OPTIMIZATION TECHNOLOGY TO DEAL WITH LACK OF DATA IN STATISTICAL ESTIMATION

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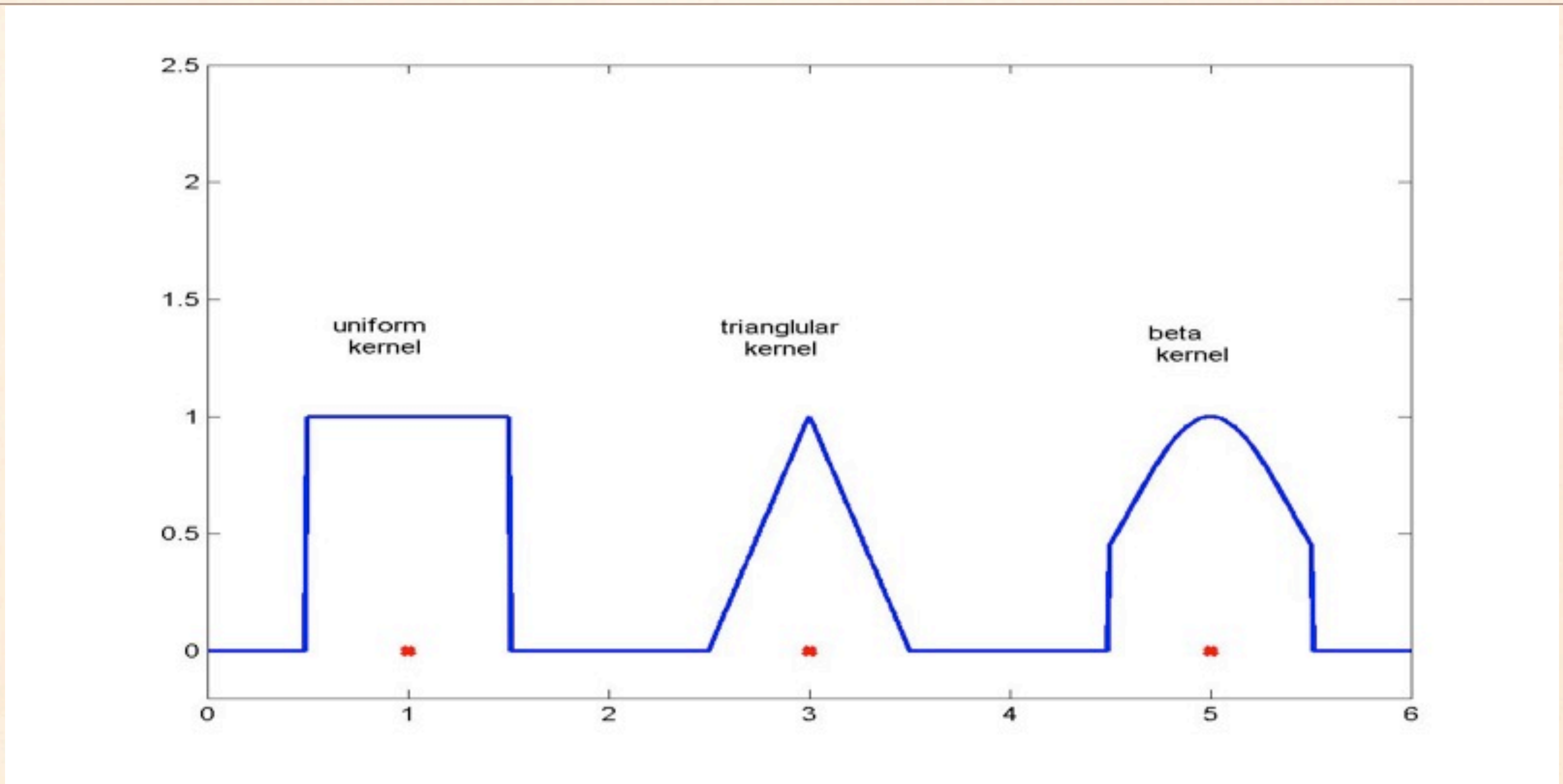
Wisconsin, Spring 2011

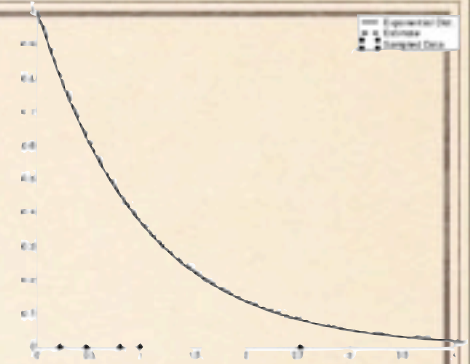
Kernel "Base" NonParametric Estimation



Information = Observations: $\xi^1, \xi^2 \ldots, \xi^l$

Optimal bandwidth = kernel support ?

Kernel Estimates



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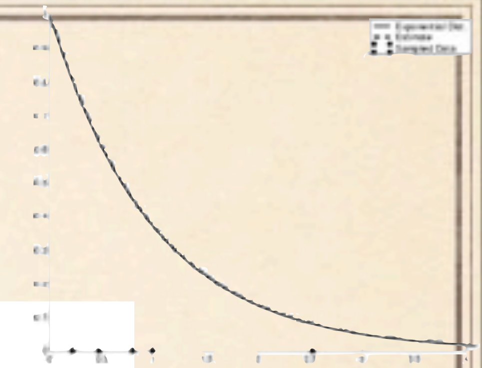
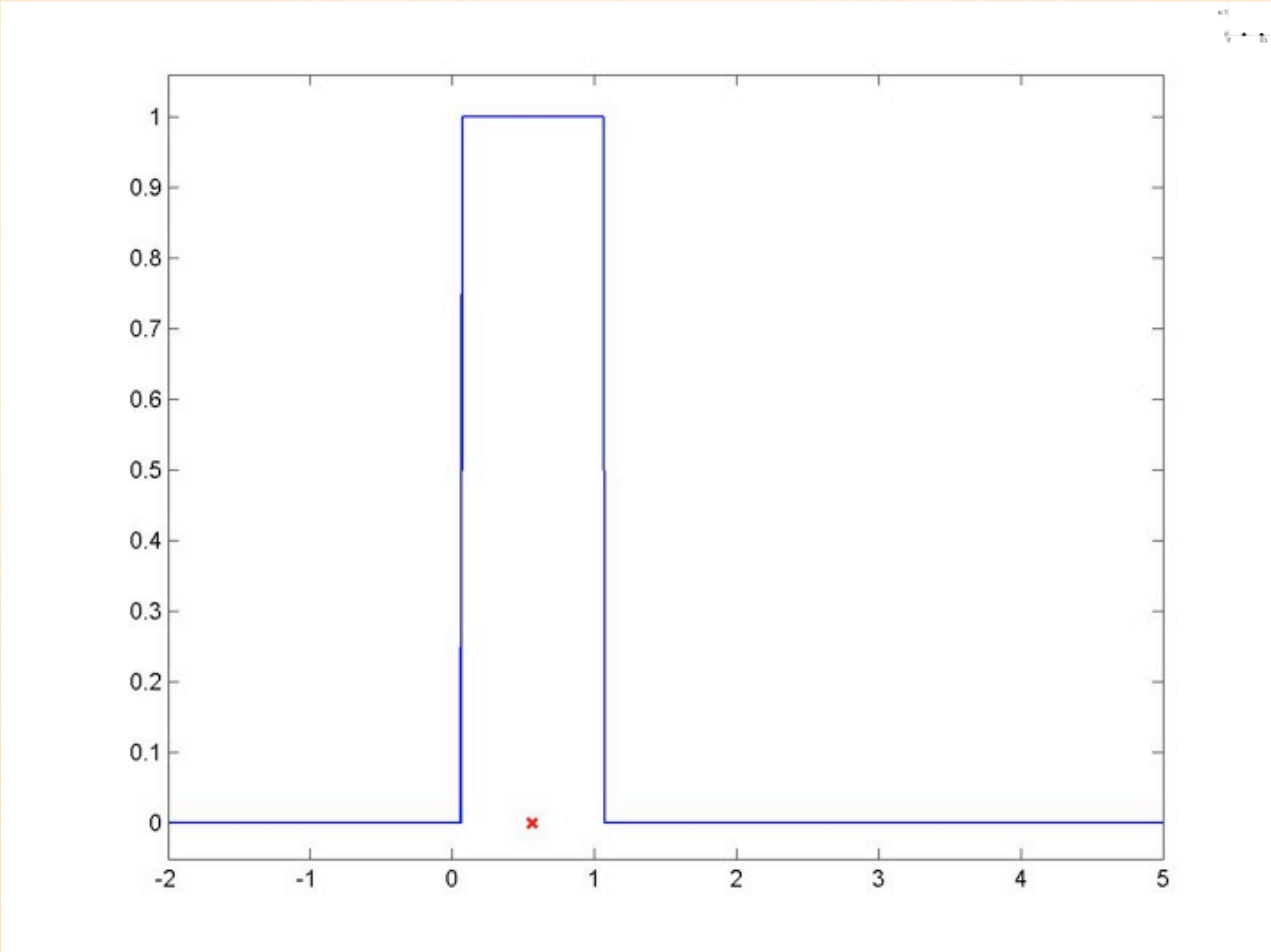
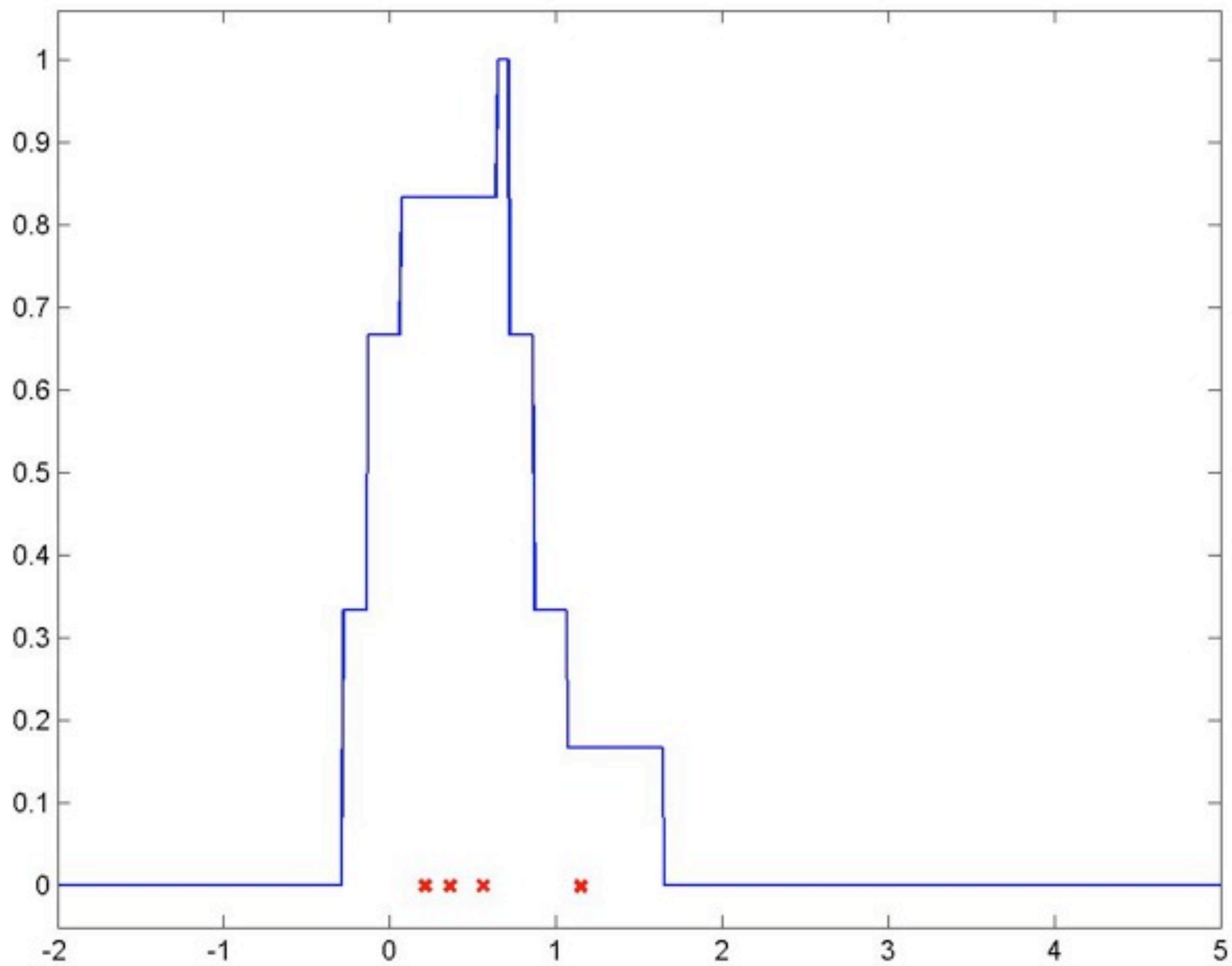
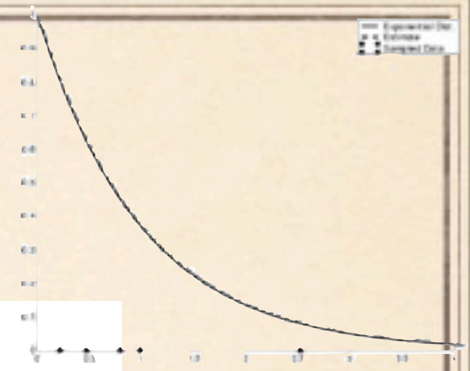


