is that although the mode locking solution is stable over most of the investigated parameter range, there are unstable ranges, particularly for large SPM and when one approaches positive dispersion. Experimentally, one can encounter the instability when using an intracavity prism sequence to tune close to zero dispersion or into the positive dispersion range. At this point one may observe, for example, either multiple pulses or chaotic behavior in which the laser power, pulse width, etc. exhibit strong fluctuations on a time scale long compared to the laser round trip time.

Figure 2.20 is another plot from [77] showing the behavior of the pulse parameters as a function of \( D_n \), for a fixed SPM coefficient \( \delta = 4 \) and various SAM coefficients \( (\gamma = 0, 0.5, 1, 2) \). Compared to Fig. 2.19, the biggest difference appears in the stability parameter plot, where unstable operation occurs over a much larger parameter range than before. In particular, the mode locked pulse is never stable when \( \gamma = 0 \); some SAM is always required to obtain a net gain window with loss on either side of the pulse! Mode locking becomes increasingly stable with stronger SAM. The other plots (chirp, pulse width, bandwidth) show similar trends to Fig. 2.19 - namely, long chirped pulses for \( D_n > 0 \) and shorter nearly chirp-free pulses with a relatively weak dependence on dispersion for \( D_n < 0 \). We notice also that for \( D_n < 0 \), the pulse parameters are quite insensitive to the strength of SAM, except for the stability. This is in contrast to the strong dependence on SPM seen in Fig. 2.19. In this range the main pulse shortening effect responsible for steady-state pulse characteristics is the interplay between SPM and negative dispersion. The main effect of SAM is to stabilize the resulting steady-state pulse dynamics; increasing \( \gamma \) allows the system to remain stable with a higher value of \( \delta \) or closer to \( D_n = 0 \), both of which tend to yield shorter pulses.

We note at this point that in some instances, actual mode locked solid-state lasers violate the assumption built into our analytical theory that all the effects
are small on a per pass basis. This may be particularly true when the pulses are very short (approaching 10 fs or below) or the material lengths are very long (e.g., fiber lasers with path lengths of meters or more); in either case there may be large nonlinearities and large dispersive effects. When the per pass effects are large, the order in which the elements are placed in the laser cavity can become important. Furthermore, when pulses are very short and $D_n \approx 0$, one may need to include higher order dispersion terms, proportional to the third or even the fourth derivative of phase with respect to frequency. However, except for the shortest pulses, the analytic theory outlined above still gives a good insight into the mode locked behavior. We will return to mode locking in chapter 7, where we will discuss some of the new effects that arise when the assumptions made above are violated.

2.4.6 Initiation of mode locking

We have mentioned already that usually Kerr lens mode locking of solid-state lasers, and in some cases additive pulse mode locking, is not self-starting, though it is self-sustaining. By self-sustaining we mean that mode locking, once initiated, can remain stably until the laser is disturbed. By self-starting we mean that the laser can mode lock spontaneously once the pump power is applied. It has also been observed that mode locking of solid-state lasers can only be achieved above a minimum average intracavity power. Mode locked lasers that are not self-starting require some external perturbation to move from continuous-wave to mode locked operation, though once mode locking is achieved the perturbation may be switched off. A number of methods have been used to initiate mode locking in such cases, including changing the cavity length by translating an end mirror or intracavity prism, regenerative mode locking, synchronous pumping with pulses from an actively mode locked laser, or simply banging on the optical table.

Unlike solid-state laser mode locking using KLM or APM, active mode locking and passive mode locking using slow saturable absorbers are typically self-starting. We can explain this difference using the concept of a pulse shortening velocity, which can be defined as the fractional change in pulse duration in a single pass through the pulse shortening element in the laser [81]. In lasers mode locked using the ultrafast optical Kerr effect, the pulse shortening velocity is strongly pulse width dependent. This is because the terms responsible for pulse shortening (e.g., SAM represented by $\gamma |a|^2 a$) are proportional to peak power, which varies inversely with pulse width (assuming fixed pulse energy). Thus, the pulse shortening velocity for the nanosecond time scale fluctuations characteristic of the unlocked multi-mode laser when it is first turned on can be more than five orders of magnitude smaller than the pulse shortening velocity for mode locked operation in the femtosecond regime. In passive mode locking using slow saturable absorbers, on the other hand, the pulse shortening velocity does not depend on pulse duration: it is equally effective for long pulses and short pulses. In active mode locking the pulse shortening velocity is actually stronger for longer pulses, since the shape of the modulation remains fixed even
as the pulses get shorter. These differences show that in the long time regime, pulse shortening effects are much weaker for mode locking using the optical Kerr effect compared to other mode locking approaches, and therefore the behavior during initiation of mode locking is also very different.

