**Finding a Stable Limit Cycle**

Consider the system

\[
\begin{align*}
\dot{x} &= x(1 - x) - \frac{xy}{\alpha + x} \\
\dot{y} &= -\beta y + \frac{\gamma xy}{\alpha + x}
\end{align*}
\]

Consider the non-zero point \((x_0, y_0)\). The following conditions hold at such a point:

\[
\begin{align*}
1 - x_0 &= \frac{y_0}{\alpha + x_0} \\
\frac{\beta}{\gamma} &= \frac{x_0}{\alpha + x_0}
\end{align*}
\]

The general Jacobian of the system is

\[
\Gamma = \begin{bmatrix}
1 - 2x - \frac{y}{\alpha + x} + \frac{xy}{(\alpha + x)^2} & -\frac{x}{\alpha + x} \\
\frac{\gamma y}{\alpha + x} - \frac{\gamma xy}{(\alpha + x)^2} & -\beta + \frac{\gamma x}{\alpha + x}
\end{bmatrix}
\]

The Jacobian evaluated at \((x_0, y_0)\) using relations (1) and (2) is

\[
\Gamma_{(x_0,y_0)} = \begin{bmatrix}
\frac{\beta}{\gamma}(1 - x_0) - x_0 & -\frac{\beta}{\gamma} \\
(\gamma - \beta)(1 - x_0) & 0
\end{bmatrix}
\]

Eigenvalues of \(\Gamma_{(x_0,y_0)}\) are solutions to its characteristic polynomial

\[
\lambda^2 + \lambda \left(\frac{\beta}{\gamma}(x_0 - 1) + x_0\right) + \frac{\beta}{\gamma}(\gamma - \beta - \alpha \beta) = 0
\]

The only possible constraints under which a stable limit cycle would appear is when \(\delta > 0\) and \(\tau = 0\). We solve \(\tau = 0\) to obtain the additional condition that

\[
x_0 = \frac{\beta}{\beta + \gamma}
\]

Solving (2) tells us that

\[
x_0 = \frac{\alpha \beta}{\gamma - \beta}
\]

Combining (3) and (4), we discover that

\[
\alpha = \frac{\gamma - \beta}{\gamma + \beta}
\]
Now since $\alpha > 0$, we know that
$$\gamma > \beta$$
(6)

Now to satisfy $\det(\Gamma(x_0, y_0)) > 0$, we want $\frac{\beta}{\gamma}(\gamma - \beta)(1 - x_0) > 0$. Since $\frac{\beta}{\gamma}(\gamma - \beta)$ is always positive, we need to satisfy only that $(1 - x_0) > 0$.

Using (4), we find another condition that
$$\alpha \beta < \gamma - \beta$$
(7)

Conditions (6) and (7) are necessary and sufficient for a stable limit cycle.

To finish off the discussion, we will leave you with the values of $x_0$ and $y_0$.

$$x_0 = \frac{\alpha \beta}{\gamma - \beta}$$
$$y_0 = \frac{\alpha \gamma (\gamma - \beta - \alpha \beta)}{(\gamma - \beta)^2}$$