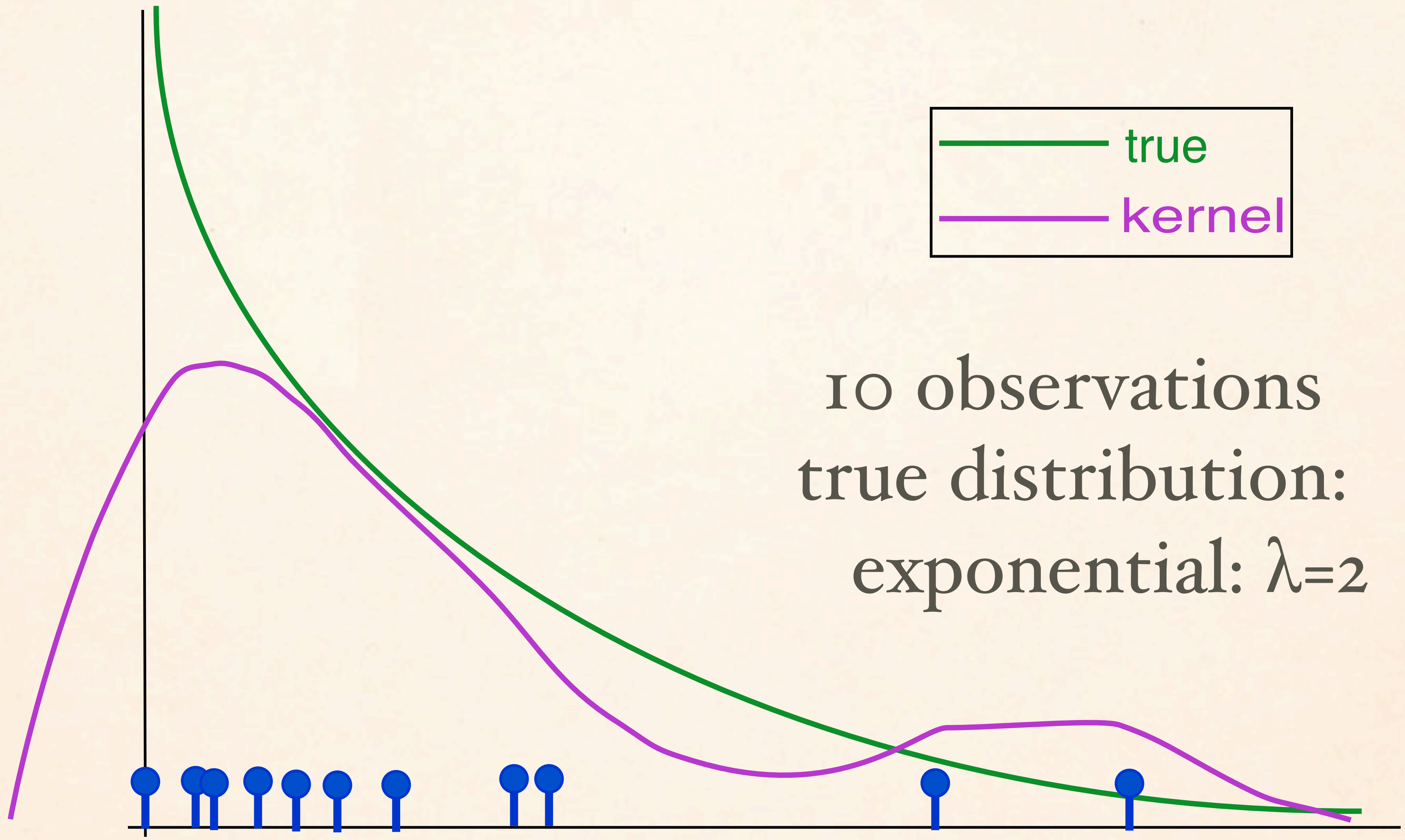


Kernel Estimates





R-STAT: KERNEL ESTIMATE



10 observations
true distribution:
exponential: $\lambda=2$

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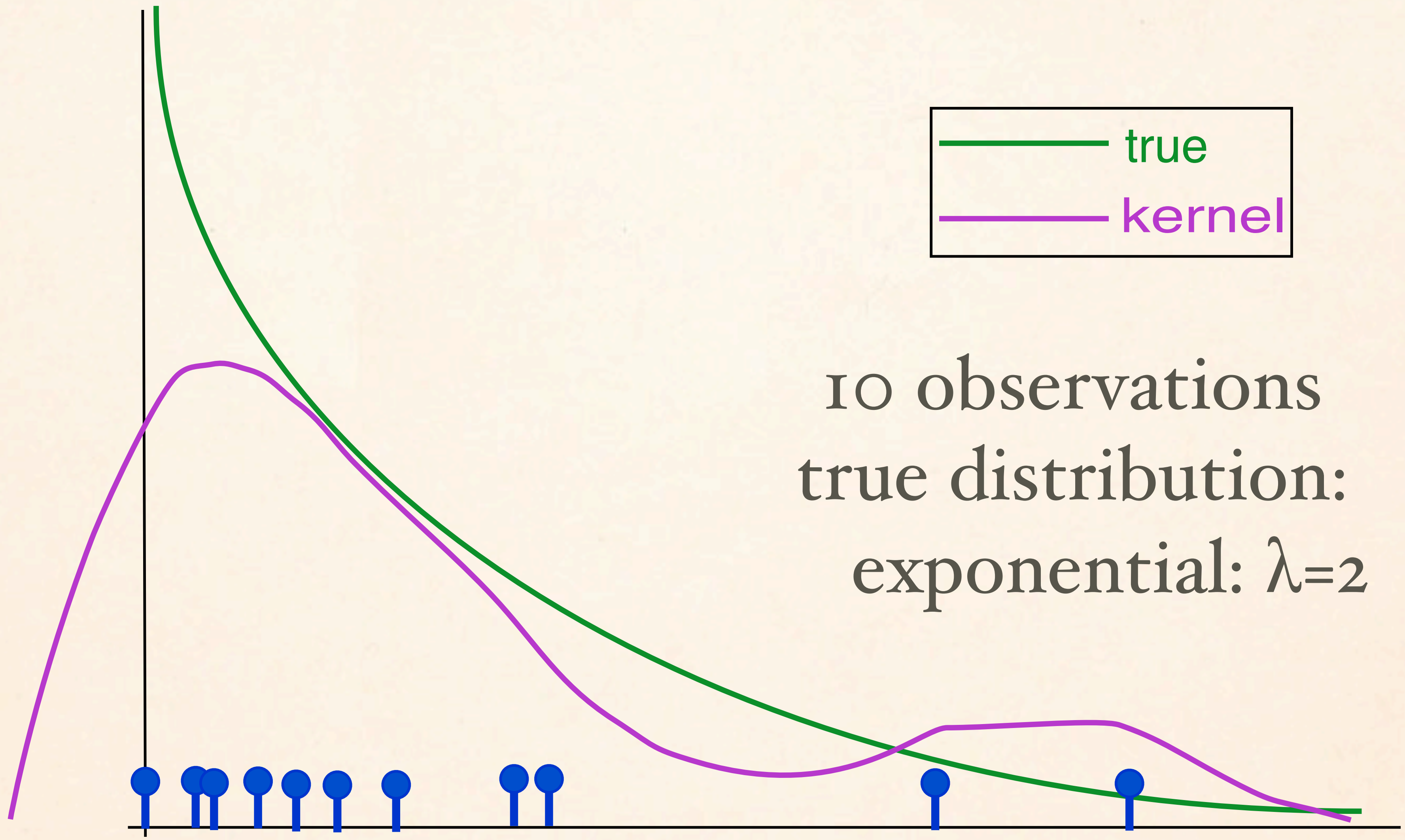

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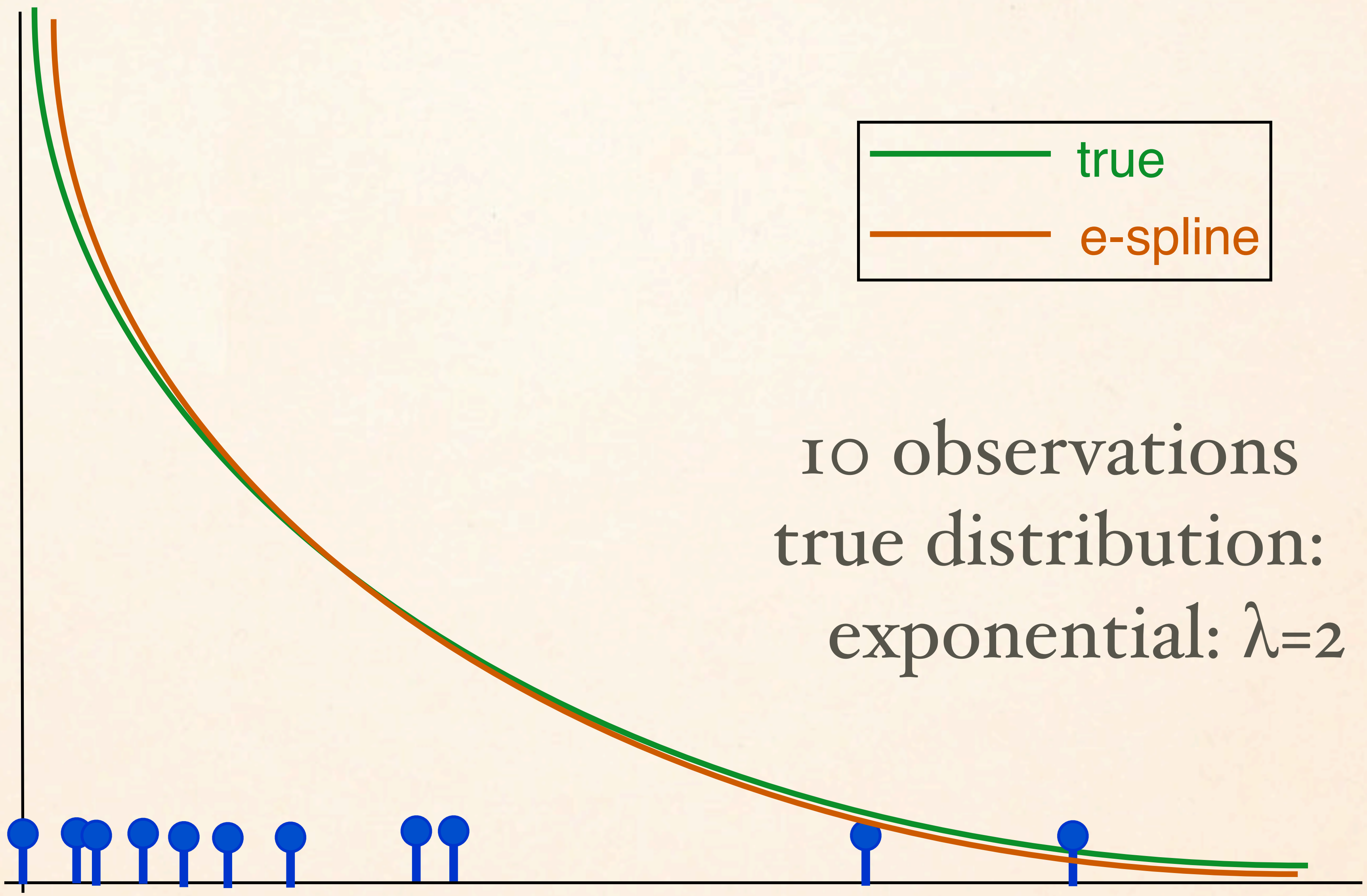
Maybe an 'optimizer's' viewpoint might help?

"OUR" ESTIMATE



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BASIC STATISTICAL ESTIMATION PROBLEM

- Find F^{est}, an estimate of the distribution of ξ given **all** the information available about this random phenomena,
- i.e. such that

$$\forall z: \ F^{est}(z) \approx F^{true}(z) = \text{prob.}[\xi \le z]$$

all information might come with inequalities ....

“ALL” INFORMATION

- Observations (hard data): $\xi^1, \xi^2, \ldots, \xi^\nu$
- Non-data facts (soft information)
  - Support: (un)bounded , density or discrete distribution,
  - bounds on expectation, moments,
  - heavy tails
  - shape: unimodal, decreasing, parametric class
- still `softer' information (modeling assumptions):
  - see above + ... level of smoothness, `Bayesian' neighborhood, ..

AN OPTIMIZATION VIEWPOINT

- Find *h in H=class-fcns*$(\mathbb{R}^n)$
- that maximizes the probability of observing $\xi^1, \xi^2, \ldots, \xi^\nu$

$$\max \boxed{\frac{1}{\nu}\sum_{l=1}^{\nu} \ln h(\xi^l) \quad \text{(likelihood)}}$$

- $$\max E^\nu \left\{ \ln h(\xi) \right\} = \max \int \ln h(\xi) P^\nu(d\xi)$$

"SOFT" INFORMATION

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'Bayesian': $\|h - h^0\| \le \beta$, objective: $\alpha \cdot \text{bayes}(h, h^0)$

RE-FORMULATION: "OPT"-VERSION

$$\max E^\nu\left\{\ln h(x)\right\} = \frac{1}{\nu}\sum_{l=1}^{\nu} \ln h(x_l)$$

$$\text{such that } \int h(x)dx = 1,$$

$$h(x) \geq 0,\ \forall x \in \mathbb{R}$$

$$h \in A^\nu \subset H$$

A^ν: soft (non-data) information constraints

$H = C^2(S), L^p(S), H^1(S), \ldots \quad S$ subset $\mathbb{R}^n$

CONSISTENCY THEOREM?

Suppose $\nu \to \infty$ (more data is acquired) and

$A^\nu \to A$ (valid information is acquired) then

estimates $h^\nu \to h^{true}$ $a.s.$ (with probability 1).

1949 A. Wald: consistency of parametric estimates (MLH-SAA)

1982-1983 Klonias & Prakasa Rao: consistency of nonparametric estimatesr

1971 Good & Gaskins: nonparametric roughness penalties (proposed)

1982 B. Silverman & 1990 J. Thompson/R.Tapia: consistency with penalization

1985 P. Groeneboom: Estimating a monotone density

==

1979 R. Wets: statistical approach to solution of stochastic program (Tech. Note)

1988 (with J. Dupacova): asymptotics of constrained estimators (parametric)

1991 (with H. Attouch): Law of Large Numbers for random lsc functions

2000 (with X. Dong): consitency of constrained estimators (non-parametric)

2006 (with M. Casey): rates of convergence

APPROXIMATION THEORY

$\min f_0(x)$ such that $x \in X \subset H$ (for our use: H Polish space)

$\min f(x)$ on $x \in X$ with $f(x) = \begin{cases} f_0(x) & \text{when } x \in X \\ \infty & \text{otherwise} \end{cases}$ lsc function: $H \to \overline{\mathbb{R}}$

$(f_0^\nu, X_0^\nu) \to (f_0, X)$ sequence of optimization problems converging(?) to, f

$\arg\min f^\nu \to \arg\min f \quad (\inf f^\nu \to \inf f)$

$\arg\min\,(f^\nu + g) \to \arg\min(f + g),\ g$ continuous perturbation

$\Rightarrow f^\nu$ epi-converges to f $\quad (\operatorname{epi} f^\nu \to \operatorname{epi} f)$: for all $x \in H$,

(a) $\forall x^\nu \to x,\ \liminf_\nu f^\nu(x^\nu) \geq f(x)$,

(b) $\exists x^\nu \to x,\ \limsup_\nu f^\nu(x^\nu) \leq f(x)$.

pointwise?, uniform?,

LLN: RANDOM LSC FUNTIONS

$f:\Xi\times H\to\overline{\mathbb{R}}$ a random lsc function, ξ values in $(\Xi,\mathcal{A},P)$

(a) lsc (lower semicontinuous) in h, $(\forall\xi\in\Xi)$

(b) (ξ,h)-measurable $\qquad (\mathcal{A}\times B_X)$-measurable

recall: $f(\xi,h)= f_0(\xi,h)$ when $h\in X(\xi)$ -- stochastic constraints

$$f^\nu(\xi,h)=\begin{cases}\frac{1}{\nu}\sum_{l=1}^{\nu}\ln h(\xi^l) \text{ if } h\geq 0,\ \int_\Xi h(\xi)d\xi=1,\ h\in A^\nu\\ \infty \text{ otherwise} \quad (\sim \text{SAA of optimisation problems})\end{cases}$$

Question: Do the $f^\nu(\xi,\cdot)$ epi-converge to $\mathbb{E}\{f(\xi,h)\}$ P-a.s.?

$$h^{true}\in\arg\min\ \mathbb{E}\{f(\xi,h)\}$$

where $\quad f(\xi,h)=\begin{cases}\ln h(\xi) \text{ if } h\geq 0,\ \int_\Xi h(\xi)d\xi=1, h\in A\\ \infty \text{ otherwise}\end{cases}$

CONSISTENCY THEOREM

Suppose $\nu \to \infty$ (more data is acquired) and

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NUMERICAL STRATEGIES

$$h \approx \sum_{k=1}^{q} u_k \phi_k(\cdot)$$

Fourier coefficients, wavelets, kernel-dictionary, …

$$h = \exp\big(s(\bullet)\big) \quad \text{exponential epi-spline}$$

$s(\bullet)$ cubic (or quadratic) epi-spline, spline-like

$\Rightarrow$ n-dimensional theory of epi-splines

TEST CASE: EXPONENTIAL

- $h^{true}(x) = \lambda e^{-\lambda x}$ if $x \geq 0$; $= 0$ if $x < 0$ $(\lambda = 1)$
- "empirical" estimate
- kernel estimate from **R-stat**
- unconstrained with support (non-negative)
- constrained (*h* decreasing)
- parametric, i.e., $h \in \exp\text{-class}$

200-OBSERVATIONS

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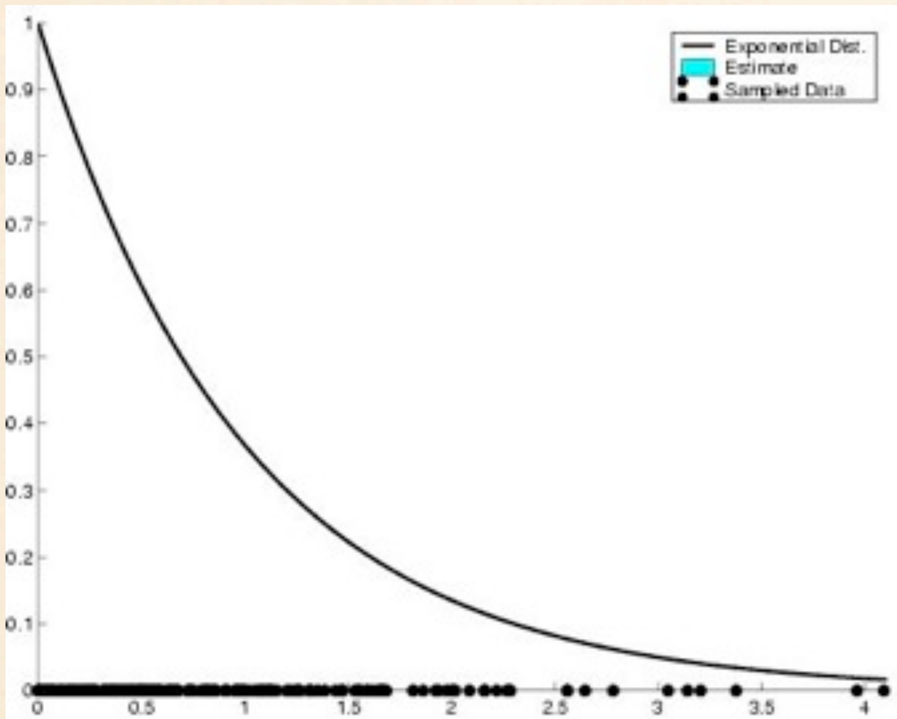

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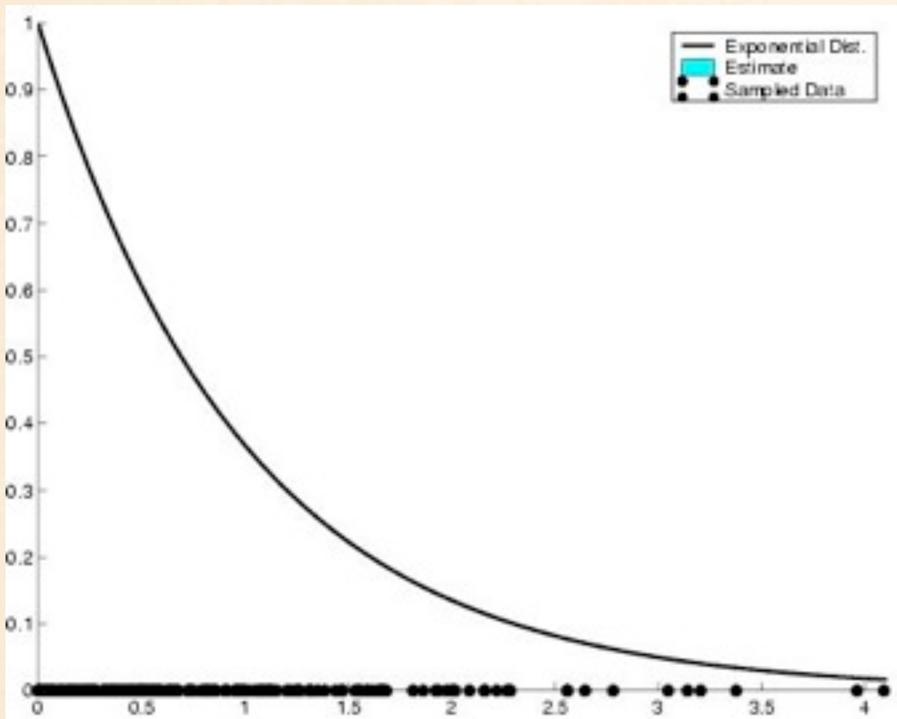

