PATH BUNDLES ON n-CUBES

MATTHEW ELDER

Abstract. A *path bundle* is a set of 2^a paths in an n-cube, denoted Q_n, such that every path has the same length, the paths partition the vertices of Q_n, the endpoints of the paths induce two subcubes of Q_n, and the endpoints of each path are complements. This paper shows that a path bundle exists if and only if $n > 0$ is odd and $0 \leq a \leq n - \lceil \log_2 (n+1) \rceil$.

1. Introduction

Path bundles are closely related to, and emerged from, the theory of binary Gray codes. Introduced as a coding mechanism in [1], binary Gray codes have proven useful in many various technical and mathematical applications. An overview of these applications, along with many generalizations and restrictions of binary Gray codes, is given in [3].

Path bundles are largely a generalization of binary Gray codes. An n-bit binary Gray code is a list of all binary words of length n where every pair of words adjacent in the list differ by only one bit. A path bundle is a set of 2^a paths in an n-cube such that every path has the same length, the paths partition the vertices of the n-cube, the endpoints of the paths induce two a-cubes of the n-cube, and the endpoints of each path are complements.

This paper shows that, for positive integers n and a, there exists in an n-cube a path bundle of 2^a paths if and only if n is odd and $a \leq n - \lceil \log_2 (n+1) \rceil$. The existence result is constructive; this paper an inductive construction of some path bundle for every possible n and a that satisfy these conditions.

Path bundles are likely useful for constructing binary Gray codes with various properties; they are the result of the author's search for a construction of a variant of the Gray codes in [2]. They encapsulate many details into a simple and predictable block; this may prove useful in other constructions of paths, circuits, or sets of paths on n-cubes.

2. Preliminaries

In Section 3 we provide conditions on path bundles necessary for their existence, and in Section 4 we show that these conditions are sufficient. First, we need some notation and definitions. We use the following definitions and notation when dealing with binary words:

- *Bits* are binary digits, with possible values `0` and `1`.
- *Words* are strings of bits; `0011`, `0110`, and `1001` are distinct words.
- The symbol $\oplus$ denotes addition in $\mathbb{Z}_2^n$ (the exclusive-or operation). For example, $\texttt{000111} \oplus \texttt{110011} = \texttt{110100}$.
- Denote a repeated bit or symbol with an exponent, like so: $\texttt{00000} = \texttt{0}^5$, and $\texttt{01}^3 = \texttt{0111}$.

- If a and b are words, ab denotes the concatenation of a and b. So, if $a = \texttt{11}$ and $b = \texttt{0}$, then $aab = \texttt{11110}$ and $b^3a = \texttt{00011}$.
- Q_n is the set of all n-bit words.
- Two words w and v are *adjacent*, denoted $w \sim v$, if they differ by exactly one bit. So, `0101` and `1101` are adjacent. `0101` and `0110` are *not* adjacent.
- Two words are *complements* if they differ in every bit. The complement of a word a is denoted $\overline{a}$. For example, $\overline{\texttt{0010}} = \texttt{1101}$.
- Counting from 0, w_j is the j^{th} bit of w. If $w = \texttt{00100}$, then $w_0 = \texttt{0}$, $w_2 = \texttt{1}$, and w_5 is undefined.

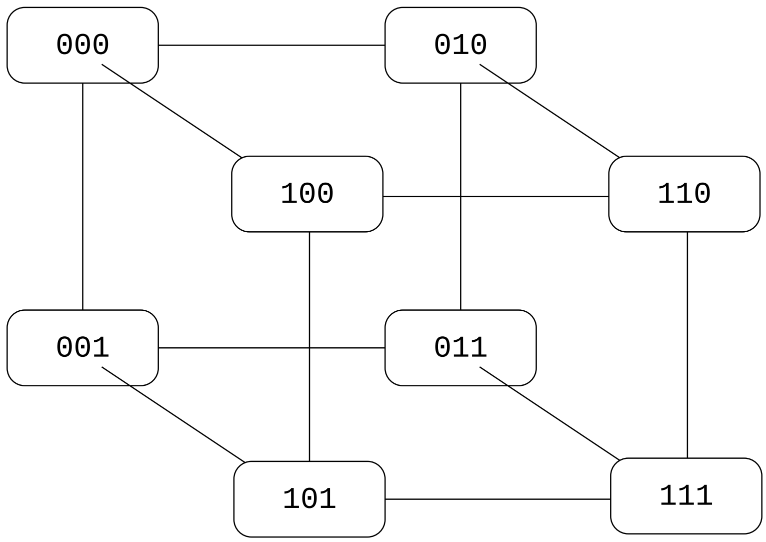


FIGURE 1. The words of Q_3 and the 3-cube are isomorphic.

Notice that Q_n and its adjacencies are isomorphic to the graph of an n-cube. Figure 1 shows an example of this isomorphism. This isomorphism will not be useful for formal mathematics, but it guides both our intuition and our illustrations.

Rather than illustrating complex n-cubes in the manner of Figure 1, we instead illustrate Q_n in the manner of Figure 2. In this illustration, adjacent elements of Q_n are placed in corresponding positions in adjacent rectangles. We consider two rectangles adjacent if they share an edge unused by a larger, bolder rectangle.

For example, in Figure 2, `0001` is adjacent to `0000` and `0101`, because they are at (trivially) corresponding positions of adjacent rectangles in the larger, upper-left square. However, `0001` is not adjacent to `0010`, because these small rectangles are not inside the same major subdivision; the edge they share is shared also by the upper pair of large rectangles. Also, `0001` is in the upper-right-hand corner of a 2-by-2 square, so it corresponds to `0011` and `1001`.

This manner of illustration profits from the recursive nature of the n-cube; the n-cube is composed of two $n-1$-cubes whose corresponding vertices are connected. Crucially for our illustrations, an n-cube is also composed of four $n-2$-cubes, connected together in a square. Figure 3 illustrates a 7-cube (Q_7) as composed of four 5-cubes (Q_5). We will illustrate cubes this way throughout this paper.

0000	0001	0010	0011
0100	0101	0110	0111
1000	1001	1010	1011
1100	1101	1110	1111

FIGURE 2. A 4-cube (Q_4) labelled with the words it represents.

Q_5	Q_5
Q_5	Q_5

FIGURE 3. A 7-cube (Q_7) shown as composed of four 5-cubes (Q_5).