A number of authors have analyzed the conditions for initiation of mode-locking in KLM and APM lasers [82][83] [84] [85] [86] [87] [88] [89]. Here we will simply summarize a few of the main concepts. Prior to mode locking the laser is usually assumed to be running in two or more longitudinal modes, which lead to intensity fluctuations, typically on the nanosecond time scale, due to the random beating between these modes. Initiation of mode locking depends on whether the nonlinearities are strong enough (over a large number of round trips through the laser) to amplify the energy within the fluctuation with the highest peak intensity and suppress the remainder of the laser radiation. Once a single fluctuation is selected, the nonlinearities must act on it repeatedly before reaching the mode-locked steady-state.

One simple rule which has been advanced provides a criterion for energy growth of an intense fluctuation as affected by two competing processes: self-amplitude-modulation which enhances the fluctuation, and dynamic gain saturation which occurs during the fluctuation and reduces its gain [82]. The fluctuation can only grow if the increase in energy due to SAM exceeds the decrease in energy due to dynamic gain saturation, which is satisfied when

$$\frac{\gamma}{g} \geq \frac{\sigma_g T_f}{\hbar \omega A_g}$$  (2.127)

Here $\sigma_g$, $A_g$, $\hbar \omega$, and $\tau_f$ are the gain cross-section, the beam area within the gain medium, the photon energy, and the duration of the fluctuation, respectively, and a pulse shape dependent numerical factor of order unity has been omitted. This relation was cited [82] as one reason why APM with Ti:S lasers ($\sigma_g = 2.7 \cdot 10^{-19} \text{ cm}^2$) and other solid-state lasers with lower gain cross-sections was observed to be self-starting, while APM with color-center lasers ($\sigma_g \sim 10^{-17} \text{ cm}^2$) was not.

Additional analyses [83] [84] [85] provide an explanation for the observation of a threshold power for mode locking, which is not predicted by eq. (2.127).

In one time domain formulation, one requires that not only must the initial fluctuation grow in energy, but this growth must occur faster than a correlation time $\tau_c$, which is taken as the lifetime of the fluctuation in the absence of nonlinearities [83]. This leads to the criterion

$$\gamma \Delta P \geq \frac{T}{\tau_c}$$  (2.128-a)

or

$$\gamma P \geq \frac{1}{\ln(N) \tau_c}$$  (2.128-b)

where $P$ is the average intracavity power, $\Delta P \approx P \ln(N)$ gives a typical value for the peak intensity of the strongest fluctuation, $N$ is the number of modes which are oscillating in the free-running laser, $T$ is the cavity round trip time,
and the effect of dynamic gain saturation has been assumed to be small. This criterion says that mode locking can be initiated only when the change in loss experienced by the initial strong fluctuation per round trip, multiplied by the number of round trips within the correlation time \( \tau_c \), is of order unity or larger. Although a few theoretical studies give insight into physical mechanisms leading to a finite \( \tau_c \) [83] [84], it is most practical to determine \( \tau_c \) experimentally. This can be achieved by measuring the laser intensity noise with a radio frequency spectrum analyzer and identifying \( \tau_c \approx 1/(\pi \Delta \nu) \), where \( \Delta \nu \) is the full-width at half-maximum width of the first beat note. Measurements yield values for \( \Delta \nu \) typically in the several kHz range for various solid-state lasers, corresponding to \( \tau_c \) on the order of 100 microseconds. Self-starting thresholds have been measured for several lasers and seem to agree at least qualitatively with the predictions of eq. (2.128-a), e.g., [83] [86] [87]. Additionally, eq. (2.128-a) also explains the greater propensity for APM lasers to self-start (compared to KLM with the same laser materials), since the self-amplitude modulation coefficient can be substantially higher in APM lasers using nonlinearities in optical fibers compared to KLM lasers using bulk nonlinearities.
Figure 2.1: (a) An actively mode locked laser arrangement, with an intracavity modulator driven at the cavity round trip period. (b) Sketch of periodic modulator transmission and resulting mode locked pulses.