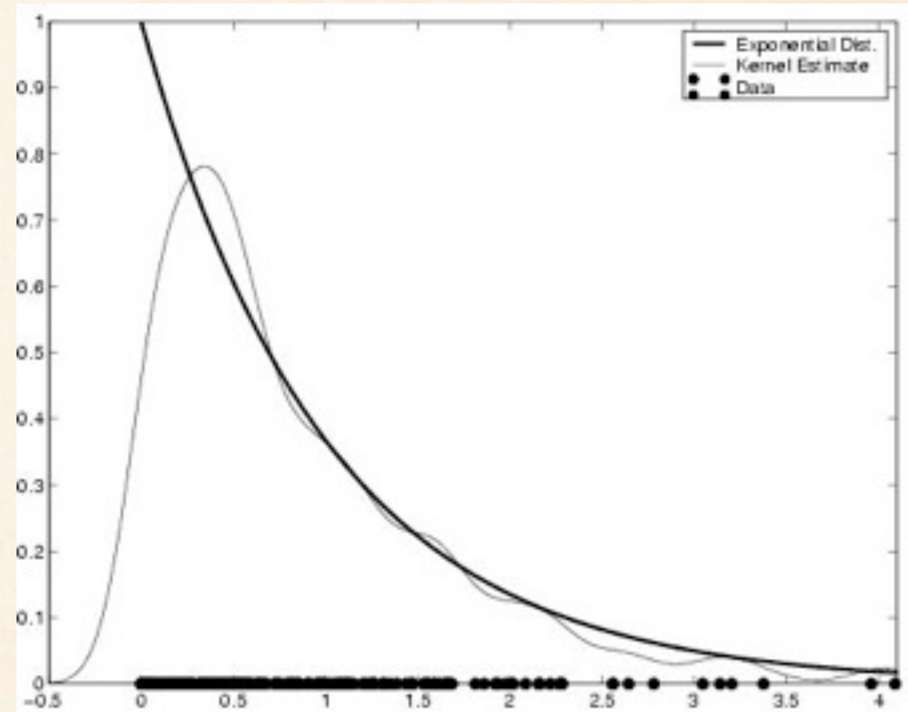

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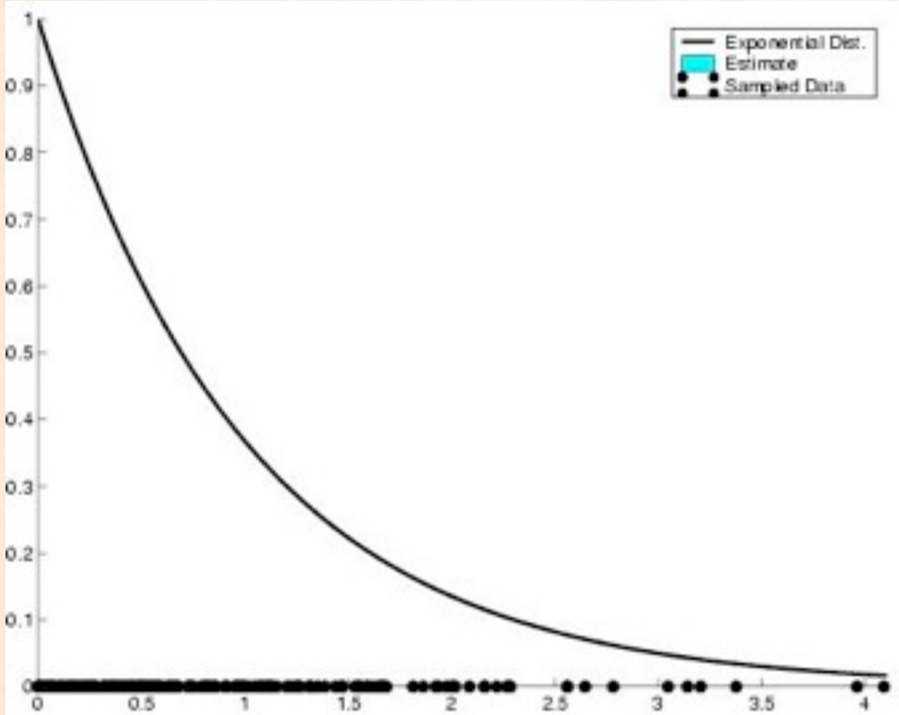

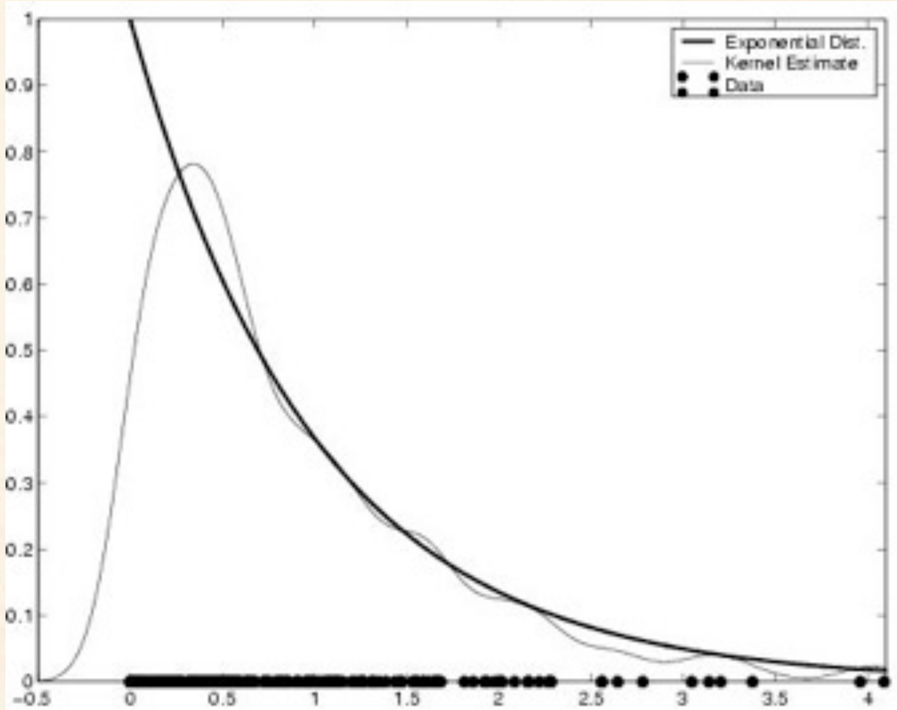

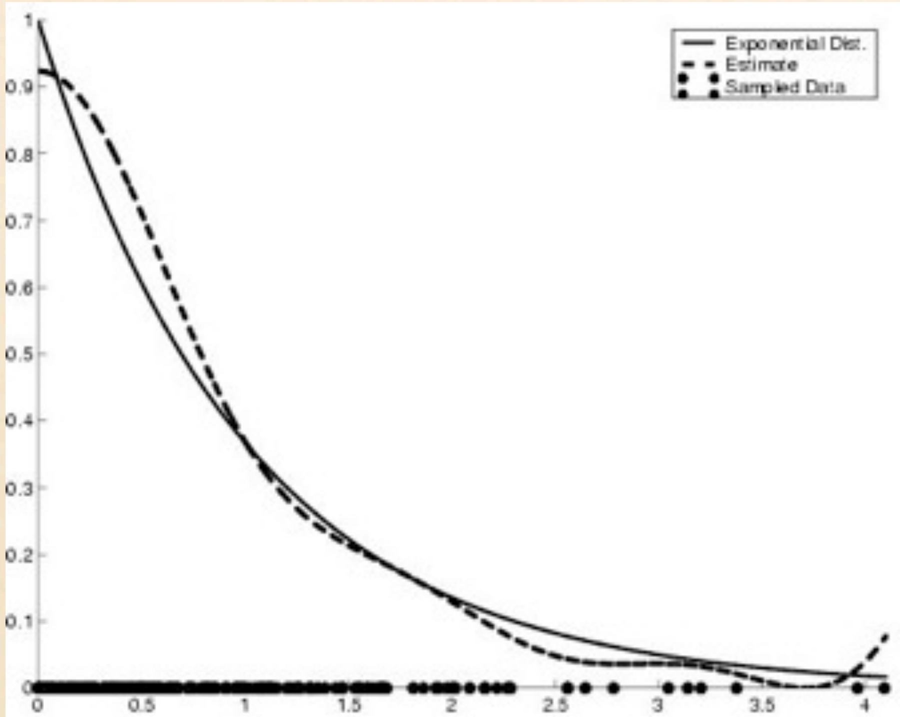

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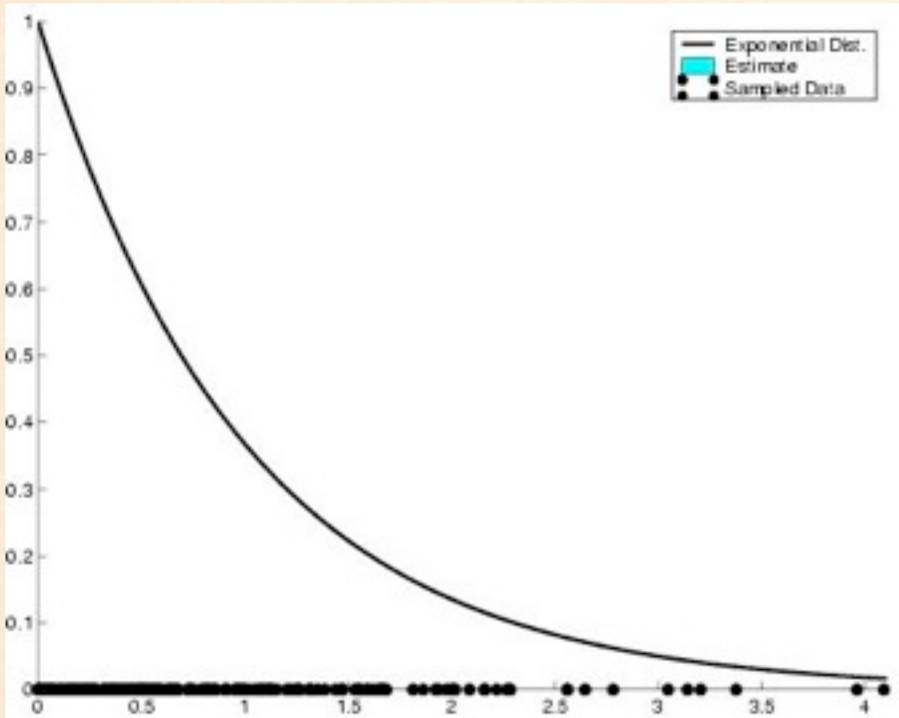