Complements are illustrated as points diagonally opposite each other inside an n-cube. In the Q_n illustrated in Figure 4, A and B are complements in the top-left Q_{n-2}, C and D are complements in the top-right Q_{n-2}, and A and E are complements in the full Q_n.

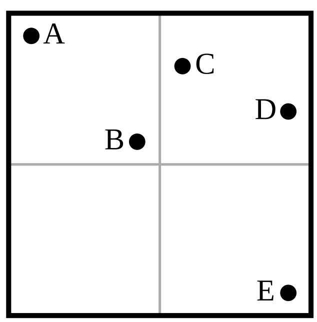


FIGURE 4. Complementary words in Q_n and Q_{n-2}.

To rigorously define path bundles, we need to rigorously define paths. The following definition should put words and notation around the most intuitive notion of a directed path in Q_n:

Definition 1 (Path)**.** *A* path P*, inside* Q_n*, is a sequence of* m *words* $P(0), P(1), P(2), \ldots P(m-1)$ *such that:*

(1) *For all* i *and* j*,* $0 \leq i < j < m$*,* $P(i) \neq P(j)$*, and*
(2) *For all* i*,* $0 \leq i < m$*,* $P(i) \sim P(i+1)$*.*

Notice that such a path is equivalent to a directed path on the graph of the n-cube, where $P(0)$ is the head vertex and $P(m-1)$ is the tail. For convenience, if we say that a path P is equal to some m-tuple of words, then $P(i)$ is the $(i+1)^{\text{th}}$ word in that tuple. If $P = (\texttt{00}, \texttt{01}, \texttt{11})$, then $P(0) = \texttt{00}$, $P(1) = \texttt{01}$, and $P(2) = \texttt{11}$.

We will tacitly apply single-word operators to entire paths. If P is a path on Q_n and w is a word in Q_n, then $R = P \oplus w$ satisfies $R(i) = P(i) \oplus w$. Similarly, if u is any word, $R = uP$ satisfies $R(i) = uP(i)$. These constructions make sense so long as only one path is involved.

Some useful notation on paths:

- The number of words in the path P is $|P|$. In graph-theoretic notation, $|P|$ is one greater than the length of P.
- For clarity and concision, we will denote $P(|P|-1)$, the last word in path P, with the notation $P(\cdot)$.
- For two paths P and R, with $P(\cdot) \sim R(0)$, let $P \rightharpoonup R$ be the path formed by following P, then following R. Formally, this concatenation operation is defined as follows:

$$(P \rightharpoonup R)(i) = \begin{cases} P(i), & 0 \leq i < |P| \\ R(i-|P|), & |P| \leq i < |P|+|R| \end{cases} \tag{1}$$

  For example, if $P = (\texttt{00}, \texttt{01})$ and $R = (\texttt{11}, \texttt{10})$, then $P \rightharpoonup R = (\texttt{00}, \texttt{01}, \texttt{11}, \texttt{10})$. Notice that the concatenation of paths is not symmetric; in fact, $P \rightharpoonup R$ and $R \rightharpoonup P$ are both well-defined only in special cases.
- We denote the reversal of path P with P'. If P is a path, then P' satisfies $P'(i) = P(|P|-1-i)$.

Finally, we can rigorously define path bundles, the objects of study in this paper:

Definition 2 (Path Bundle)**.** *A* path bundle $\mathfrak{B}(n,a)$ *is a set of* 2^a *paths in* Q_n*,* $n > 0$ *and* $a \geq 0$*. For each word* $w \in Q_a$*, the bundle contains the path* P_w*, which has the following properties:*

(1) $|P_w| = 2^{n-a}$*,*
(2) $P_w(0) = \texttt{0}^{n-a}w$*,*
(3) $P_w(\cdot) = \texttt{1}^{n-a}\overline{w}$*, and*
(4) *every word of* Q_n *is in exactly one path of* $\mathfrak{B}(n,a)$*.*

Each path in the set $\mathfrak{B}(n,a)$ starts at a specified point in one 'corner' of the n-cube, and ends at the complementary point in the opposite 'corner' of the n-cube. Taken together, the paths partition the words of Q_n into 2^{n-a} sets of size 2^a. Figure 5 illustrates an example of a path bundle where $n = 5$ and $a = 1$.

3. Restrictions

Lemma 3. *If* $\mathfrak{B}(n,a)$ *exists, then* n *is odd.*

Proof. Call a word *even* if it contains an even number of $\texttt{1}$s, and *odd* if it contains an odd number of $\texttt{1}$s. Since exactly one bit differs between any two consecutive

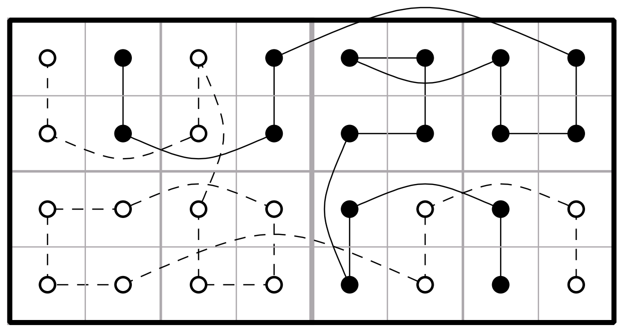


FIGURE 5. $\mathfrak{B}\,(5,1)$: a path bundle in Q_5 with two paths.

words in a path, then $P\,(i) \oplus P\,(i+1)$ is odd. More generally, $P\,(i) \oplus P\,(i+k)$ is odd if and only if k is odd.

If P is a path in a path bundle, then $|P| = 2^{n-a}$. Because $2^{n-a} - 1$ is odd, $P\,(0) \oplus P\,(\cdot)$ is odd. The endpoints of a path in a path bundle are complements, so $P\,(0) \oplus P\,(\cdot) = \mathtt{1}^n$. Thus $\mathtt{1}^n$ is odd, so n must be odd. $\square$

Lemma 4. *If $\mathfrak{B}\,(n,a)$ exists, then $a \leq \lceil \log_2 (n+1) \rceil$.*

Proof. The endpoints of each path are complements, and only one bit may differ between $P(i)$ and $P(i+1)$, so each path must contain at least $n+1$ words. To accomodate that many words in each of the 2^a paths, $2^n \geq 2^a\,(n+1)$. Hence, $a \leq n - \lceil \log_2 (n+1) \rceil$. $\square$

4. CONSTRUCTIONS

By the end of this section, we will prove the following theorem.

