CS 547 Lecture 41: Solving Recurrence Relations

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Consider the following sequence.

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

This is, of course, the famous Fibonacci sequence, where each term is the sum of the previous two terms. It's common to define this sequence as a *recurrence relation*, along with initial conditions for the first two elements.

$$f_n = f_{n-1} + f_{n-2}$$
$$f_0 = 0$$
$$f_1 = 1$$

Now, suppose we want to know a *closed-form* method for calculating Fibonacci numbers. That is, is there a single formula that we will allow us to calculate the n^{th} Fibonacci number, f_n, without needing to calculate all of the previous numbers in the sequence?

We can use generating functions to derive the closed-form solution. The basic procedure:

1. Derive a generating function from the recurrence relation
2. Manipulate the generating function into an invertible form (for us, this will be a sum of basic geometric generating functions like we used in the last lecture)
3. Invert the generating function and recover the formula for f_n

Deriving the Initial Generating Function

The first step in the solution process is to derive a generating function associated with the recurrence relation $f_n = f_{n-1} + f_{n-2}$. The following six step procedure will allow us to do this in a mostly mechanical way. The same basic approach will work on other simple recurrences.

Shift the subscripts so that the smallest subscript is n
We'll rewrite the recurrence relation as

$$f_{n+2} = f_{n+1} + f_n$$

This transformation shifts us away from the initial conditions, so that the relationship is now true for all n from zero to ∞.

Multiply by the power of z corresponding to the left-hand side subscript
Multiply both sides of the relation by z^{n+2}.

$$f_{n+2}z^{n+2} = f_{n+1}z^{n+2} + f_n z^{n+2}$$

Sum over all values of n, from zero to ∞

$$\sum_{n=0}^{\infty} f_{n+2}z^{n+2} = \sum_{n=0}^{\infty} f_{n+1}z^{n+2} + \sum_{n=0}^{\infty} f_n z^{n+2}$$

Factor the extra z values from each summation on the right-hand side

$$\sum_{n=0}^{\infty} f_{n+2}z^{n+2} = z\sum_{n=0}^{\infty} f_{n+1}z^{n+1} + z^2\sum_{n=0}^{\infty} f_n z^{n}$$

Rewrite the summations in terms of the generating function $F(z)$
First, observe that

$$\sum_{n=0}^{\infty} f_n z^n = F(z)$$

is simply the defintion of the generation function.

The other sum on the right-hand side is

$$\sum_{n=0}^{\infty} f_{n+1}z^{n+1} = f_1 z + f_2 z^2 + f_3 z^3 + \dots$$

This is the defintion of the generating function with the first term omitted. Therefore,

$$\sum_{n=0}^{\infty} f_{n+1}z^{n+1} = F(z) - f_0$$

Similarly,

$$\sum_{n=0}^{\infty} f_{n+2}z^{n+2} = F(z) - f_1 z - f_0$$

Putting everything together,

$$F(z) - f_1 z - f_0 = z(F(z) - f_0) + z^2 F(z)$$

Solve for F(z) and use the initial conditions to simplify

$$F(z) = \frac{f_0 + f_1 z - f_0 z}{1 - z - z^2}$$

The intial conditions of the Fibonacci sequence are $f_0 = 0$ and $f_1 = 1$, so the final result is

$$F(z) = \frac{z}{1 - z - z^2}$$

Note that this entire process was mostly mechanical. The only "clever" part was in writing the summations in terms of $F(z)$, but similar transformations will appear in other recurrence relations, so we can reuse that strategy in the future. More complicated recurrences might require more manipulations to get into a simplified final form.

Get the Generating Function into an Invertible Form

$F(z)$ is already in a rational form, with a second degree polynomial in the denominator. To begin the inversion process, We would like to factor the denominator into

$$F(z) = \frac{z}{(1-az)(1-bz)}$$

for some values of a and b that depend on the roots of the polynomial.

If we re-expand the factored polynomial, we'd have

$$1 - (a+b)z + abz^2 = 1 - z - z^2$$

By equating the coefficients, we know

$$\begin{aligned} a + b &= 1 \\ ab &= -1 \end{aligned}$$

Substituting $a = 1 - b$ and using the quadratic formula, we can find the two possible values of a and b.

$$\begin{aligned} a &= \frac{1+\sqrt{5}}{2} \\ b &= \frac{1-\sqrt{5}}{2} \end{aligned}$$

The number $\frac{1+\sqrt{5}}{2}$ is the famous *golden ratio*, the most aesthetically pleasing of all proportions. It's customary to use the letter ϕ to denote the ratio, after Phidias, who allegedly used it in planning the structure of the Parthenon. Let $\hat{\phi}$ denote the other root, $\frac{1-\sqrt{5}}{2}$. We now have

$$F(z) = \frac{z}{(1-\phi z)(1-\hat{\phi} z)}$$

All that remains is to use partial fractions to split $F(z)$ in two.

$$F(z) = \frac{c}{1-\phi z} + \frac{d}{1-\hat{\phi} z}$$

Combining the two fractions over a common denominator gives

$$c(1-\hat{\phi} z) + d(1-\phi z) = z$$

Setting $z = \frac{1}{\phi}$ and $z = \frac{1}{\hat{\phi}}$, we can solve and obtain

$$\begin{aligned} c &= \frac{1}{\phi - \hat{\phi}} \\ d &= \frac{-1}{\phi - \hat{\phi}} \end{aligned}$$

The final, invertible form of the generating function is

$$F(z) = \frac{\frac{1}{\phi-\hat{\phi}}}{1-\phi z} - \frac{\frac{1}{\phi-\hat{\phi}}}{1-\hat{\phi} z}$$

Invert the Generating Function and Obtain the Solution

Using the basic geometric series generating function, we can write $F(z)$ as as summation.

$$F(z) = \sum_{n=0}^{\infty} \frac{1}{\phi - \hat{\phi}} (\phi^n - \hat{\phi}^n) z^n$$

Therefore, the n^{th} term of the Fibonacci sequence can be calculated using

$$f_n = \frac{1}{\phi - \hat{\phi}} (\phi^n - \hat{\phi}^n)$$