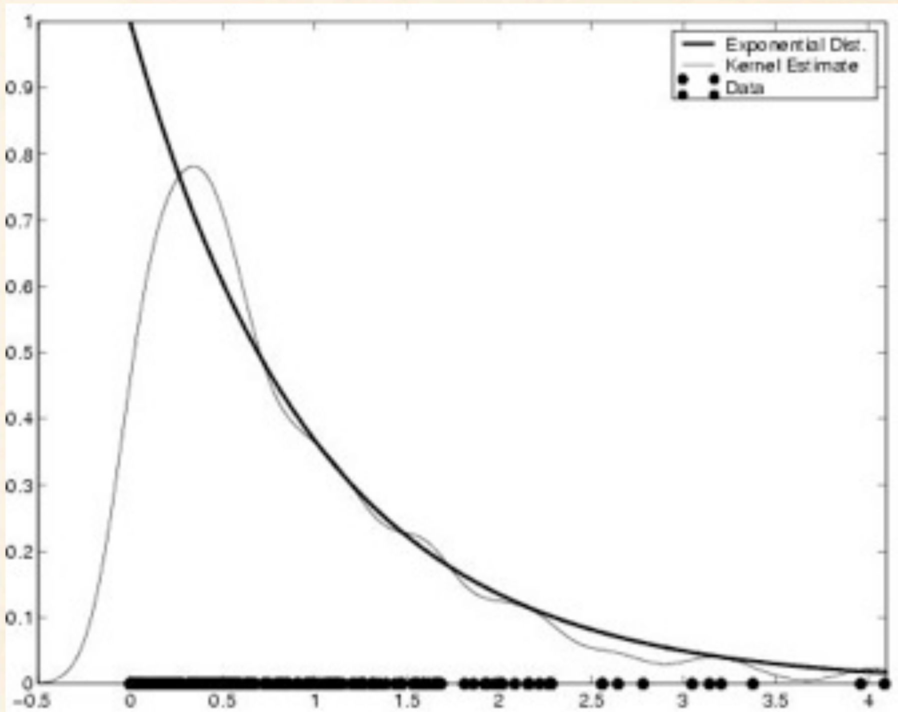

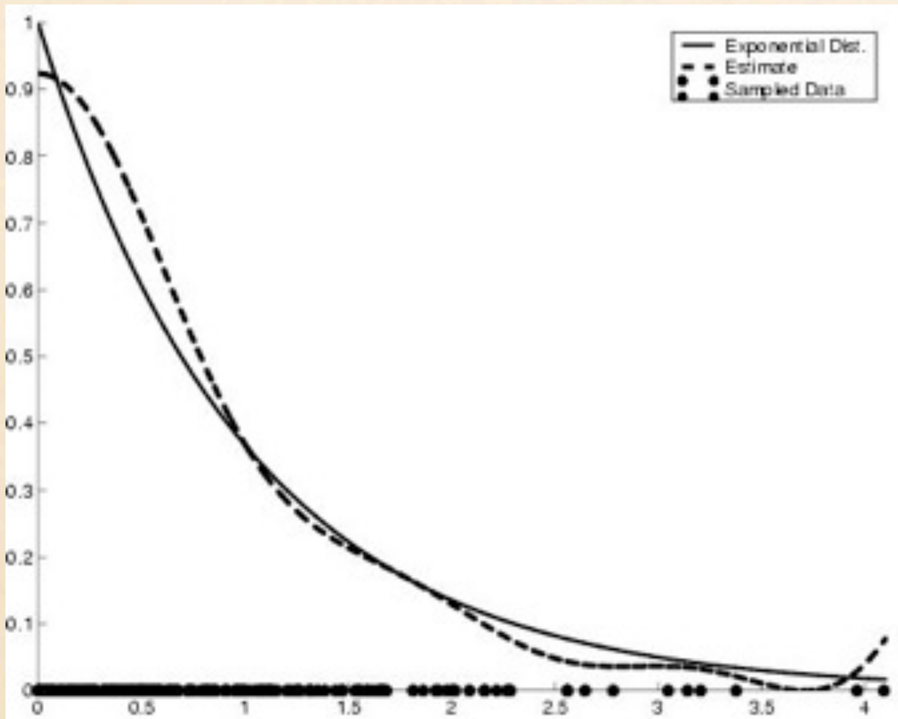

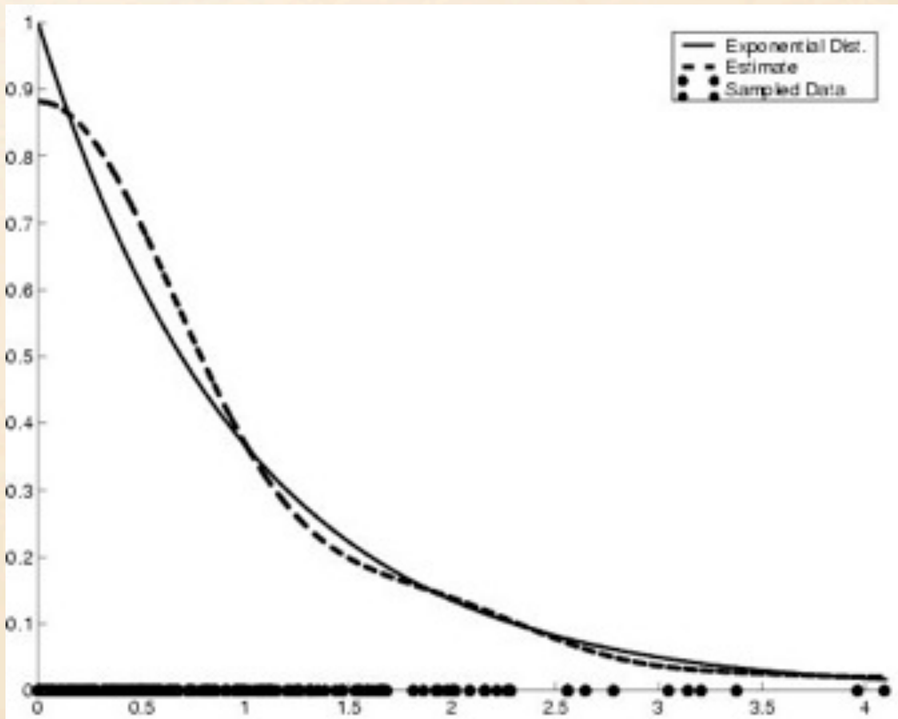

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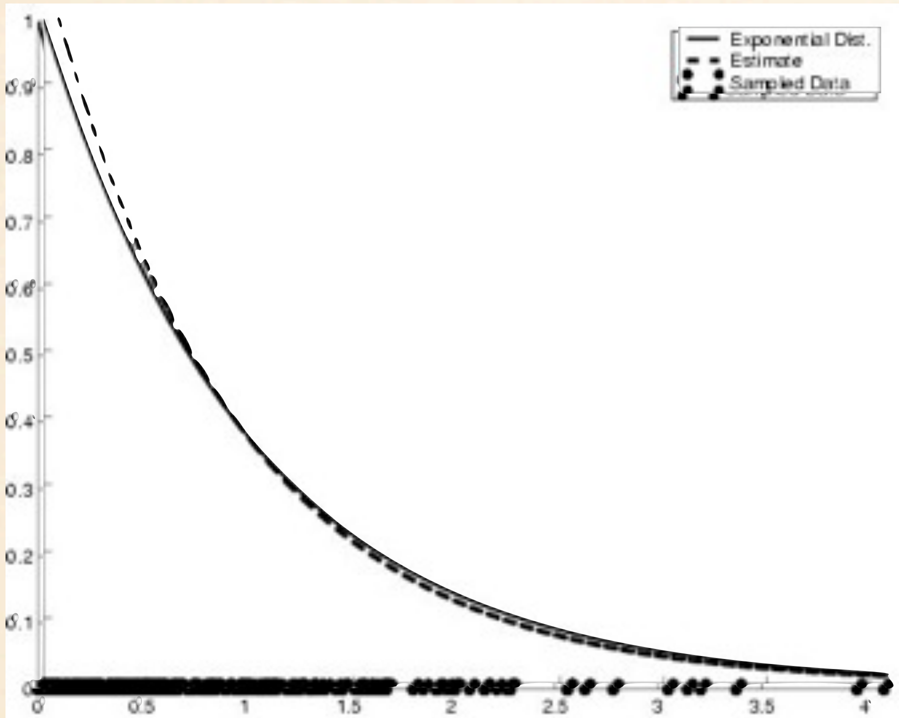
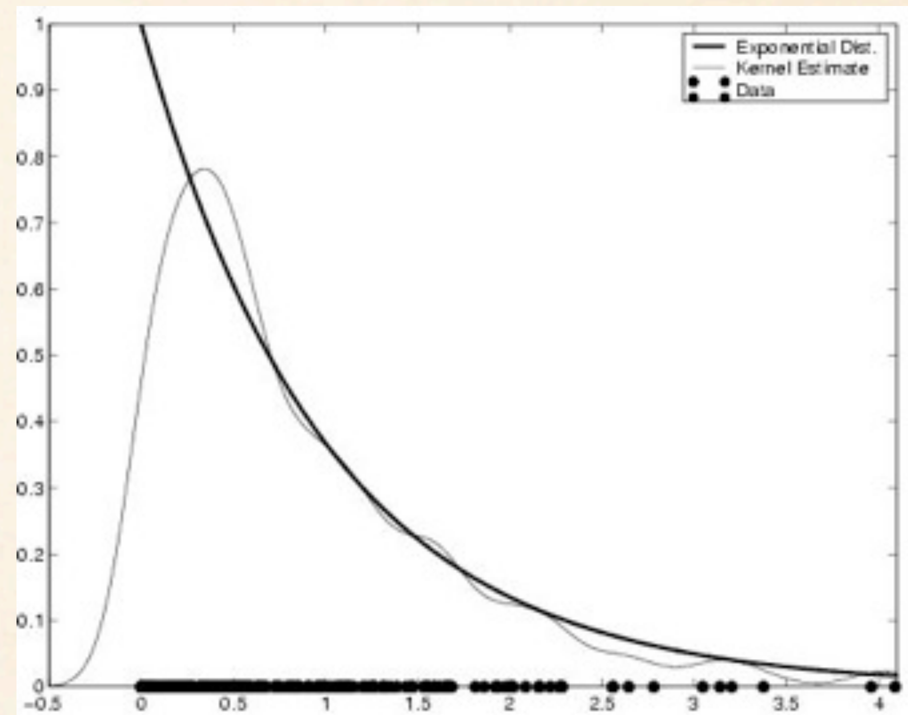
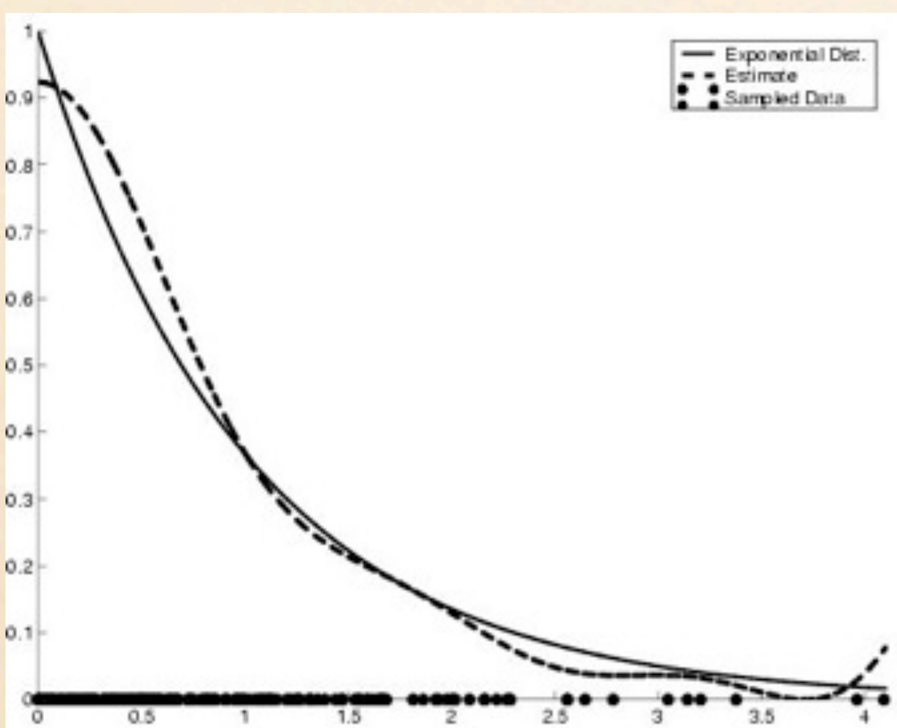
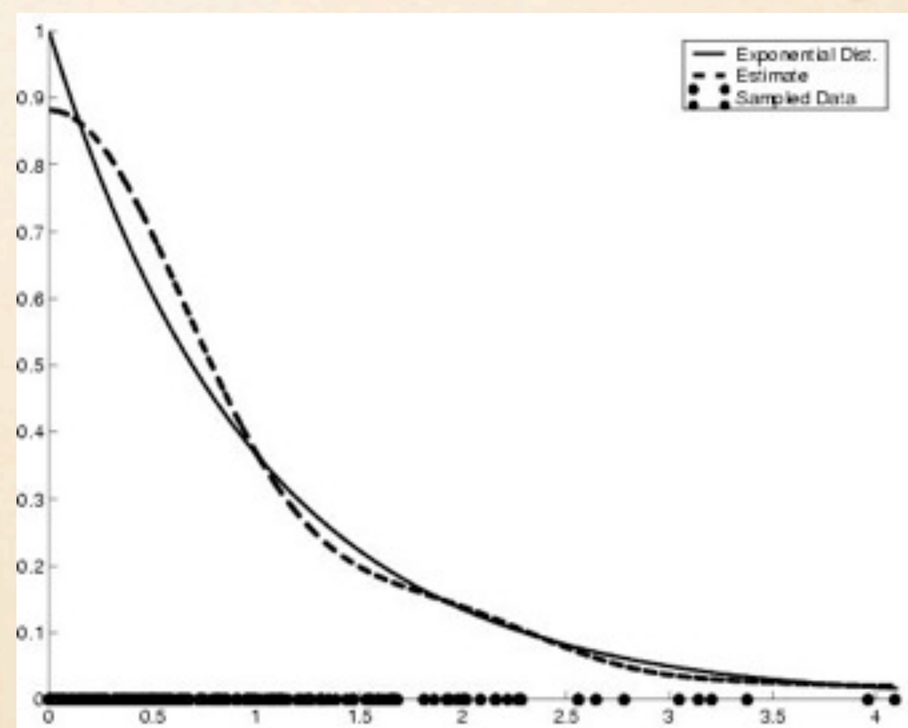


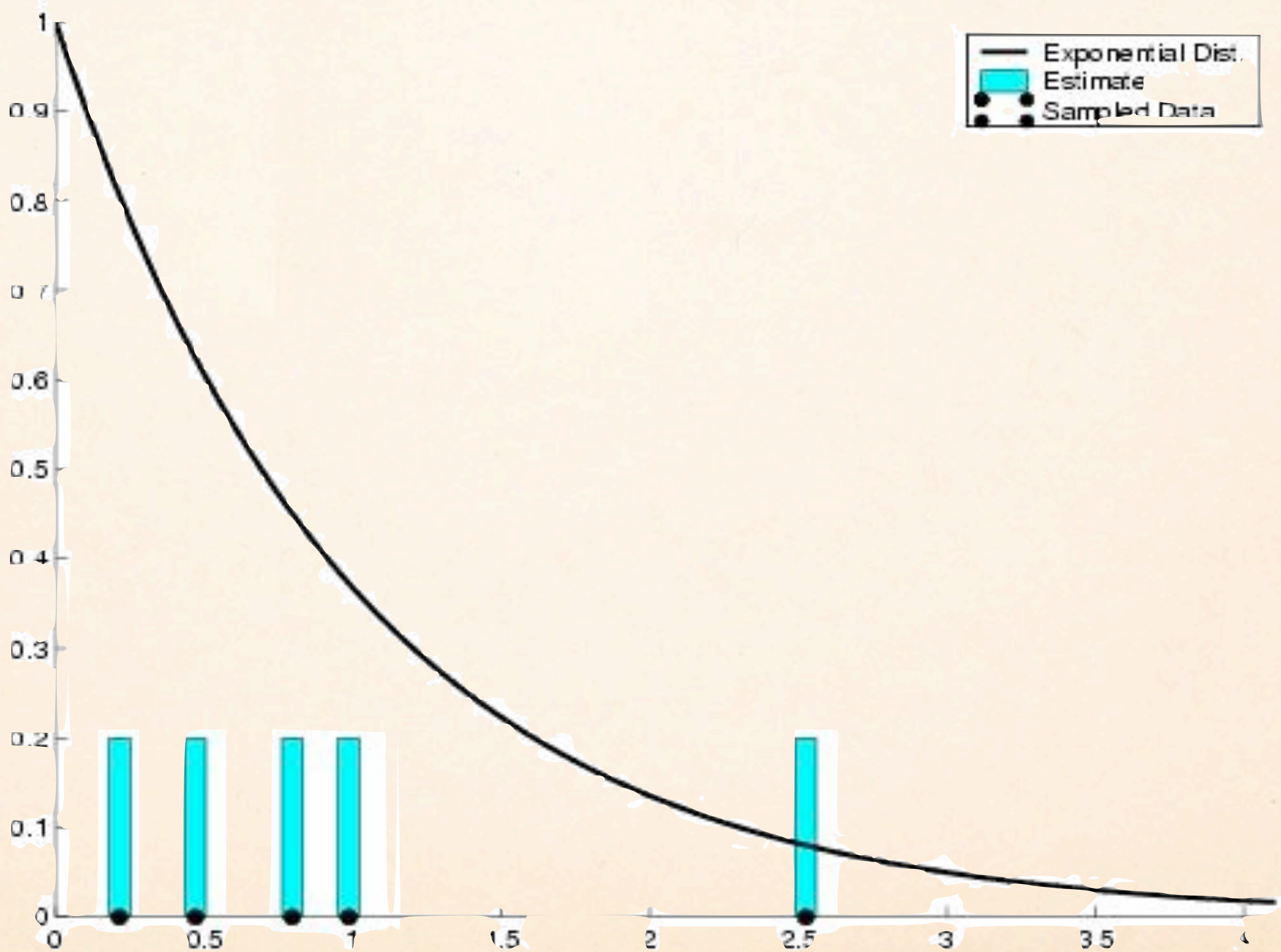




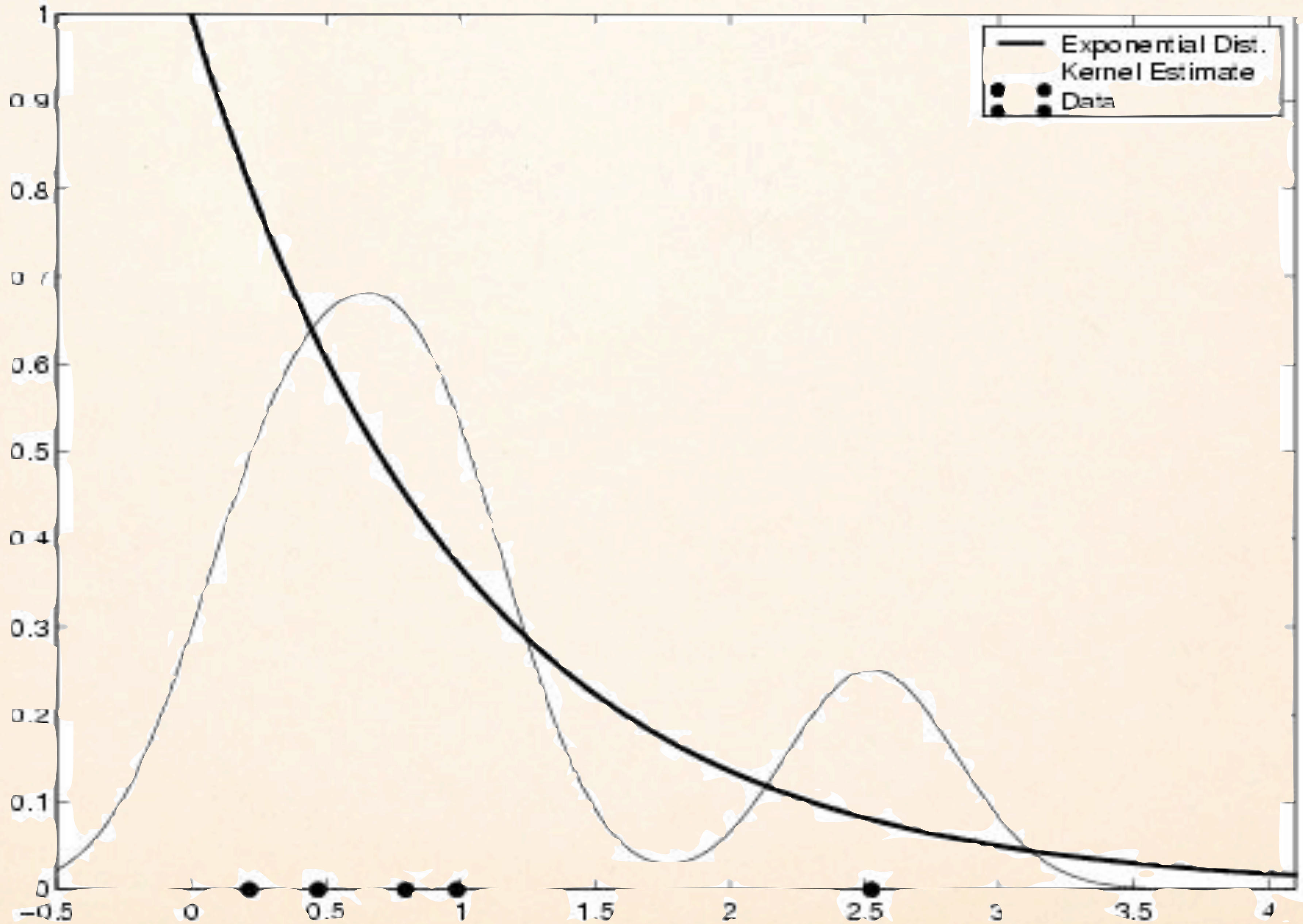


But what about 5 sample points?