Theorem 5. *If n is an odd, positive integer and a is a nonnegative integer satisfying $2^{n-a} \geq n+1$, then a path bundle $\mathfrak{B}\,(n,a)$ exists.*

After the following series of lemmas, we will be able to prove Theorem 5 inductively. The base case is trivial.

The next two lemmas construct all bundles except where a is less than its maximum possible value. For these constructions, we will need the well-known binary-reflected Gray code, first defined in [1]. Let G_n denote the binary-reflected Gray code on n bits. This is a path that traverses every element in Q_n where the first element and the last element are adjacent. We can permute these endpoints as we like without changing any important properties, so let us further specify that $G_n(0) = \mathtt{0}^n$ and $G_n\,(\cdot) = G_n(2^n - 1) = \mathtt{0}^{n-1}\mathtt{1}$.

Lemma 6. *If $\mathfrak{B}\,(n-2,0)$ exists, then $\mathfrak{B}\,(n,0)$ exists.*

Proof. Since a is 0, both $\mathfrak{B}\,(n-2,0)$ and $\mathfrak{B}\,(n,0)$ contain just one path. Let P be the single path contained in $\mathfrak{B}\,(n-2,0)$. By definition, $P(0) = \mathtt{0}^{n-2}$ and $P\,(\cdot) = \mathtt{1}^{n-2}$.

The path $R = \mathtt{0}G_{n-1} \rightharpoonup \mathtt{10}G'_{n-2} \rightharpoonup \mathtt{11}P$ is the single path we seek. This construction is illustrated by Figure 6.

To prove this, we show that R is indeed a path, and that it satisfies the definition of $\mathfrak{B}\,(n,0)$. The expression for R is well-defined if the tail of each piece of R is

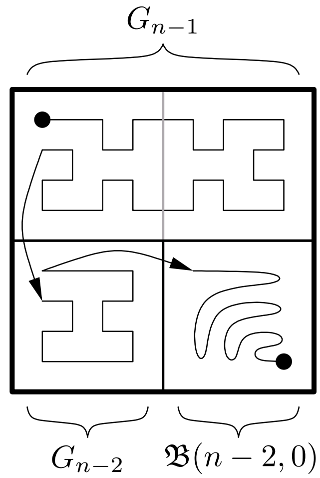


FIGURE 6. A $\mathfrak{B}(n,0)$ constructed from $\mathfrak{B}(n-2,0)$.

adjacent to the head of the next piece. We check these adjacencies, and find that

$$0G_{n-1}(\cdot) = 0^{n-1}1 \sim 10^{n-2}1 = 10G'_{n-2}(0)\,, \text{ and} \tag{2}$$

$$10G'_{n-2}(\cdot) = 10^{n-1} \sim 110^{n-2} = 11P(0)\,. \tag{3}$$

Thus, R is a well-defined path.

Clearly, R has size 2^n, and it's easy to check that $R(0) = 0^n$ and $R(\cdot) = 1^n$. Every word in Q_n is in R exactly once, because G_{n-1} contains every word in Q_{n-1}, G_{n-2} contains every word in Q_{n-2}, and P contains every word in Q_{n-2} by its definition. So, the singleton set containing R forms $\mathfrak{B}(n,0)$. $\square$

To clarify this construction, let us examine how $\mathfrak{B}(5,0)$ is constructed from $\mathfrak{B}(3,0)$. The path $0G_4$ begins with 00000, ends with 00001, and contains every word in Q_5 that begins with 0. Next, the path $10G'_3$ begins with 10001, ends with 10000, and contains every word in Q_5 that begins with 10. Finally, if P is the single path in $\mathfrak{B}(3,0)$, then $11P$ begins with 11000, ends with 11111, and contains every word in Q_5 that begins with 11. The construction of $\mathfrak{B}(5,0)$ is shown in Figure 7.

Lemma 7. *If $\mathfrak{B}(n-2,a-1)$ exists, then $\mathfrak{B}(n,a)$ exists.*

Proof. For each $v \in Q_{a-1}$, let P_v be the path in $\mathfrak{B}(n-2,a-1)$ that starts with $0^{n-a+1}v$.

Define R_w for each $w \in Q_a$ as follows:

$$R_w = 00G_{n-a-2}w \rightharpoonup 01P_{w_{1:|w|}} \oplus 0^{n-a-1}1w_0 0^{a-1} \rightharpoonup 11\overline{G'_{n-a-2}w}. \tag{4}$$

This construction is illustrated in Figure 8. To see that each R_w is a path, we need to check the endpoints around the concatenation of paths; it is straightforward, if tedious, to do so.

To see that the set $\{R_w : w \in Q_a\}$ form a $\mathfrak{B}(n,a)$, note that each word in Q_n is used because G_{n-a-2} fills up those subcubes of Q_n where words begin with 00 or 11, and $P_{w_{1:|w|}}$ fills the others. Also, the size of each R_w is $|G_{n-a-2}| + \left|P_{w_{1:|w|}}\right| + \left|G'_{n-a-2}\right| = 2^{n-a}$, and it is easy to see that $R_w(0) = 0^{n-a}w$ and that $R_w(\cdot) = 1^{n-a}\overline{w}$. So, the set $\{R_w : w \in Q_a\}$ is a $\mathfrak{B}(n,a)$. $\square$

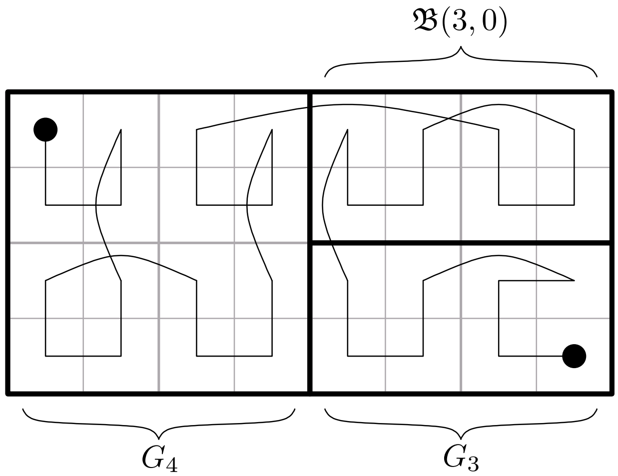


FIGURE 7. A $\mathfrak{B}\,(5,0)$ constructed from $\mathfrak{B}\,(3,0)$.