Figure 2.2: Model of an actively mode locked laser

Figure 2.3: Complex refractive index plotted versus normalized frequency for a Lorentzian gain medium. $n'$ and $n''$ are normalized by the factor $(eg/\omega g)$. 
Figure 2.4: A frequency domain view of active mode locking. (a) Unsaturated gain spectrum and saturated gain spectrum for free running and mode locked laser operation. (b) Resulting mode locked spectrum (from [47]).

Figure 2.5: Model of a passively mode locked laser.
Figure 2.6: Model of 4-level saturable absorber.

Figure 2.7: Pulse shortening process in slow saturable absorber mode locking. The shaded region indicates net positive gain.
Figure 2.8: Time-dependent loss $\ell(t)$, gain $g(t)$, and net gain $g_T(t)$ plotted as a function of pulse energy $U(t)$. The shaded regions indicate net positive gain.
Figure 2.9: Colliding-pulse mode locked (CPM) ring dye laser. OC: output coupler.

Figure 2.10: Gain and loss dynamics in fast saturable absorber mode locking. The shaded region corresponds to positive net gain.
Figure 2.11: Graphical solution of eq. (2.69), in which the left and right hand sides of the equation are plotted vs. pulse amplitude $a_o$. Two cases are plotted: the case of a relatively large $g_o$ where no single pulse solutions are found, and the case of a smaller $g_o$ where two solutions are found (marked by circles). Of these, only the longer pulse, lower peak power solution is stable.

Figure 2.12: Schematic layout of an additive pulse mode locked or coupled cavity mode locked laser. The lengths of the main and auxiliary cavities are closely matched, although the exact lengths may differ slightly to provide the required phase bias.
Figure 2.13: Variations of internal power, output power, and second-harmonic generation (SHG) as a function of the length of the fiber cavity. Exact cavity match is at $\tau = 0$ (from [39]).
Figure 2.14: (a) Schematic of the cavity configuration for original self mode locked Ti:S laser. The inset shows the intracavity prism sequence for dispersion compensation (from [48]). (b) Schematic for typical sub-10-fs Ti:S lasers (from [51]). In many such lasers, the prism pair is replaced by chirped, dispersion compensating mirrors. G: gain medium (Ti:sapphire crystal). OC: output coupler. F: birefringent filter.

Figure 2.15: Measured horizontal mode profiles from a KLM Ti:S laser. (a) CW operation. (b) mode locked. The profiles were recorded at the prism closest to the laser rod. For both (a) and (b) the right peak corresponds to the beam coming from the crystal, while the left peak is the beam returning toward the crystal after reflecting from the end mirror (from [50]).
Figure 2.16: (a) Internal lens model for analysis of KLM laser resonator. (b) Equivalent two mirror resonator.

Figure 2.17: Stability regions and calculated low power beam diameters for a KLM resonator, where $2w_1$ and $2w_2$ are the beam diameters at mirrors $M_1$ and $M_2$ in Fig. 2.16(a), respectively. Cavity parameters are given in the text (from [51]).
Figure 2.18: Contour plots showing calculated values of $\delta_w$ as a function of spherical mirror spacing and laser rod position. Filled circles indicate resonator configurations for which self-sustaining hard aperture KLM was experimentally observed. (a) $s_1 = 50$ cm, $s_2 = 110$ cm. (b) $s_1 = 70$ cm, $s_2 = 90$ cm. In both cases $f_1 = f_2 = 5$ cm and the slit for hard aperture KLM was placed in the short arm of the laser cavity (from [72]).
Figure 2.19: Pulse parameters versus normalized dispersion $D_n$ for $\gamma = 1$ and different SPM parameters $\delta$: (a) the chirp parameter $\beta$, (b) the normalized pulse width $t_{p,n}$, (c) the stability parameter $\ell - g$, (d) the normalized bandwidth $(1/t_{p,n})\sqrt{1 + \beta^2}$ (from [77]).
Figure 2.20: Pulse parameters versus normalized dispersion $D_n$, for $\delta = 4$ and different SAM parameters $\gamma$: (a) the chirp parameter $\beta$, (b) the normalized pulse width $t_{p,n}$, (c) the stability parameter $\ell - g$, (d) the normalized bandwidth $(1/t_{p,n})\sqrt{1 + \beta^2}$ (from [77]).
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