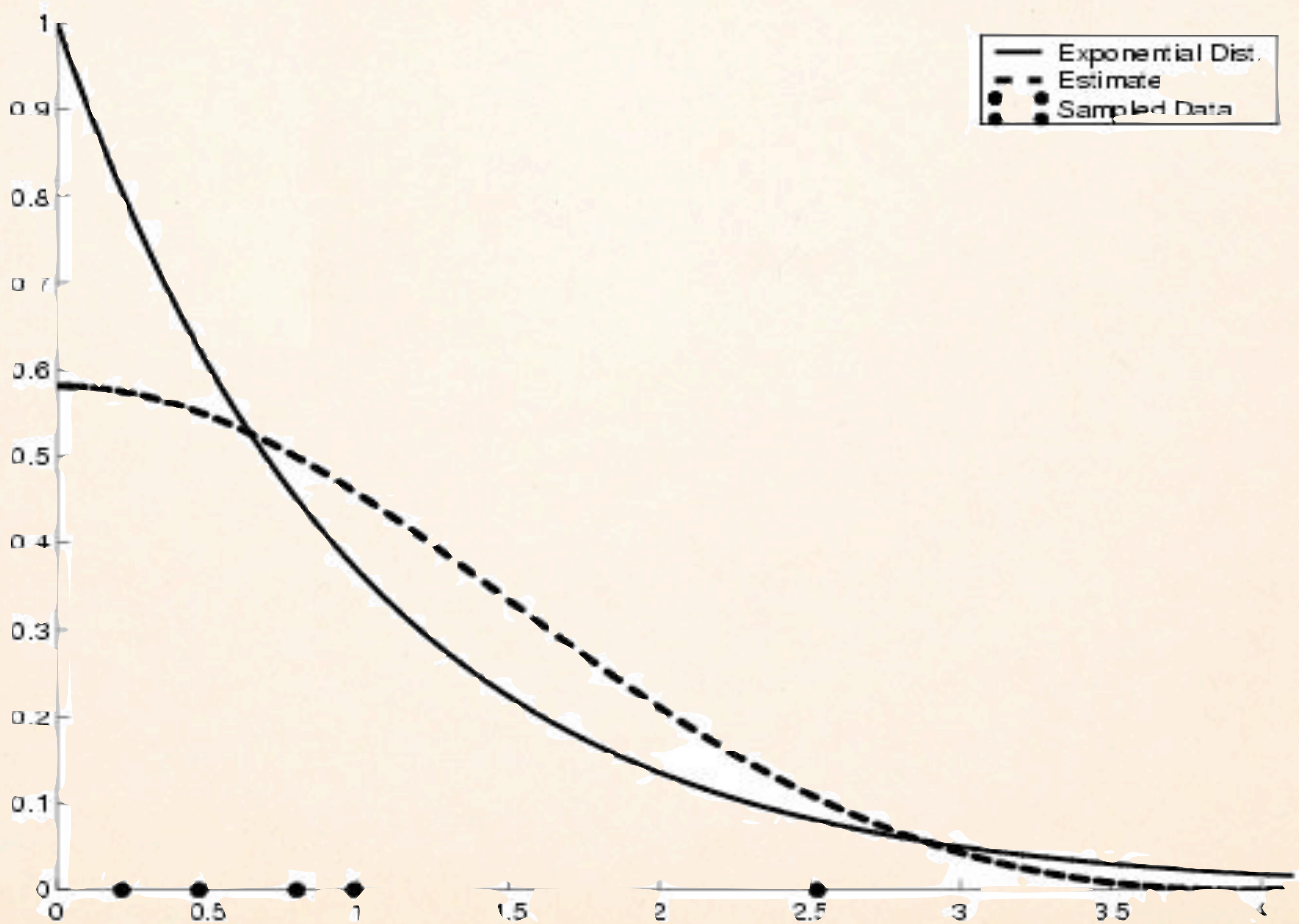
h^{est} : EMPIRICAL ESTIMATE



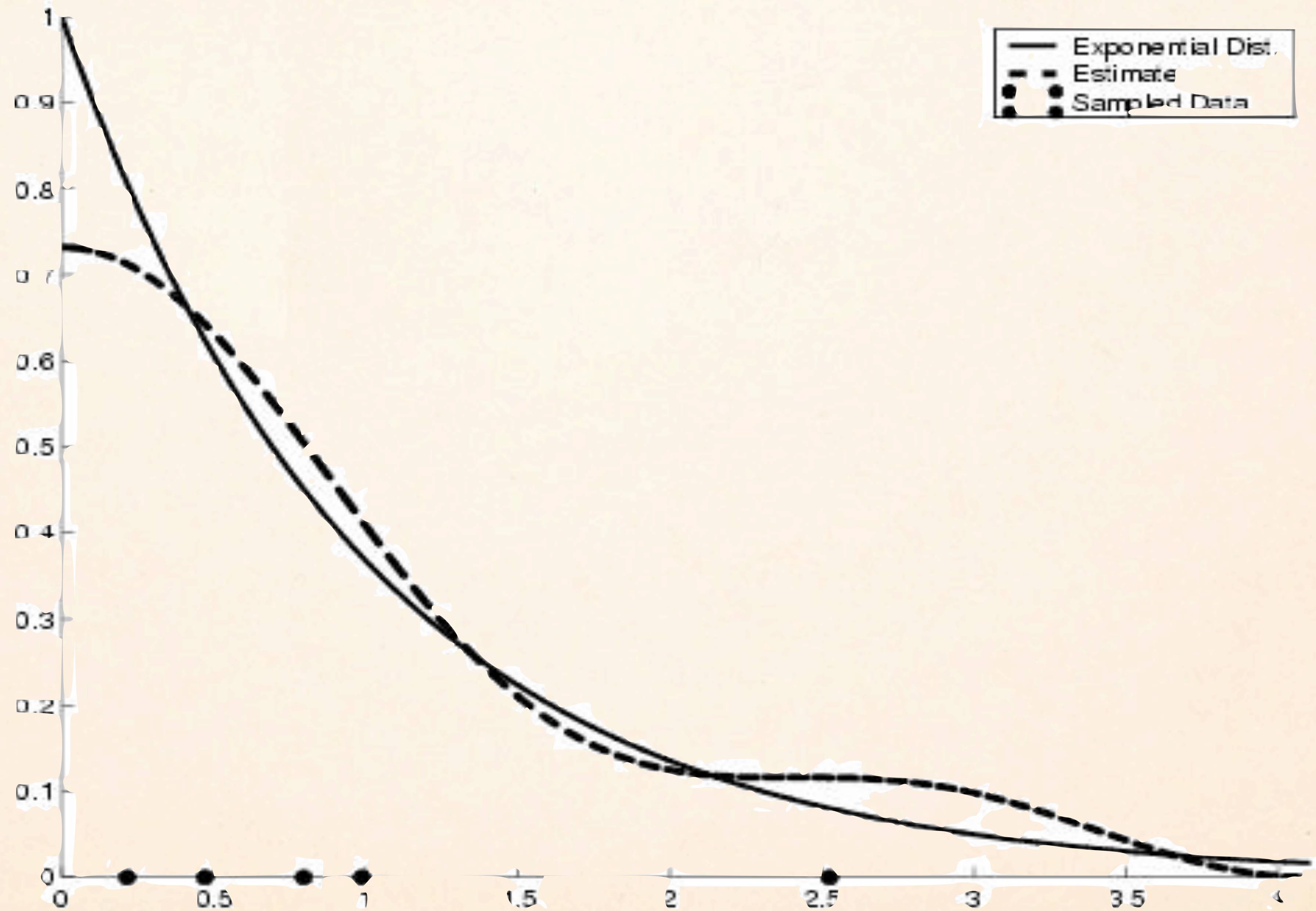
h^{est}: KERNEL ESTIMATE (R-STAT)



h^{est}: ..WITH KNOWN SUPPORT



h^{est} : DECREASING DENSITY



DOES IT PAY OFF? ERROR ANALYSIS

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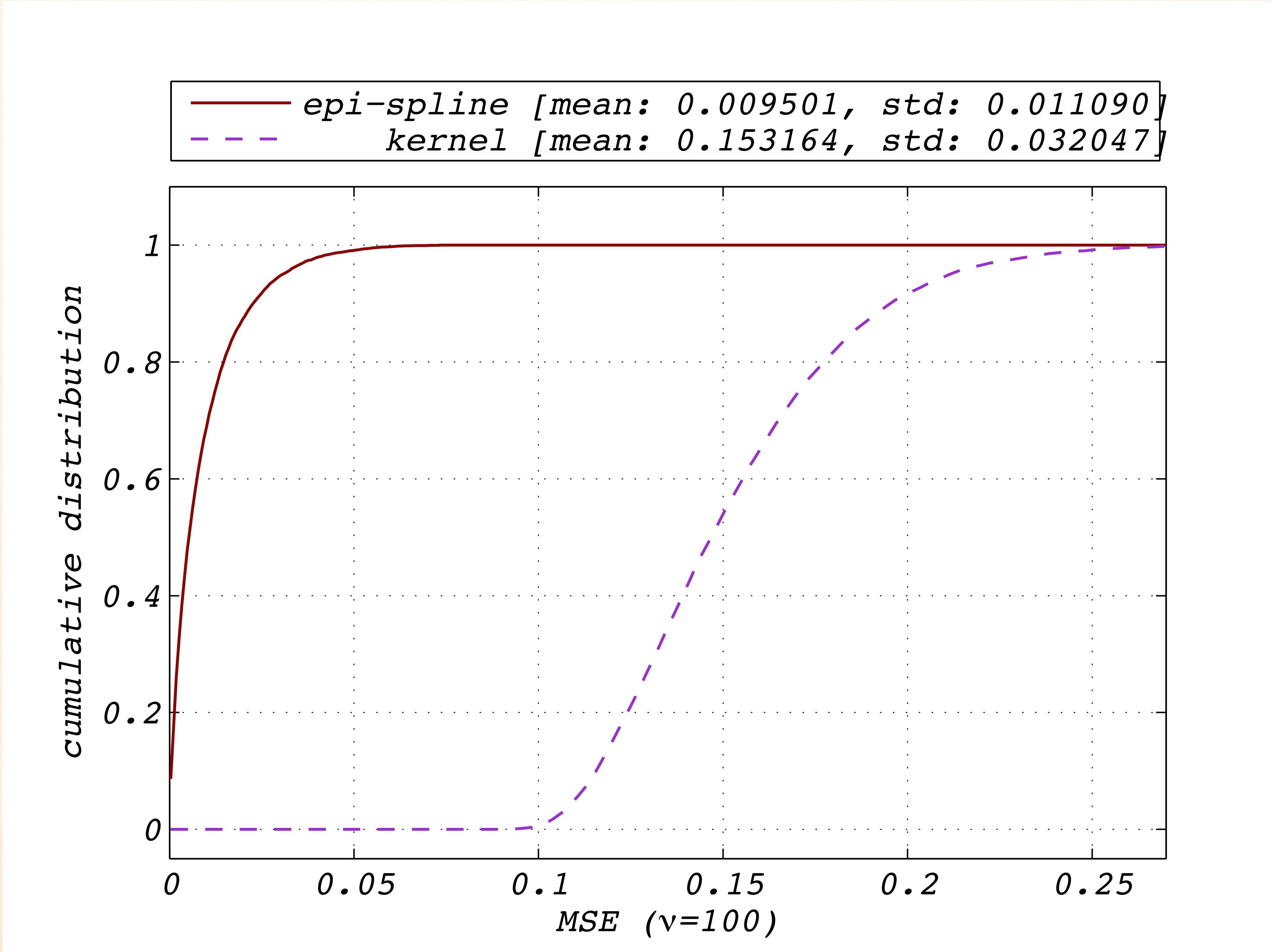

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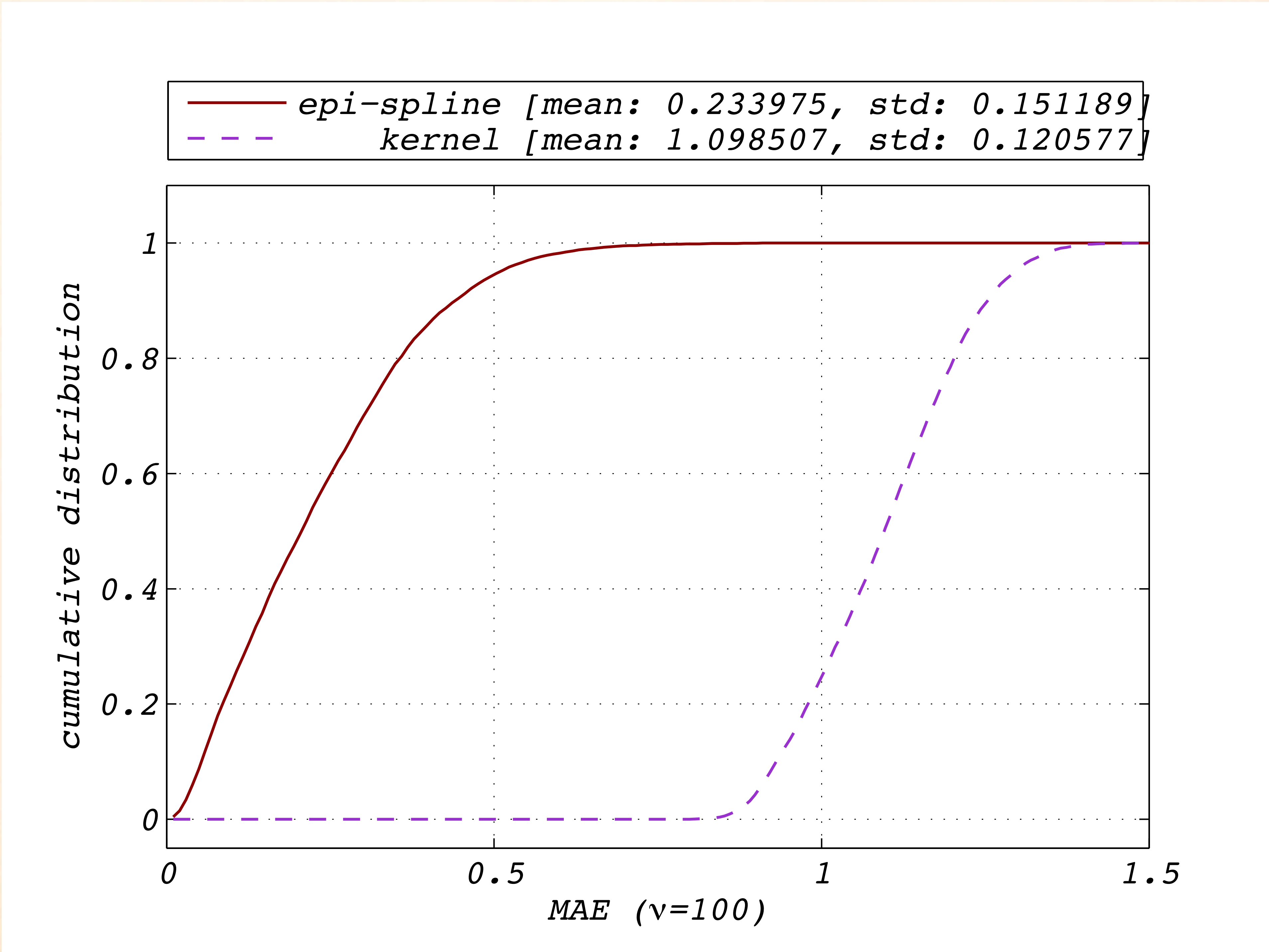

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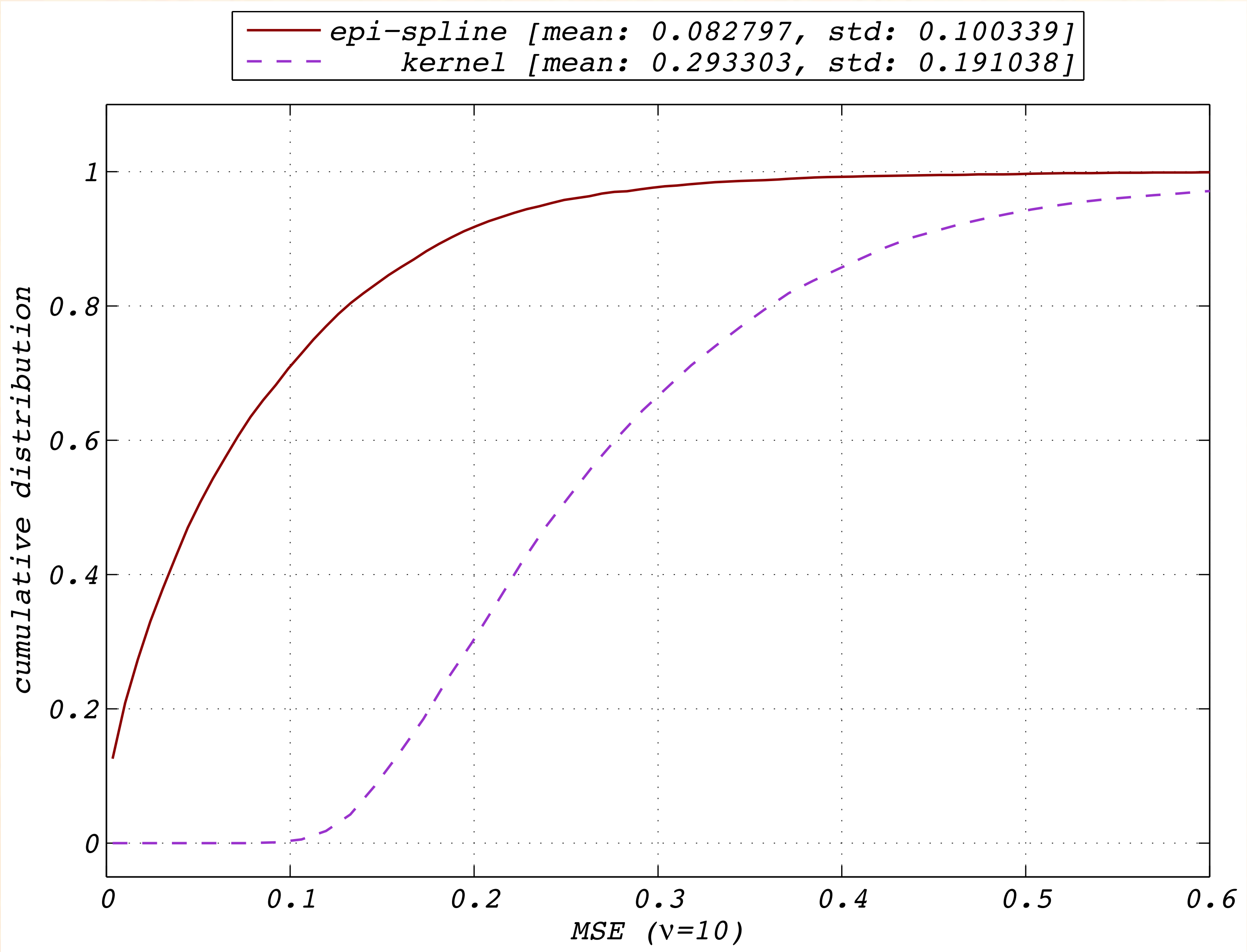

Mean square error
samples: 10
runs: 10,000
kernel estim: 0.293
epi-spline est.: 0.083

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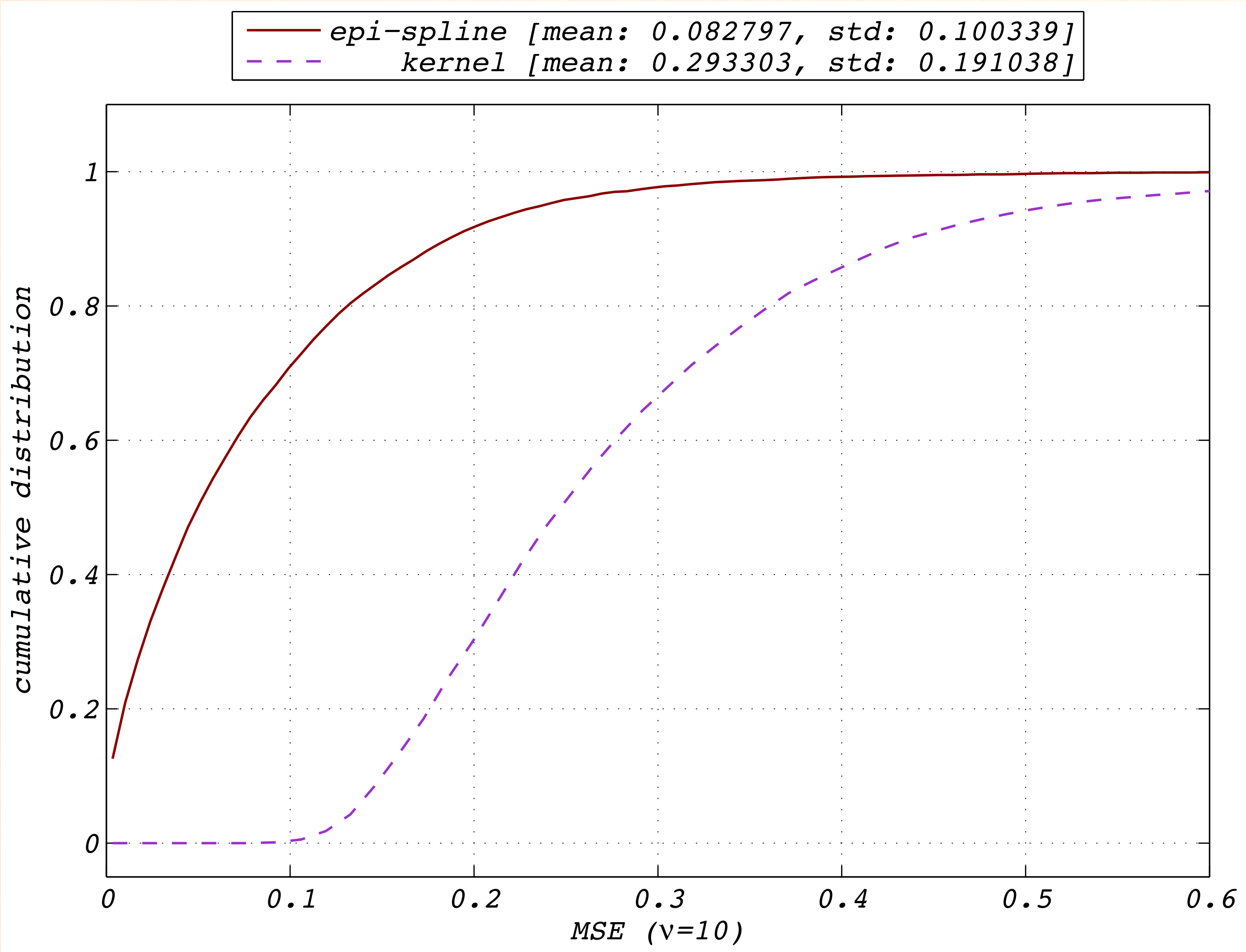