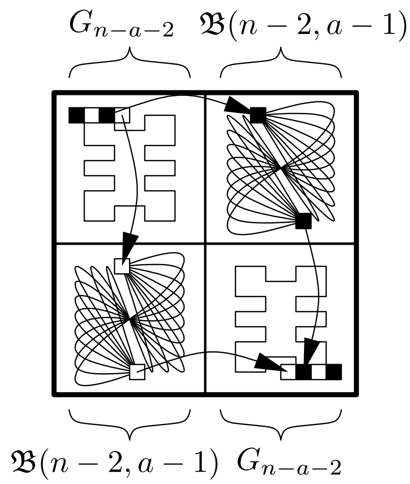


FIGURE 8. A $\mathfrak{B}\,(n,a)$ constructed from $\mathfrak{B}\,(n-2,a-1)$.

Before we proceed to the final lemma, we need to introduce the concept of near-bundles.

Definition 8 (Near-Bundle)**.** *The notation* $\mathfrak{B}'\,(n,a)$ *denotes a* near-bundle*. A near-bundle is a set of paths,* P_w*, where:*

(1) $|P_w| = 2^{n-a}$,
(2) $P_w\,(0) = \mathtt{0}^{n-a}w$,
(3) $P_w\,(\cdot) = \mathtt{01}^{n-a-1}w$*, and*
(4) *Every word or* Q_n *is in exactly one of these paths.*

Near-bundles differ from path bundles only in that the endpoints of each path in a near-bundle have the same leftmost bit. We could construct these structures in the same depth as we construct path bundles, but we need only the following lemma about them.

Lemma 9. *If* $\mathfrak{B}(n-1,a)$ *exists, then* $\mathfrak{B}'(n,a)$ *also exists.*

Proof. Let P_w be the path in $\mathfrak{B}(n-1,a)$ with the head $\mathtt{0}^{n-a-1}w$. For each $w \in Q_a$, we construct R_w as follows:

$$R_w = \mathtt{00}G_{n-a-2}w \rightharpoonup \mathtt{1}P_w \oplus \mathtt{0}^{n-a-1}\mathtt{1}\mathtt{0}^{a} \rightharpoonup \mathtt{01}\overline{G'_{n-a-2}w} \tag{5}$$

Again, checking that R_w is a well-defined path is a technical exercise. Every word in Q_n that starts with $\mathtt{0}$ is used by the G_{n-a-2} or G'_{n-a-2} pieces of R_w, and every word in Q_n that starts with $\mathtt{1}$ is used by the P_w piece of R_w. The size of R_w is $|G_{n-a-2}| + |P_w| + |G'_{n-a-2}| = 2^{n-a}$. Finally, it is clear that $R_w(0) = \mathtt{0}^{n-a}w$ and $R_w(\cdot) = \mathtt{01}^{n-a-1}\overline{w}$. So, the set $\{R_w : w \in Q_a\}$ forms the near-bundle $\mathfrak{B}'(n,a)$. $\square$

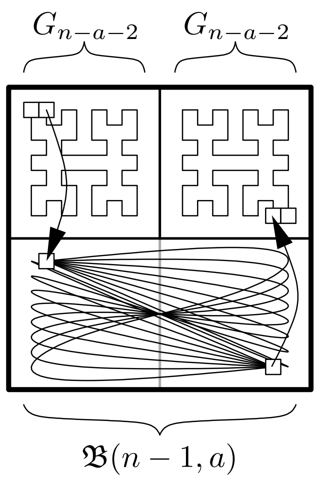


FIGURE 9. A $\mathfrak{B}'(n,a)$ constructed from $\mathfrak{B}(n-1,a)$.

Figure 9 illustrates this method for constructing $\mathfrak{B}'(n,a)$: the binary reflected Gray paths fill one half of the n-cube, and the $\mathfrak{B}(n-1,a)$ fills the other half. The second Gray path undoes the system-wide bit flip caused by the first Gray path, so the two Gray paths taken together merely fill space.

Also, note that, although the definition of near-bundle stipulates that the left-most bit remains the same between the head and tail of any path, this position is merely a matter of notational convenience. A path remains a path after its bits are permuted, as long as the same permutation is applied to every word. Thus, we can choose any bit i to be that unchanged position.

The following theorem will yield the path bundle on Q_n containing the greatest possible number of paths.

Theorem 10. *For odd* n*, if* $\mathfrak{B}(m,k)$ *exists for all* m *and* k *where* m *is odd,* $0 < m < n$ *and* $0 \le k \le \lceil \log_2(m+1) \rceil$*, then the bundle* $\mathfrak{B}(n, n - \lceil \log_2(n+1) \rceil)$ *exists.*

Proof. We will prove Theorem 10 by construction. For convenience, let $a = n - \lceil \log_2(n+1) \rceil$ and $b = \lceil \log_2(n+1) \rceil$. In the bundle we aim to construct the head of every path is $\mathtt{0}^b w$, where w is a bits long. In this construction, we will build two sets of paths, R and S. The heads of the paths in R are of the form $\mathtt{0}^b w$, and the heads of the paths in S are of the form $\mathtt{1}^b w$. From these starting points, the two paths will simultaneously grow towards each other in a twisty, complex manner. When they fuse, the result will be a $\mathfrak{B}(n,a)$.

Define c by

$$c = 2^b - n - 1,$$

and let c_i be the i^{th} bit in the $(b-1)$-bit binary representation of c, starting with c_0 as the rightmost bit and padding leftmost bits with zeroes as necessary.

Define n_i and a_i as follows. For $0 \le i < b-1$:

$$n_i = 2^{i+1-c_i} + (c_i - 1)\,, \tag{6}$$

$$a_i = 2^{i+1-c_i} + (c_i - 1) - (i+1)\,. \tag{7}$$

So, for example, if n is 37, then:

$$\begin{aligned} a &= 31, \\ b &= 6, \\ c &= 26 = 11010_2, \end{aligned}$$

$$\begin{aligned} &c_0 = 0,\ \ c_1 = 1,\ \ c_2 = 0,\ \ c_3 = 1,\ \ c_4 = 1, \\ &n_0 = 1,\ n_1 = 2,\ n_2 = 7,\ n_3 = 8,\ n_4 = 16, \\ &a_0 = 0,\ n_1 = 0,\ n_2 = 4,\ n_3 = 4,\ n_4 = 11. \end{aligned}$$

It is reasonably straightforward to see that the binary representation of c fits into $b-1$ bits, that $n_i - a_i = i+1$, and that $\sum_{i=0}^{b-2}(n_i - c_i) = a$. Since c is always even, $c_0 = 0$, $n_0 = 1$ and $a_0 = 0$.