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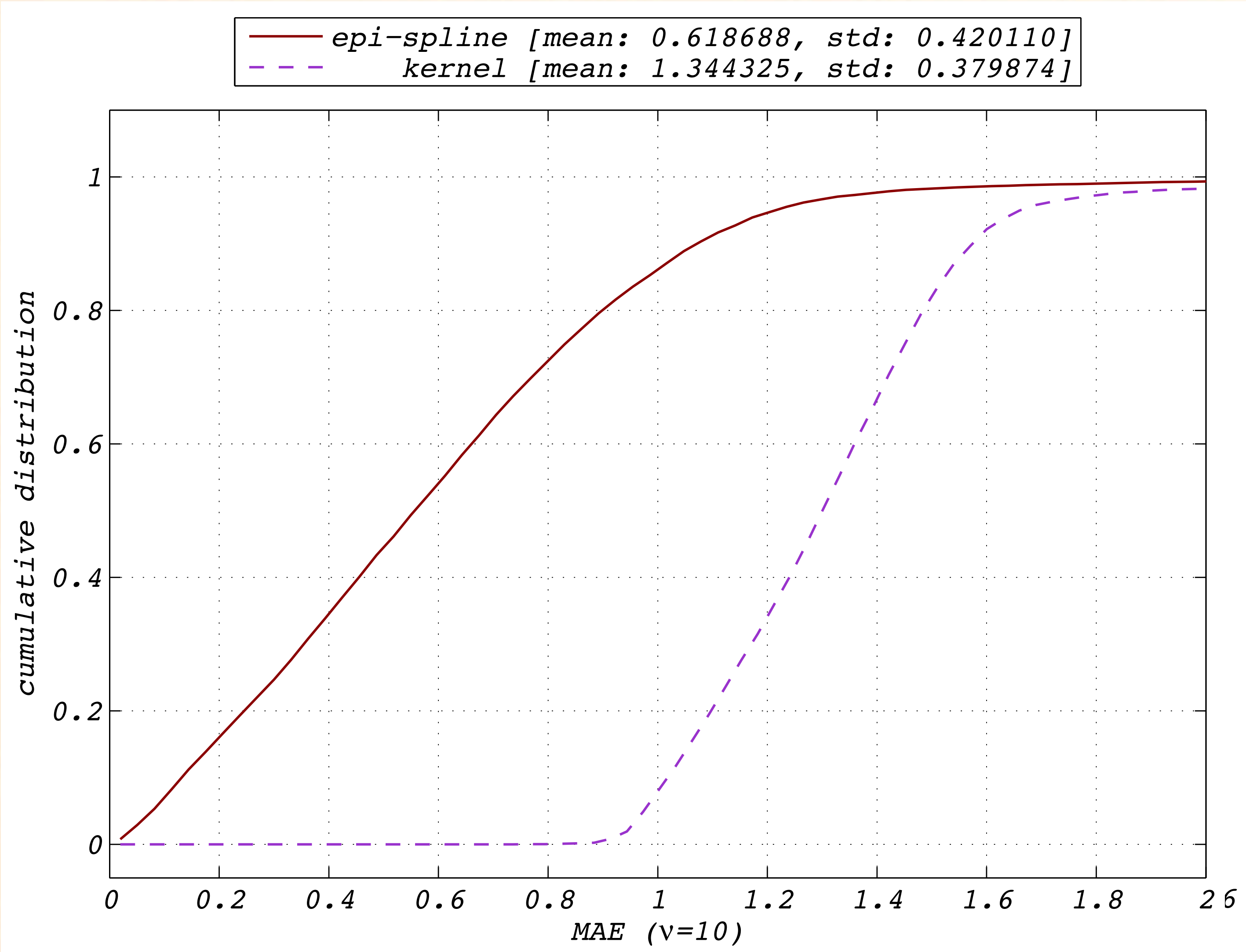

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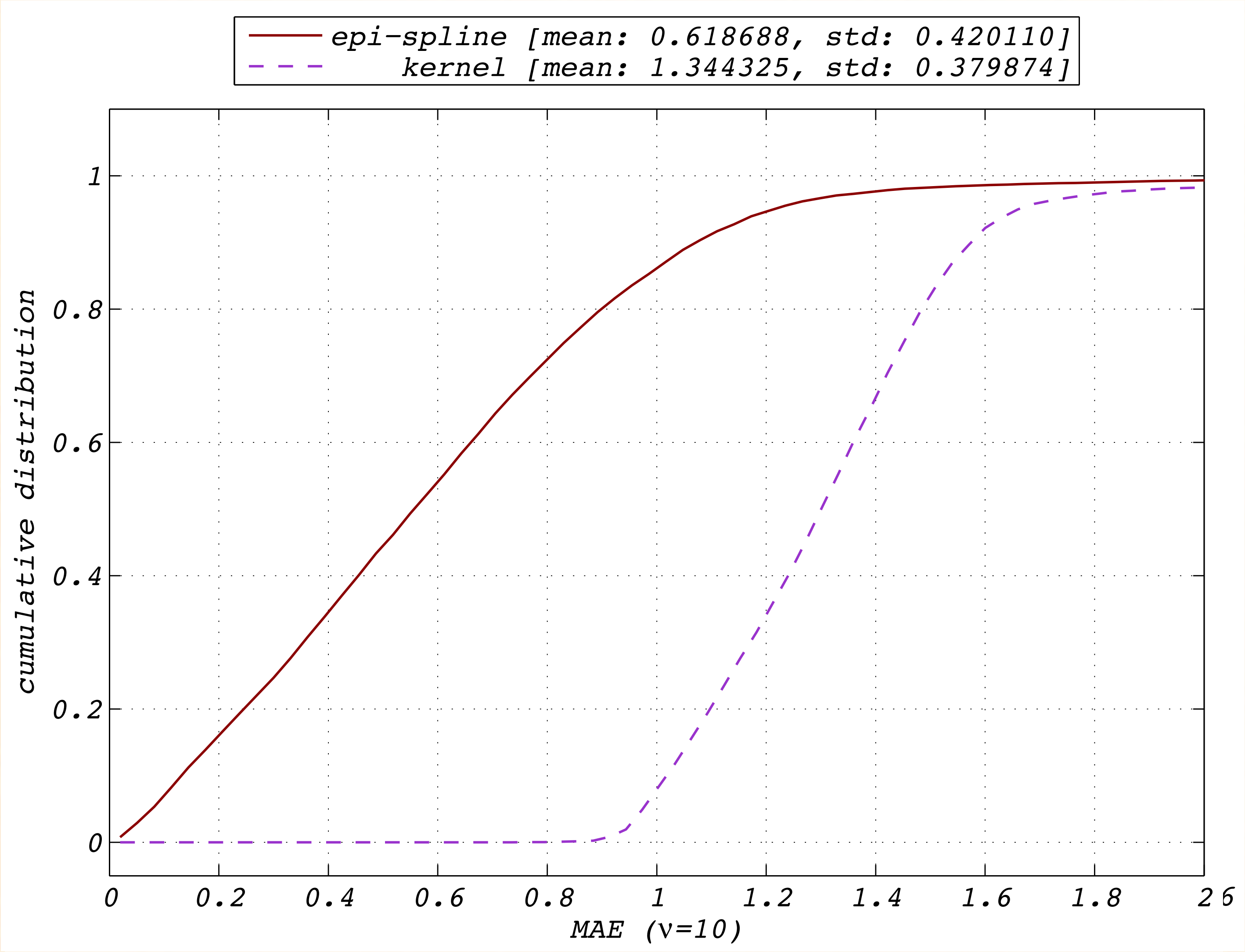
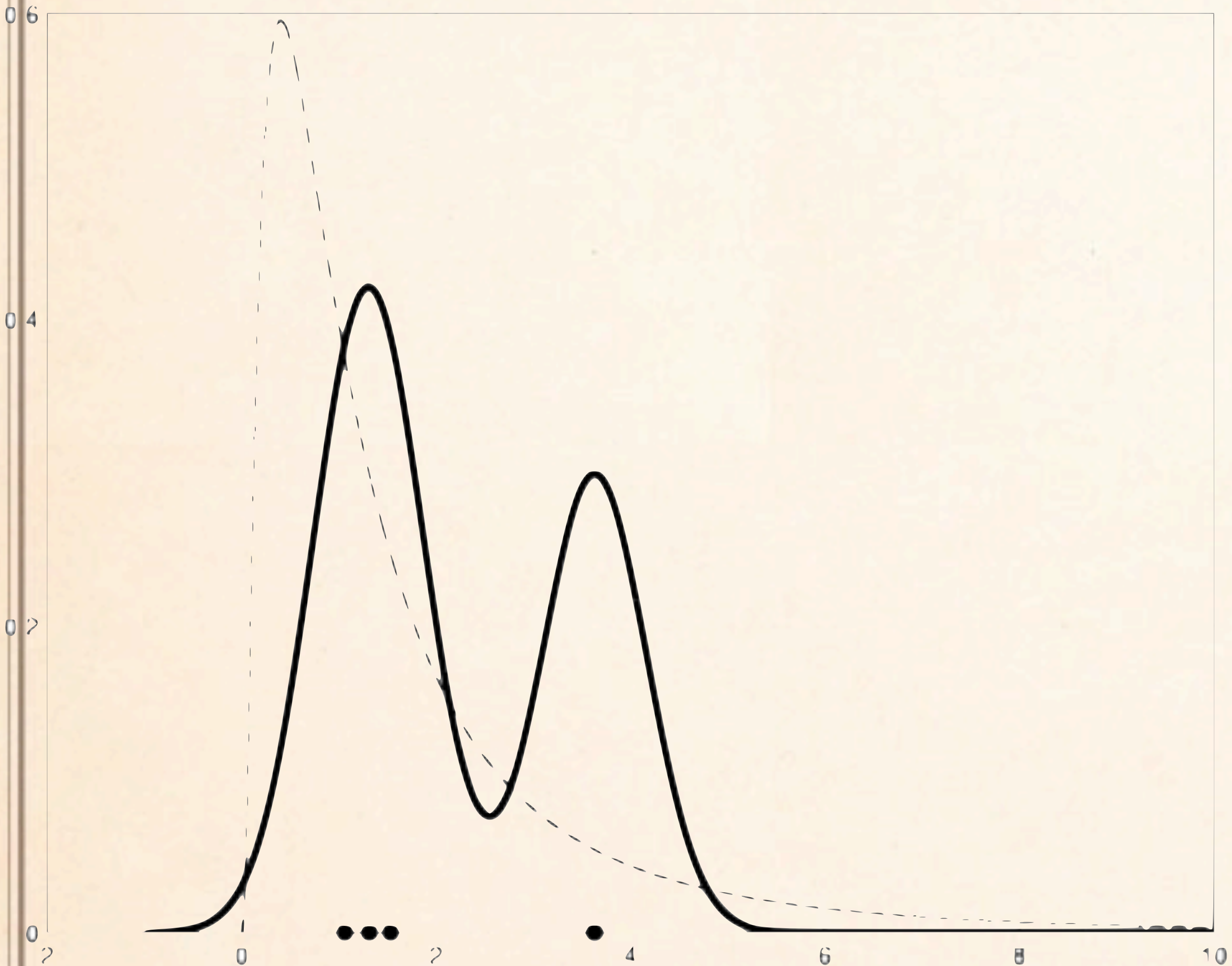


Max. absolute error
samples: 10
runs: 10,000
kernel estim: 1.344
epi-spline est.: 0.619

LOG-GAUSSIAN, N = 4!



R-Stat (kernel) estimate

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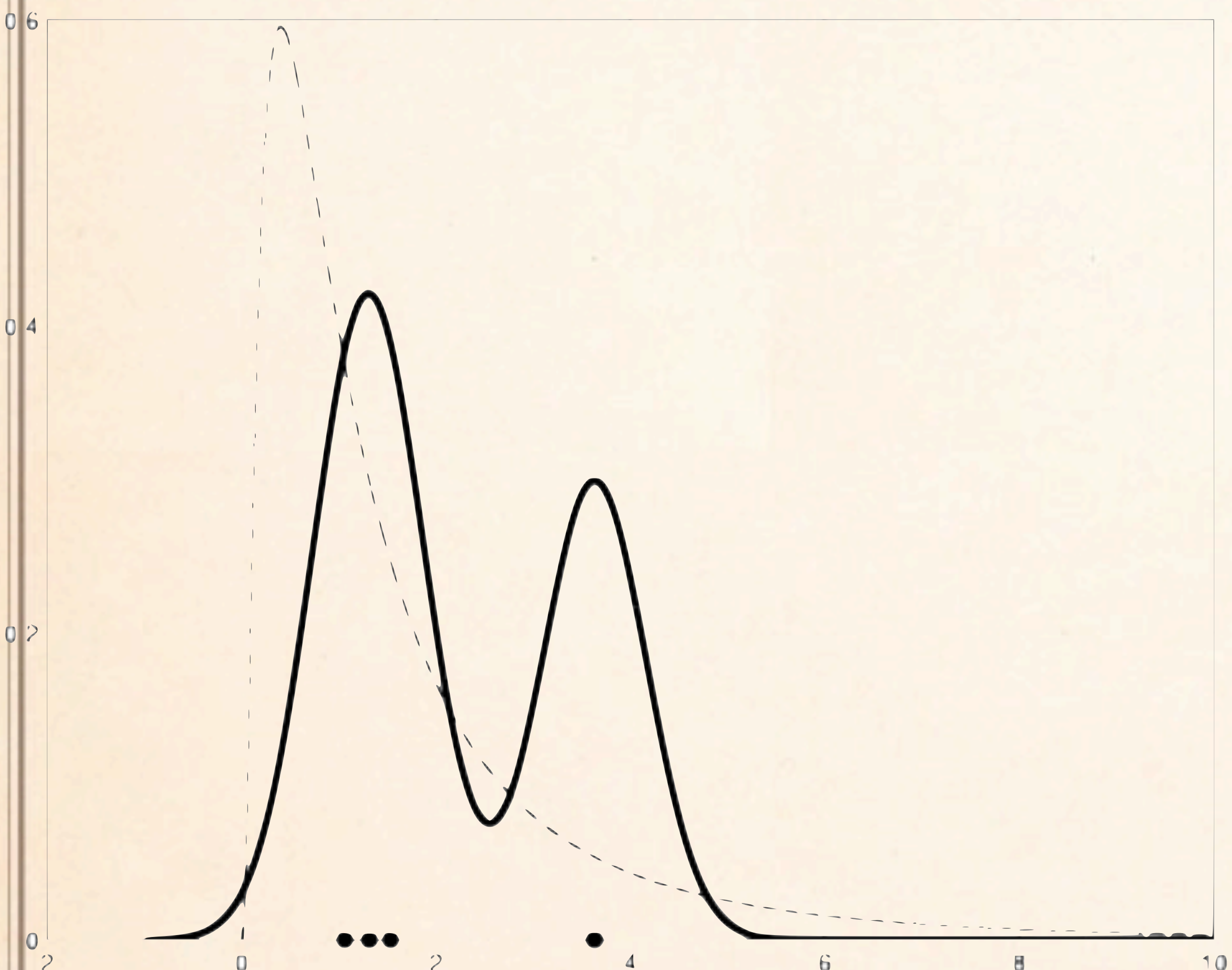