We will need to consider the words of Q_n in sections. Define the *front*, *visible*, and *hidden* subwords of w, denoted by $F(w)$, $V_i(w)$, and $H_i(w)$, as follows:

$$w = F(w)\,V_0(w)\cdots V_{b-2}(w)\,H_1(w)\cdots H_{b-2}(w)\,,\ \text{where} \tag{8}$$

$$|F(w)| = b, \tag{9}$$

$$|V_i(w)| = i + 1 - c_i,\ \text{and} \tag{10}$$

$$|H_i(w)| = a_i. \tag{11}$$

The hidden subword $H_0(w)$ is omitted because $|H_0(w)| = 0$.

We will also need the *signature* of $w \in Q_a$, which is denoted $\operatorname{sig}(w)$.

$$\operatorname{sig}(w) = \bigoplus_{i=0}^{b-2} \mathtt{0}^{b-2-i} V_i\left(\mathtt{0}^b w\right) \mathtt{0}^{c_i} \tag{12}$$

Since $|V_i(w)| = i + 1 - c_i$, each term in this exclusive-or summation is a word of length $b-1$.

Finally, we extend the bundle notation in this construction. Define $\mathfrak{B}(x,y)(w)$ to be the path in $\mathfrak{B}(x,y)$ that starts with the word w. Similarly extend the notation of near-bundles.

Now, the actual construction may proceed. For each $w \in Q_a$, let $R_{w,0} = \left(\mathtt{0}^b w\right)$ and $S_{w,0} = \left(\mathtt{1}^b \overline{w}\right)$.

We define $x(w,i)$ to be a word in Q_n as follows:

$$x(w,i) = \begin{cases} S_{w,i}(\cdot) \oplus \mathtt{0}^{b-i+1}\mathtt{1}\mathtt{0}^{i+a}, & \operatorname{sig}(w)_i = \mathtt{0}, \\ R_{w,i}(\cdot) \oplus \mathtt{0}^{b-i+1}\mathtt{1}\mathtt{0}^{i+a}, & \operatorname{sig}(w)_i = \mathtt{1}. \end{cases} \tag{13}$$

For convenience, we also define $y(w,i)$ to be the word in Q_n such that $H_i(y(w,i)) = H_i(x(w,i))$, and every other subword of $y(w,i)$ is all zeroes.

Next, we define $P_{w,i}$, a short path. The front, visible, and hidden subwords of every word in $P_{w,i}$ are composed entirely of zeroes except for $V_i(P_{w,i})$, $H_i(P_{w,i})$, and possibly $V_0(P_{w,i})$.

If $c_i = 0$, then:

$$V_i(P_{w,i})\, H_i(P_{w,i}) = \mathfrak{B}(n_i, a_i)\left(\mathtt{0}^{i+1} H_i(x(w,i))\right) \oplus y(w,i)\,. \tag{14}$$

$$V_0(P_{w,i}) = \mathtt{0}, \text{ unless } i = 0. \tag{15}$$

If $c_i = 1$ instead, then:

$$V_0(P_{w,i})\, V_i(P_{w,i})\, H_i(P_{w,i}) = \mathfrak{B}'(n_i, a_i)\left(\mathtt{0}^{i+1} H_i(x(w,i))\right) \oplus y(w,i)\,. \tag{16}$$

Define $r_{w,i}$ and $s_{w,i}$, the *extensions* of $R_{w,i}$ and $S_{w,i}$, as follows:

$$r_{w,i} = \begin{cases} ()\,, & \operatorname{sig}(w)_i = \mathtt{0}. \\ x(w,i) \oplus P_{w,i}, & \operatorname{sig}(w)_i = \mathtt{1}. \end{cases} \tag{17}$$

$$s_{w,i} = \begin{cases} x(w,i) \oplus P_{w,i}, & \operatorname{sig}(w)_i = \mathtt{0}. \\ ()\,, & \operatorname{sig}(w)_i = \mathtt{1}. \end{cases} \tag{18}$$

Finally, we set the values of $R_{w,i+1}$ and $S_{w,i+1}$ by:

$$R_{w,i+1} = R_{w,i} \rightharpoonup r_{w,i}. \tag{19}$$

$$S_{w,i+1} = S_{w,i} \rightharpoonup s_{w,i}. \tag{20}$$

To complete the construction, we create the set of paths T_w as follows:

$$T_w = R_{w,b-1} \rightharpoonup S'_{w,b-1}. \tag{21}$$

As we will soon show, this set is a $\mathfrak{B}(n,a)$. Notice that Equations 13, 17, and 18 maintain a sort of symmetry between R and S and their signatures. This symmetry means we can abbreviate the proofs of the following claims, as the same logic that applies to R will also apply to S.

Figure 10 gives a high-level perspective on what this path bundle looks like for $n = 15$ and $a = 11$. The 16 large squares are each a Q_{11}; connected, they form the Q_{15}. The front subwords of the words in each of these cubes are constant, and every possible selection of visible and hidden words is represented somewhere in each. Similarly, the small white and gray squares are slices of groups of $2^5 = 32$ paths each. The front and visible subwords within each square are constant, while all possible hidden words are represented. Except for the beginning and ending Q_{11}, every small gray square is paired with a small white square in a larger box. This box represents those words traversed by the sets of path bundles that connect the gray square to the white square. Finally, the key at the bottom of Figure 10 shows how groups of paths move from one Q_{11} to another. An arrow pointing from a source Q_{11} to a destination Q_{11} in the key means that some (or all) white squares in the source Q_{11} are connected along paths in the bundle to corresponding gray squares in the destination Q_{11}.

To help illustrate the diagram itself, one set of 32 paths is traced through the entire bundle. Every small square marked with $\times$ lies on the same set of paths. It may help to note that, other than the first and last Q_{11}, there are three different types of Q_{11}, each type subdivided differently. Every path passes through exactly one Q_{11} of each type.

To prove that the set of paths T_w is actually a $\mathfrak{B}(n,a)$, we need to show that all of the operations in its construction are well-formed and that it satisfies the path bundle definition.

Claim 11. *When used in this construction, $\mathfrak{B}(n_i,a_i)$ and $\mathfrak{B}'(n_i,a_i)$ exist.*

Proof. The statement of this theorem includes the assumption that $\mathfrak{B}(m,k)$ exists for all m and k where m is odd, $0<m<n$, and $0 \le k \le \lceil \log_2(m+1) \rceil$. It is clear that $0<n_i<n$ for all n_i.

When $c_i = 0$, Equation 16 the bundle $\mathfrak{B}(n_i,a_i)$. In these cases, $n_i = 2^{i+1}-1$ and $a_i = 2^{i+1}-1-(i+1)$. Clearly, n_i is odd. In this construction, i is never greater than $b-2$, so $n_i \le 2^{b-1}-1 = 2^{\lceil \log_2(n+1)\rceil -1}-1 < n$. Finally, a little manipulation shows that $0 \le a_i = n_i - \lceil \log_2(n_i+1)\rceil$. So, $\mathfrak{B}(n_i,a_i)$ is well defined in these cases.

When $c_i = 1$, Equation 14 references the near-bundle $\mathfrak{B}'(n_i,a_i)$. In these cases, $n_i = 2^i$ and $a_i = 2^i-(i+1)$. Lemma 9 shows that this near-bundle exists as long as $\mathfrak{B}(n_i-1,a_i)$ exists. We already know that $c_0 = 0$, so in this case $0 < i \le b-2$. For these values of i, $n_i - 1$ is clearly odd, $n_i - 1 < n$, and some manipulation shows, again, that $0 \le a_i = \lceil \log_2(n_i-1+1)\rceil$. Thus, in these cases $\mathfrak{B}(n_i-1,a_i)$ exists, so $\mathfrak{B}'(n_i,a_i)$ exists. □

Definition 12 ($w_{j:k}$)**.** *The subword of bits in w that starts at bit j and ends just before bit k is $w_{j:k}$. If $w = \mathtt{00100}$, then $w_{0:2} = \mathtt{00}$, $w_{2:5} = \mathtt{100}$, and $w_{3:3}$ is the empty word. Note the identity $w_{0:|w|} = w$.*

With this subword notation, we can clearly state the following claim.

Claim 13. *We know the following about the front subword of the tail of each path:*

$$F(R_{w,i}(\cdot)) = 0^{b-i}\mathit{sig}(w)_{b-i:b}\,, \text{ and} \tag{22}$$

$$F(S_{w,i}(\cdot)) = 1^{b-i}\overline{\mathit{sig}(w)_{b-i:b}}. \tag{23}$$

Proof. When $i=0$, $F(R_{w,0}(\cdot)) = \mathtt{0}^b$ and $F(S_{w,0}(\cdot)) = \mathtt{1}^b$. Thus, the claim holds for $i=0$. Inductively assume that the claim is true for some $i<b$.

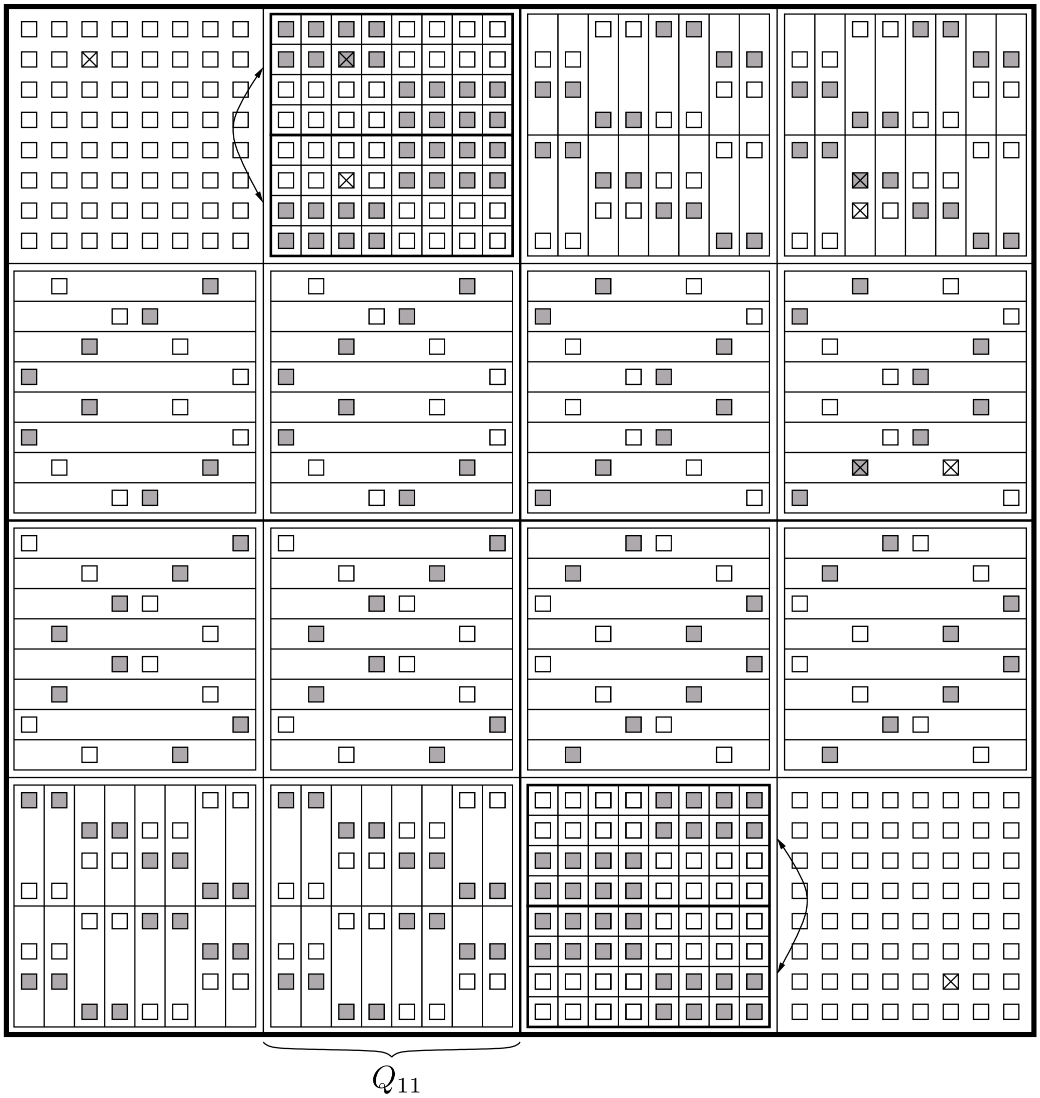
Suppose $\operatorname{sig}(w)_i = \mathtt{0}$. Then, $R_{w,i+1}(\cdot) = R_{w,i}(\cdot)$, which is satisfactory. The S paths are more complicated;