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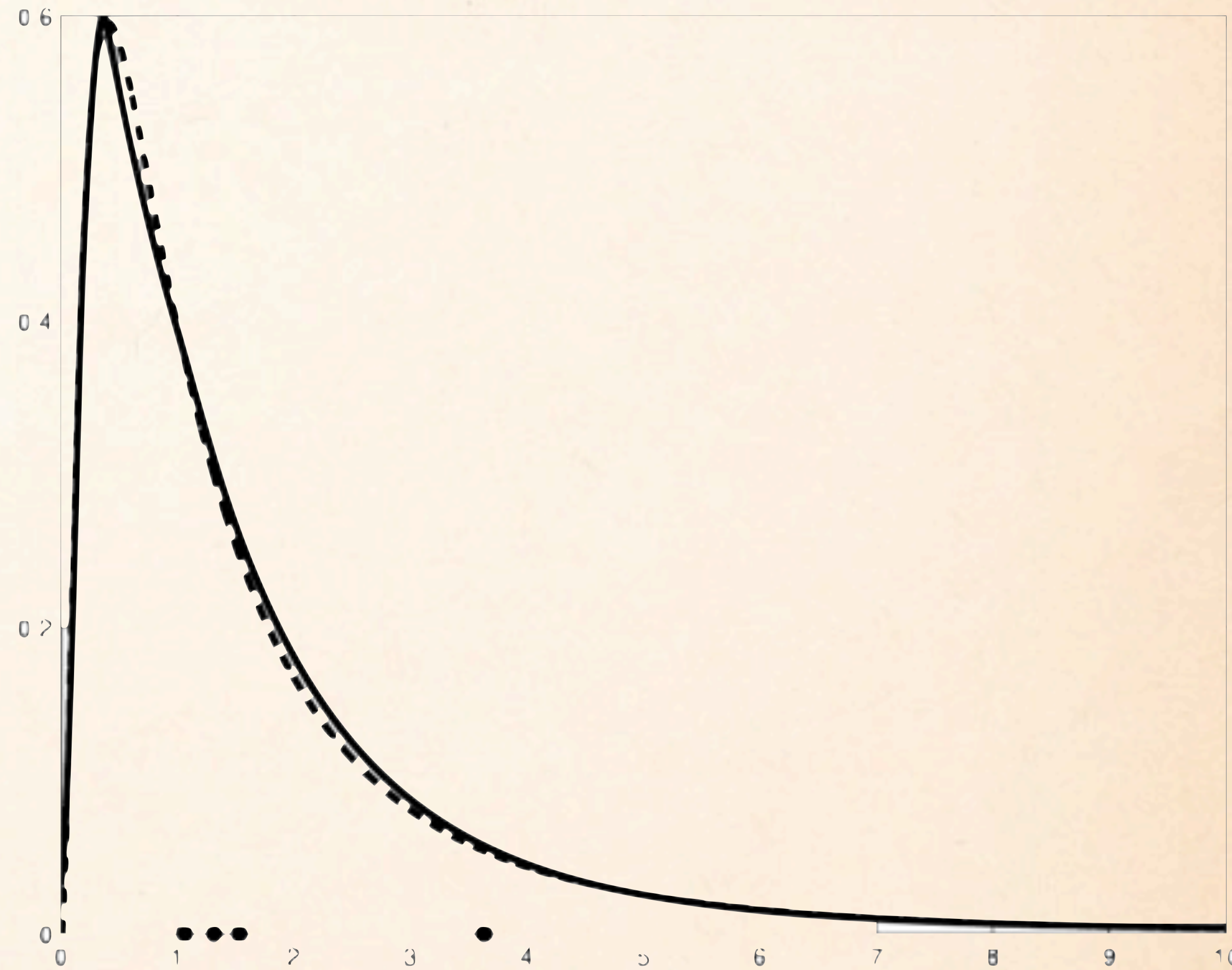


support + unimodal

EXTENSIVE EXPERIMENTATION

www.math.ucdavis.edu/~prop01

EPI-SPLINES
An Approximation Tool

EPI-SPLINE: 2nd Order

$c:(a,b] \to \mathbb{R}$ twice differentiable (not C^2)

$c'':(a,b] \to \mathbb{R}$ $2nd$ derivative approzimated by $z:\mathbb{R} \to \mathbb{R}$

split (a,b]: $\{(x_{k-1},x_k],k=1,\dots,N\}$, N relatively large

fix $z(t)=z_k$ (constant) for $t \in (x_{k-1},x_k]$

$2nd$ order epi-spline (1-dim.) +++ mesh = $\max_{k=1,\dots,N} |x_k - x_{k-1}| = m$

$$s_m(x) = s_0 + v_0 x + \int_0^x dr \int_0^r dt\, z(t), \quad z(t) \equiv z_k \text{ on } (x_k, x_{k+1}]$$

$$= s_0 + v_0 x + \sum_{j=1}^{k} a_{kj} z_j \quad \text{when } x \in (x_k, x_{k+1}]$$

$s_m \in C^{1,+pl}$, linear w.r.t. $s_0, v_0, z_1, \dots, z_N$ (finite # parameters)

as mesh $m \searrow 0$, $\max \|s_m - c\|^2 \to 0$ and s_m epi-converges to c

EPI-SPLINES

originally (Wets, Bianchi & Yang, 2002):

derive financial curves,

later, also stochastic volatility

Epi-splines of k*th* order:

piece-constant k*th* derivative

still linear w.r.t. its parameters

and epi-convergence to *c*

EXPONENTIAL EPI-SPLINE
1-dimensional, 2nd order

$$h(x) = e^{-s(x)}$$

$$s(x) = s_0 + v_0 x + \int_0^x dr \int_0^r dt\, z(t), \quad z(t) \equiv z_k \text{ on } (x_k, x_{k+1}]$$

$$= s_0 + v_0 x + \sum_{j=1}^{k} a_{kj} z_j \quad \text{when } x \in (x_k, x_{k+1}]$$

$$\max E^\nu \left[\ln h(\xi) \right] = \frac{1}{\nu} \sum_{l=1}^{\nu} \ln h(\xi^l) = \min \frac{1}{\nu} \sum_{l=1}^{\nu} s(\xi^l)$$

$$\text{such that } \int e^{-s(\xi)}\, d\xi \le 1. \qquad (h \ge 0)$$

$$z_k \in \left[-\kappa_l, \kappa_u \right] \quad \text{'constrained' curvature}$$

$$\text{unimodal: } \kappa_l = 0 \Rightarrow s(\cdot) \text{ convex}$$

HIGHER-DIMENSIONAL EPI-SPLINES
2-DIMENSIONAL - FIRST VERSION

$$s(x,y) = z_0 + v_1 x + v_2 y + \int_0^x d\tau \int_0^\tau d\theta\, a(\theta)$$

$$+ \int_0^y d\tau \int_0^\tau d\theta\, b(\theta) + \int_0^x d\tau \int_0^y d\theta\, c(\tau,\theta)$$

on $(x_{k-1}, x_k] : a(x) = a_k$,

on $(y_{k-1}, y_k] : b(y) = b_k$,

on $(x_{k-1}, x_k] \times (y_{l-1}, y_l) : c(x,y) = c_{kl}$

requires bouundary continuity properties

estimation : a_k and $b_k \Rightarrow$ marginal distributions

c_{kl} correlation coefficients (locally)

HIGHER-DIMENSIONAL EPI-SPLINES
2-DIMENSIONAL - FIRST VERSION

$$\nabla^2 s(x) = \begin{pmatrix} a_{k-1,l-1} & a_{k-1,l} \\ a_{k,l-1} & a_{k,l} \end{pmatrix} \text{ Hessian}$$

on open rectangle $(x_{k-1}, x_k) \times (y_{l-1}, y_l)$

unimodal $h(x) = e^{-s(x)} \Rightarrow s$ convex (globally)

$\nabla^2 s(x)$ positive semidefinite, symmetric

(3 parameters)

rectangle boundary values: (at a mesh)

via monotonicity of $\nabla s(x)$

all conditions included in the optimization problem

EXAMPLE: NORMAL DENSITY

mean = (0,0) ... data samples correlated

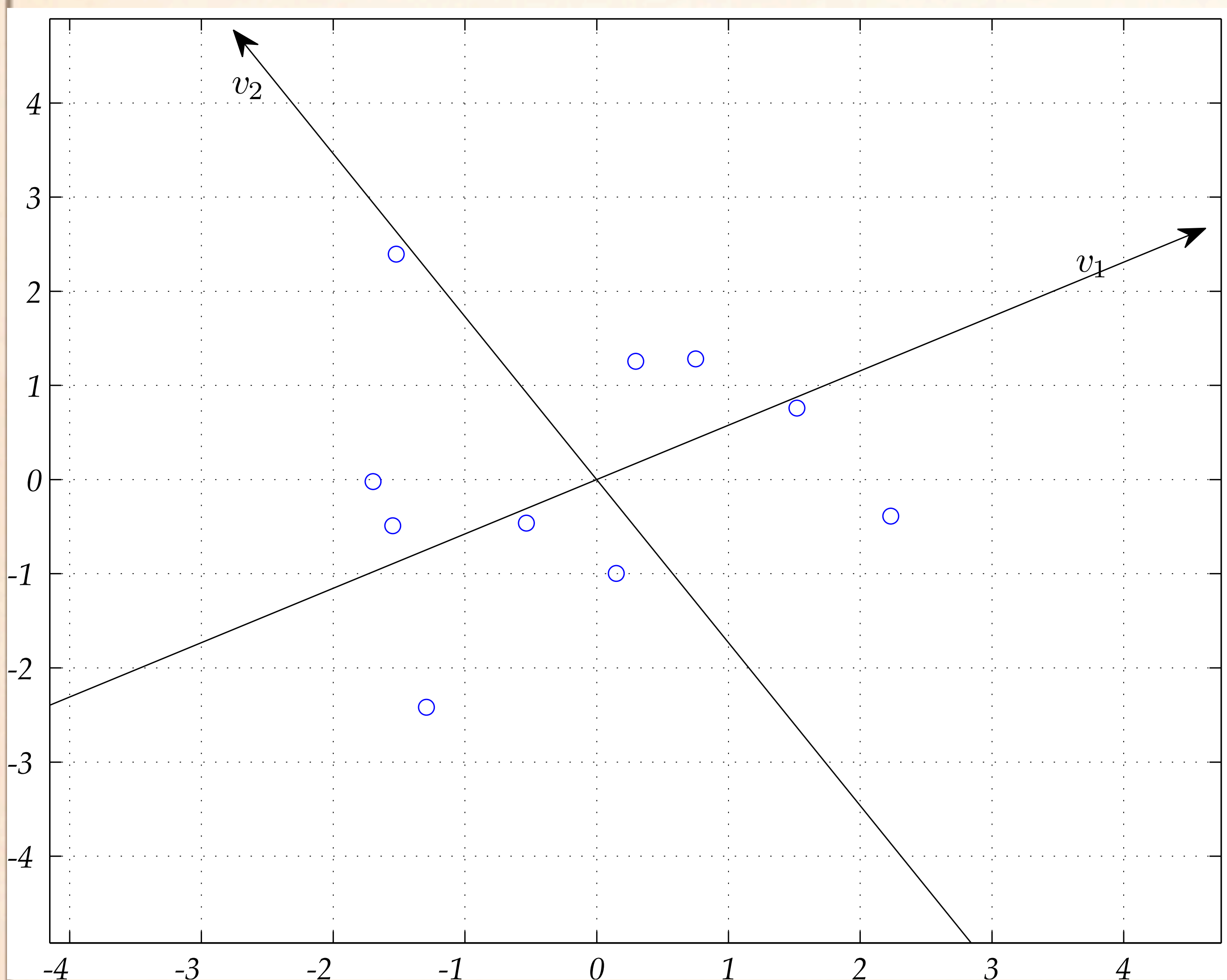
covariance: MDM^T, $D=$ diag(4,1), $M=\begin{pmatrix} \cos(\pi/6) & \cos(2\pi/3) \\ \sin(\pi/6) & \sin(2\pi/3) \end{pmatrix}$

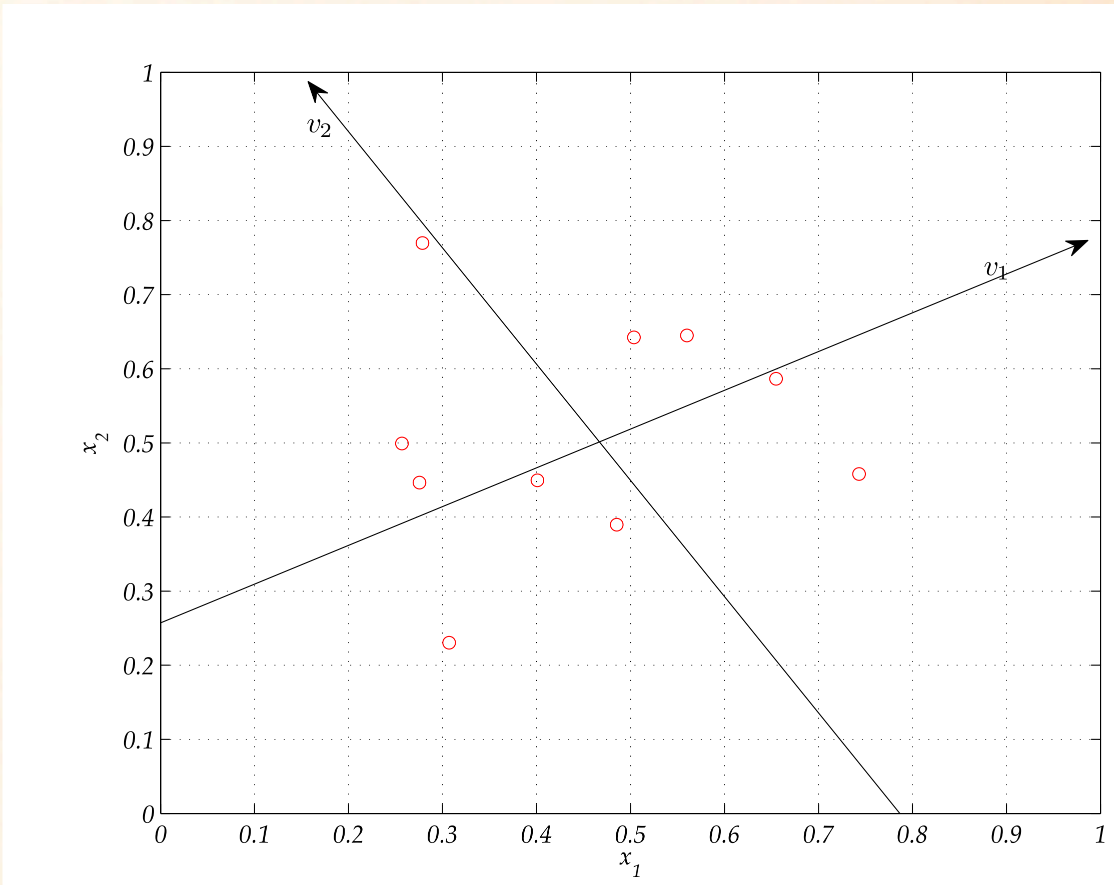
samples: $\nu = 10$, "soft" information: h unimodal

Results:

$$\left\| h^{true} - h^{est} \right\|_2^2 = 0.028, \quad \left\| h^{true} - h^{est} \right\|_\infty = 0.006$$