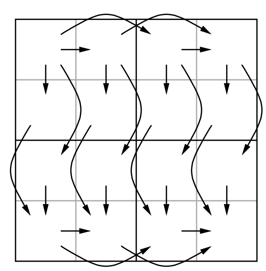
$$\begin{aligned} F(S_{w,i+1}(\cdot)) &= F(x(w,i)) \oplus F(P_{w,i}) \\ &= F(S_{w,i}(\cdot)) \oplus F(P_{w,i}) \\ &= 1^{b-i}\overline{\operatorname{sig}(w)_{b-i:b}} \oplus \mathtt{0}^{b-i+1}\mathtt{1}\mathtt{0}^{i+a} \\ &= 1^{b-i-1}\overline{\operatorname{sig}(w)_{b-i-1:b}}. \end{aligned} \tag{24}$$

If $\operatorname{sig}(w)_i = \mathtt{1}$, the same reasoning applies symetrically. So, this claim is true by induction. □

Claim 14. *For any $0 \le i < b-1$, no two paths in $\{R_{w,i} : w \in Q_a\}$ or $\{S_{w,i} : w \in Q_a\}$ include the same word.*

Proof. We prove this claim by induction on i. The claim is satisfied for $i=0$, because $R_{w,0}$ and $S_{w,0}$ are singleton paths, distinct for each w.





Connections between Q_{11}s

FIGURE 10. A $\mathfrak{B}\,(15, 11)$, made by this construction.

For any $w \in Q_a$ and any $0 \le k < b-1$, the front subwords in $P_{w,k}$ are all $\mathtt{0}^b$, and the leftmost bit of $x(w,k)$ is $\mathtt{0}$. This means that nothing ever changes the leftmost bit in $R_{w,k}$ or $S_{w,k}$, so the leftmost bit of every word in $R_{w,k}$ is $\mathtt{0}$, and the leftmost bit of every word in $S_{w,k}$ is $\mathtt{1}$. Clearly, then, no word is in both an R path and an S path.

Assume that the claim is true for i, and focus just on the R paths. If $\operatorname{sig}(w)_i = 0$, then $R_{w,i+1} = R_{w,i}$. By our inductive assumption, $R_{w,i+1}$ intersects no $R_{v,i}$ for any $v \in Q_a$. If, instead, $\operatorname{sig}(w)_i = 1$, then there are words in $r_{w,i}$. By Claim 13 and the definition of $P_{w,i}$, the front subwords of $r_{w,i}$ are constant and different from the front subwords in every $R_{v,i}$ for any $v \in Q_n$.

Suppose that $R_{w,i+1}$ and $R_{v,i+1}$ intersect somewhere, with $w \neq v$. From the above reasoning, this is possible only if $\operatorname{sig}(w)_i = \operatorname{sig}(v)_i = 1$, and their intersection is in $r_{w,i}$ and $r_{v,i}$. By the inductive assumption we know that $R_{w,i}(\cdot) \neq R_{v,i}(\cdot)$, which implies that $r_{w,i}(0) \neq r_{v,i}(0)$. By the construction of $P_{w,i}$, the only subwords that vary in these extensions are V_i, H_i, and V_0, so if these two extensions intersect, all other subwords must be the same between both extensions. That is, $F(r_{w,i}) = F(r_{v,i}), V_j(r_{w,i}) = V_j(r_{v,i})$, and $H_j(r_{w,i}) = H_j(r_{v,i})$ for all j not equal to 0 or i.

Equation 13 is the only place in the entire construction where front subwords vary, so the front subwords of these extensions are determined by the signatures of w and v. Since $F(r_{w,i}) = F(r_{v,i})$, we know $\operatorname{sig}(w)_{0:i+1} = \operatorname{sig}(v)_{0:i+1}$.

From this, we know that $\bigoplus_{j=0}^{i-j} V_j\left(\mathtt{0}^b w\right)\mathtt{0}^{c_j} = \bigoplus_{j=0}^{i} \mathtt{0}^{i-j} V_j\left(\mathtt{0}^b v\right)\mathtt{0}^{c_j}$. We already determined that most of these visible subwords are equal; factoring them away leaves

$$V_i\left(\mathtt{0}^b w\right)\mathtt{0}^{c_i} \oplus V_0\left(\mathtt{0}^b w\right)\mathtt{0}^{c_i} = V_i\left(\mathtt{0}^b v\right)\mathtt{0}^{c_i} \oplus V_0\left(\mathtt{0}^b v\right)\mathtt{0}^{c_i}. \tag{25}$$

If $c_i = 0$, then Equation 15 implies that the zeroth visible subword is constant within both $r_{w,i}$ and $r_{v,i}$. Again, since we assumed that these extensions intersect somewhere, this implies that $V_0\left(\mathtt{0}^b w\right) = V_0\left(\mathtt{0}^b v\right)$. So, Equation 25 factors further, resulting in $V_i\left(\mathtt{0}^b w\right) = V_i\left(\mathtt{0}^b v\right)$. So, every visible subword in w and v are the same.

If $c_i = 1$, then Equation 25 factors into $V_i\left(\mathtt{0}^b w\right) V_0\left(\mathtt{0}^b w\right) = V_i\left(\mathtt{0}^b v\right) V_0\left(\mathtt{0}^b v\right)$, which also implies that every visible subword in w and v are the same.

So, w and v differ only in their i^{th} hidden subword. But, P is constructed so that if w and v differ only by the i^{th} hidden subword, then $r_{w,i}$ and $r_{v,i}$ are different paths in the same bundle or near-bundle. Since the non-intersection of its component paths is among the defining characteristics of bundles and near-bundles, $r_{w,i}$ and $r_{v,i}$ cannot intersect at all.

This contradicts the previous supposition that $R_{w,i+1}$ and $R_{v,i+1}$ intersect for some $w \neq v$. Thus, no paths of the form $R_{w,i+1}$ intersect each other. By induction, no two paths of the form $R_{w,i}$ intersect for any i, and the claim is proven for all R paths.

The case for S paths is symmetrical, so the claim is true for paths in S and R. □

Claim 15. *For any $w \in Q_a$ and any $0 \leq i \leq b-2$, let $D_i = R_{w,i}(\cdot) \oplus S_{w,i}(\cdot)$. D_i satisfies:*

$$\begin{aligned} F(D_i) &= \mathtt{1}^{b-i}\mathtt{0}^i \\ V_j(D_i) &= \mathtt{1}^{j+1-c_j}, j < i \\ V_j(D_i) &= \mathtt{0}^{j+1-c_j}, j \geq i \\ H_j(D_i) &= \mathtt{1}^{a_j}, j < i \\ H_j(D_i) &= \mathtt{0}^{a_j}, j \geq i. \end{aligned}$$

Proof. We prove this claim by induction on i. It is easy to see that the claim is satisfied for $i = 0$, because $D_0 = \mathtt{1}^n$. So, we assume that the claim is true for all $i < j$, and we will show that the claim is true for $i = j$.

If $\operatorname{sig}(w)_k = 0$, then $R_{w,i} = R_{w,i-1}$, so $D_i = R_{w,i-1}(\cdot) \oplus S_{w,i}(\cdot) = D_{i-1} \oplus S_{w,i-1}(\cdot) \oplus S_{w,i}(\cdot)$. By our construction of S, we can see that $S_{w,i-1}(\cdot) \oplus S_{w,i}(\cdot) = t$ consists entirely of zeroes, except that:

$$\begin{aligned} F(t) &= \mathtt{0}^{b-i}\mathtt{1}\mathtt{0}^{i-1} \\ V_i(t) &= 1^{i+1}, \text{ and} \\ H_i(t) &= 1^{a_i}. \end{aligned}$$

So, $D_i = D_{i-1} \oplus t$ can easily be seen to satisfy our claim.

Symmetrically, the same logic applies if $\operatorname{sig}(w)_k = 1$; we need only interchange R and S. So, by induction, we know that D_i will always satisfy the claim. □

The importance of Claim 15 is that it implies that $R_{w,b-2}(\cdot) \oplus S_{w,b-2}(\cdot) = \mathtt{1}\mathtt{0}^{n-1}$. This means that $R_{w,b-2}(\cdot) \sim S_{w,b-2}(\cdot)$. By Claim 14, $R_{w,b-2}$ and $S_{w,b-2}$ have no words in common, so we can form the path $T_w = R_{w,b-2} \rightharpoonup S'_{w,b-2}$.

We will now show that the set $T = \{T_w : w \in Q_a\}$ is a $\mathfrak{B}(n, a)$.

We can see that the size of T_w is 2^b by a simple induction on the construction. We know that $|R_{w,0}| + |S_{w,0}| = 2$. Assume that $|R_{w,i}| + |S_{w,i}| = 2^{i+1}$. By the construction, we can see that

$$\begin{aligned} (|R_{w,i+1}| + |S_{w,i+1}|) - (|R_{w,i}| + |S_{w,i}|) &= |r_{w,i}| + |s_{w,i}| \\ &= |P_{w,i}| = 2^{n_i - a_i} = 2^{i+1}, \text{ so} \\ |R_{w,i+1}| + |S_{w,i+1}| &= |R_{w,i}| + |S_{w,i}| + |P_{w,i}| \\ &= 2^{i+1} + 2^{i+1} = 2^{i+2}. \end{aligned} \tag{26}$$

So by induction, $|T_w| = |R_{w,b-1}| + \left|S'_{w,b-1}\right| = |R_{w,b-1}| + |S_{w,b-1}| = 2^b$.

Since there are 2^a paths in T, each path contains 2^b words, and by Claim 14 no word is included twice in any two paths of T, we know that every word in Q_n is in exactly one path in T.

For every $w \in Q_a$, $T_w(0) = R_{w,0}(0) = \mathtt{0}^b w$. And finally, $T_w(0) = \mathtt{0}^b w = \overline{\mathtt{1}^b \overline{w}} = \overline{T_w(\cdot)}$. So, T is a $\mathfrak{B}(n, n - \lceil \log_2(n+1) \rceil)$. □

We can now prove our main result.

Theorem 16. *The path bundle $\mathfrak{B}(n, a)$ exists if and only if $n > 0$ is odd and $0 \leq a \leq n - \lceil \log_2(n+1) \rceil$.*

Proof. We proceed by induction on n. Trivially, $\mathfrak{B}(1,0)$ exists.

We assume that the theorem is true for all odd $n < m$. By Lemma 6, we know that $\mathfrak{B}(m,0)$ exists. By Lemma 7, we know that $\mathfrak{B}(m,a)$ exists if $a > 0$ and $a \leq n - 1 - \lceil \log_2(n-1) \rceil$. If $a = n - \lceil \log_2(n-1) \rceil = n - \lceil \log_2(n+1) \rceil$, then $\mathfrak{B}(m,a)$ exists by Theorem 10.

So, by induction, $\mathfrak{B}(n,a)$ exists if $n > 0$ is odd and $0 \leq a \leq n - \lceil \log_2(n+1) \rceil$. If these bounds aren't satisfied, $\mathfrak{B}(n,a)$ does not exist, by Lemmas 3 and 4. Therefore, $\mathfrak{B}(n,a)$ exists if and only if $n > 0$ is odd and $0 \leq a \leq n - \lceil \log_2(n+1) \rceil$. □

5. Conclusion

Theorem 16 is the end result of this paper: there exists a path bundle on the n-cube containing 2^a paths if and only if $n > 0$ is odd and $0 \leq a \leq n - \lceil \log_2(n+1) \rceil$. This gives a complete categorization of the existence of path bundles over the parameters n and a.

Other open questions about path bundles remain. How many distinct path bundles exist for a given n and a? Instead of forcing paths to begin and end in single subcubes, what other sets of words can the endpoints of the paths form? Do analogues to this result exist in bases higher than 2?

Thanks to Joe Gallian and the Duluth Mathematics REU both for revisions and advice about this paper, and for introducing me to a similar problem (and, thus, mathematical research). Thanks to David Sumner and Matt Boylan at the University of South Carolina for helping revise early versions of this paper, and to the anonymous reviewers for their suggestions for clarification.

This research was supported in part by grants from the NSF (DMS0447070-001) and the NSA (H98230-06-01-0013).

References

[1] F. Gray. Pulse code communication. US Patent 2632058, March 1958.

[2] Charles E. Killian and Carla D. Savage. Antipodal gray codes. *Discrete Mathematics*, 281(1–3):221–236, 2004.

[3] Carla D. Savage. A survey of combinatorial Gray codes. *SIAM Review*, 39(4):605–629, 1997.

U. Wisconsin-Madison, 1210 W. Dayton St., Madison, WI 53706-1685, elder@cs.wisc.edu