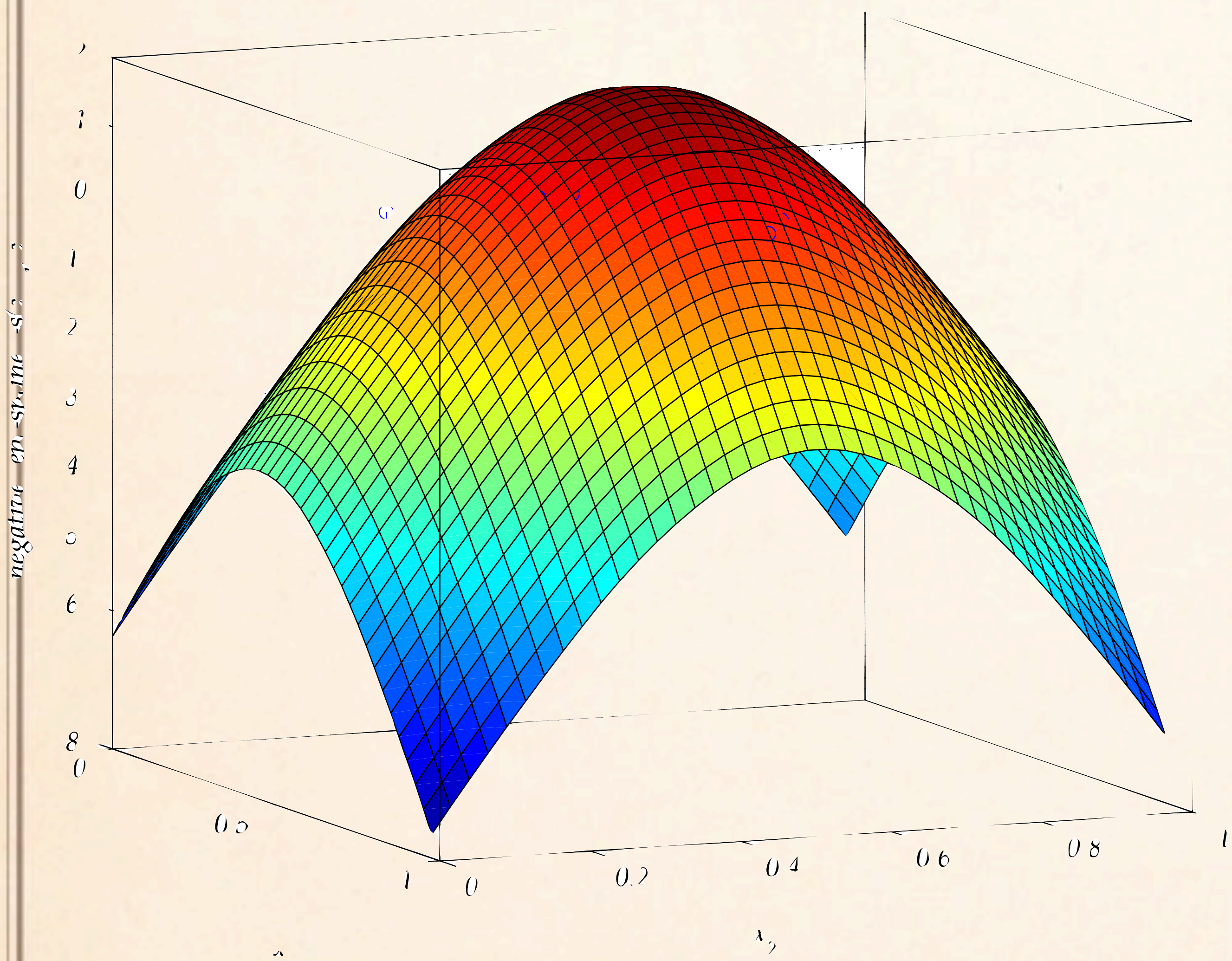
SAMPLED DATA

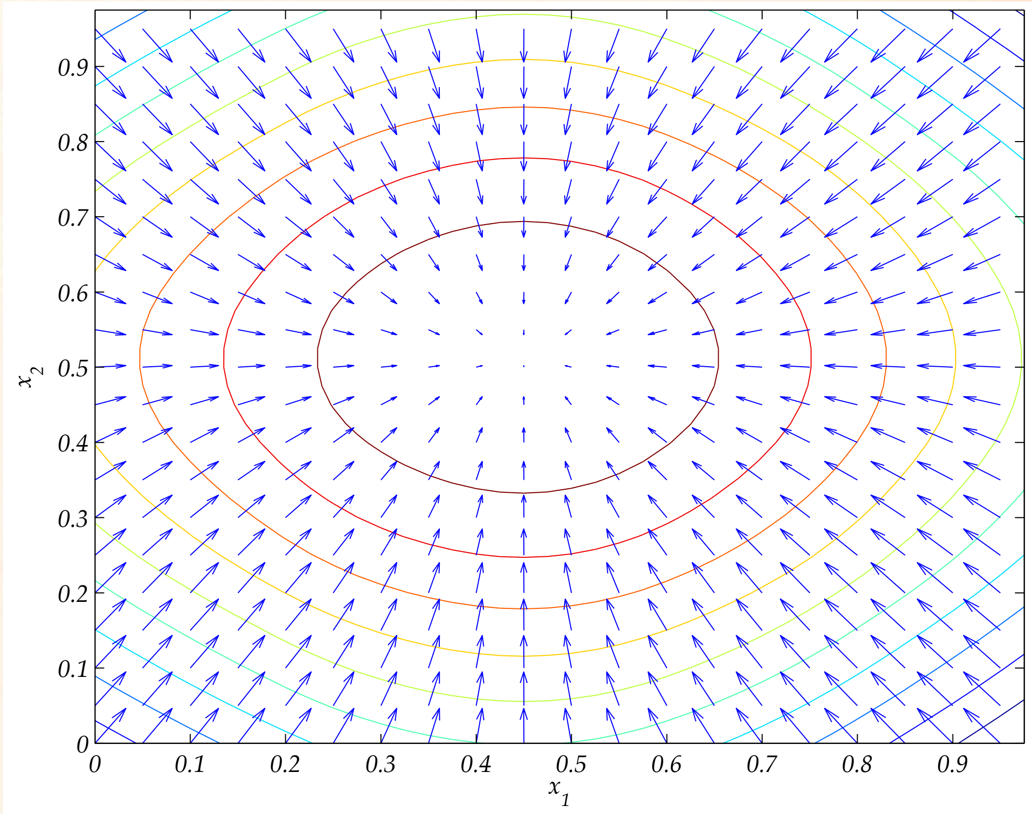




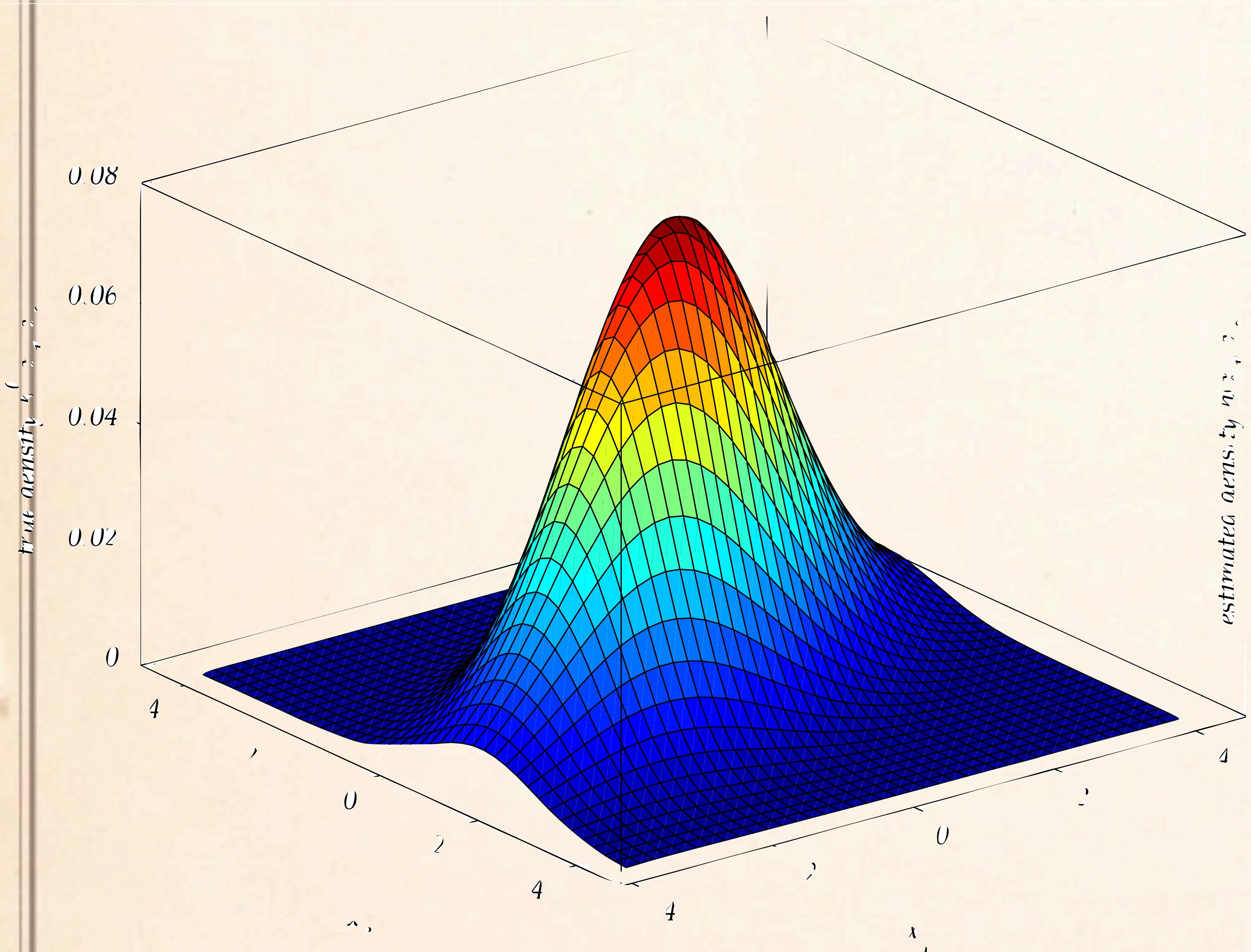
normalized

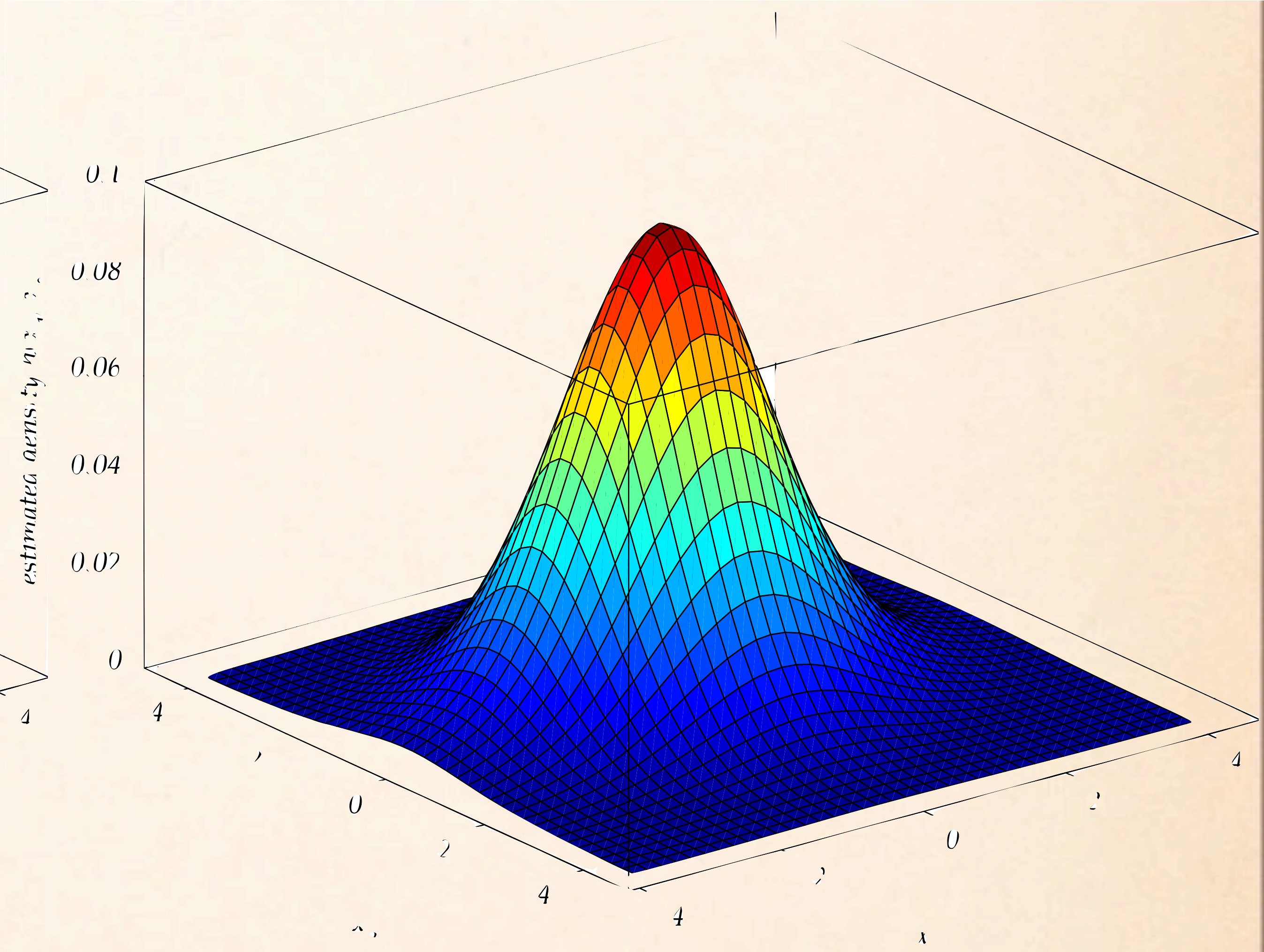
EPI-SPINE & VECTOR FIELD



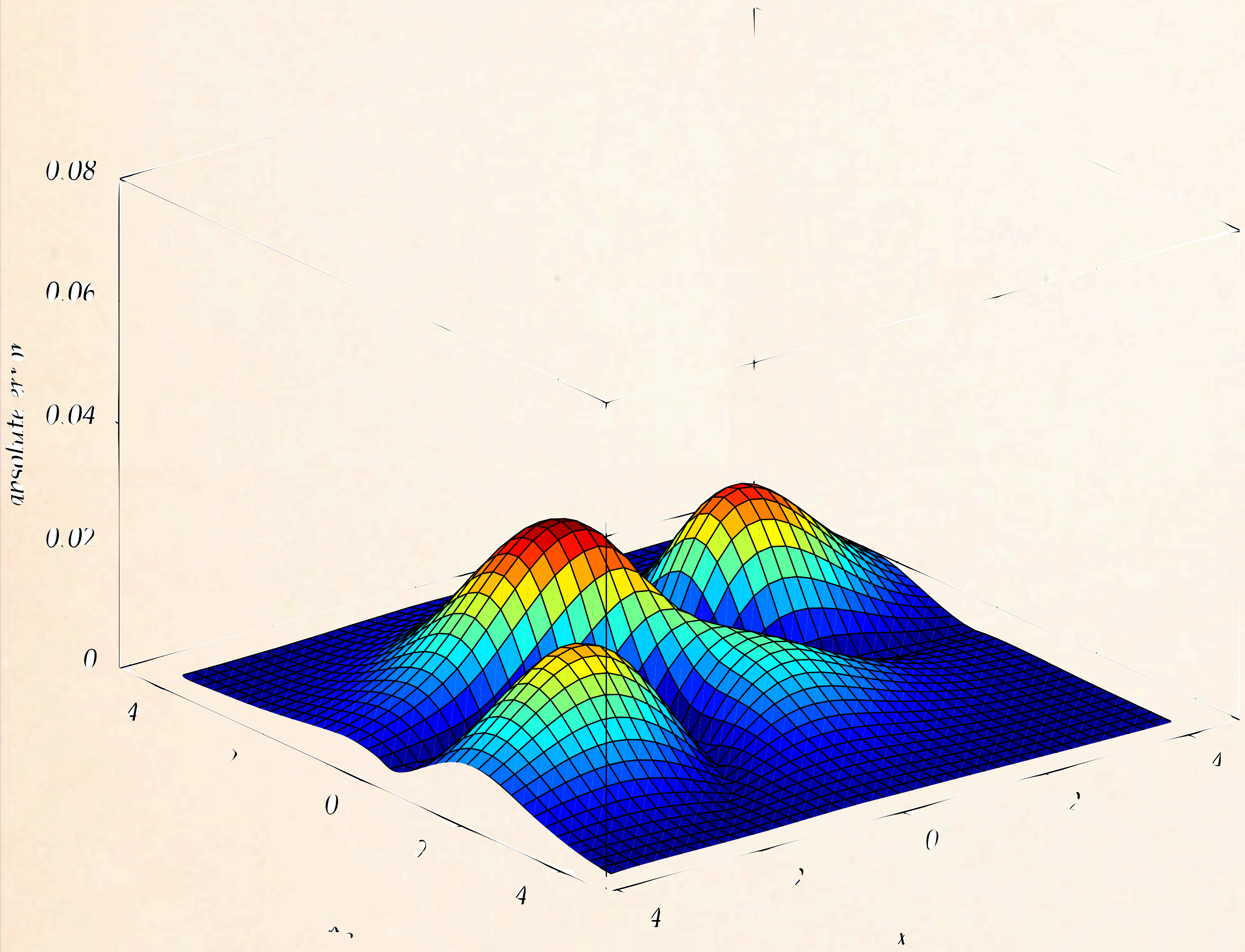


TRUE & ESTIMATED DENSITY

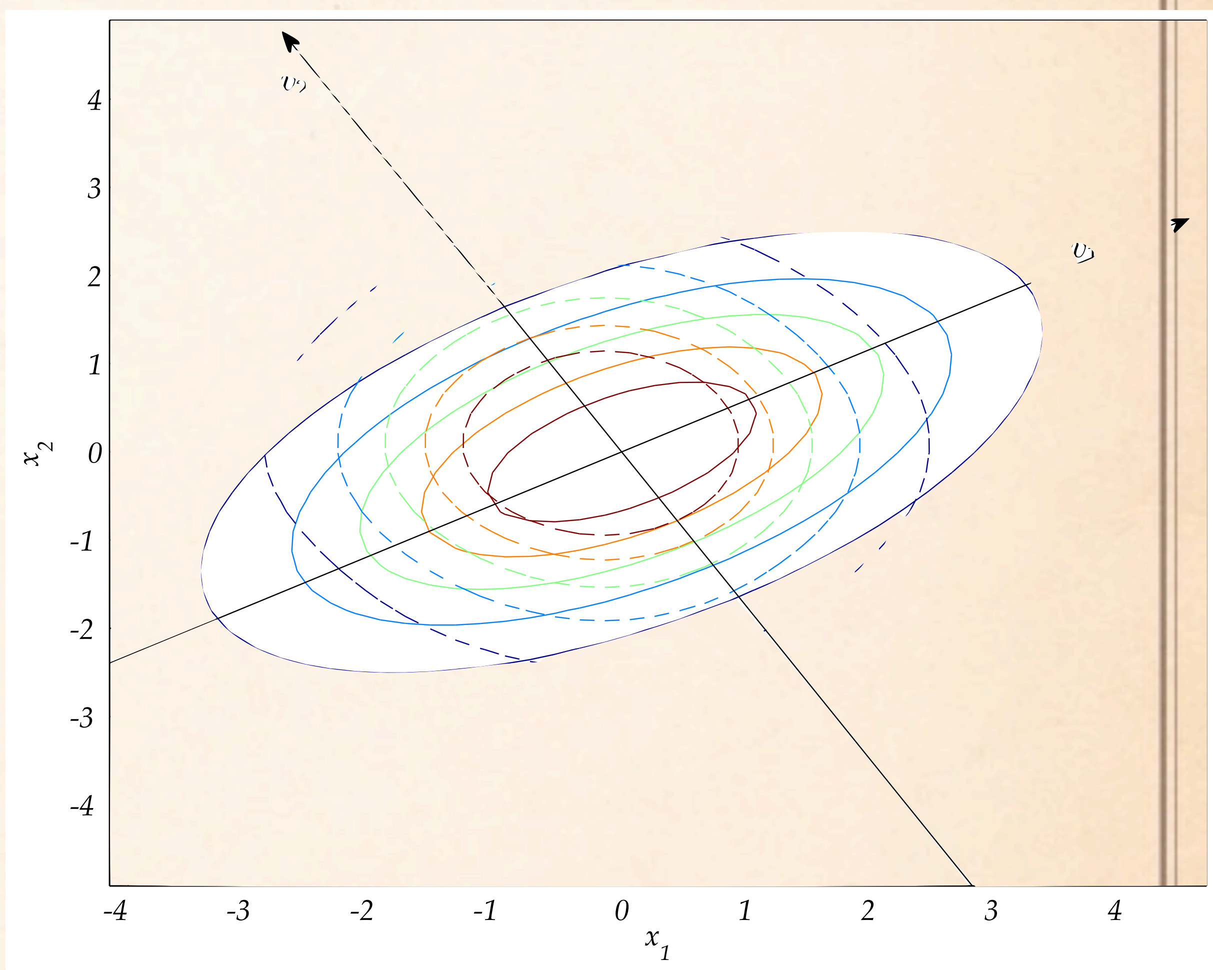




MEASUREMENT ERRORS

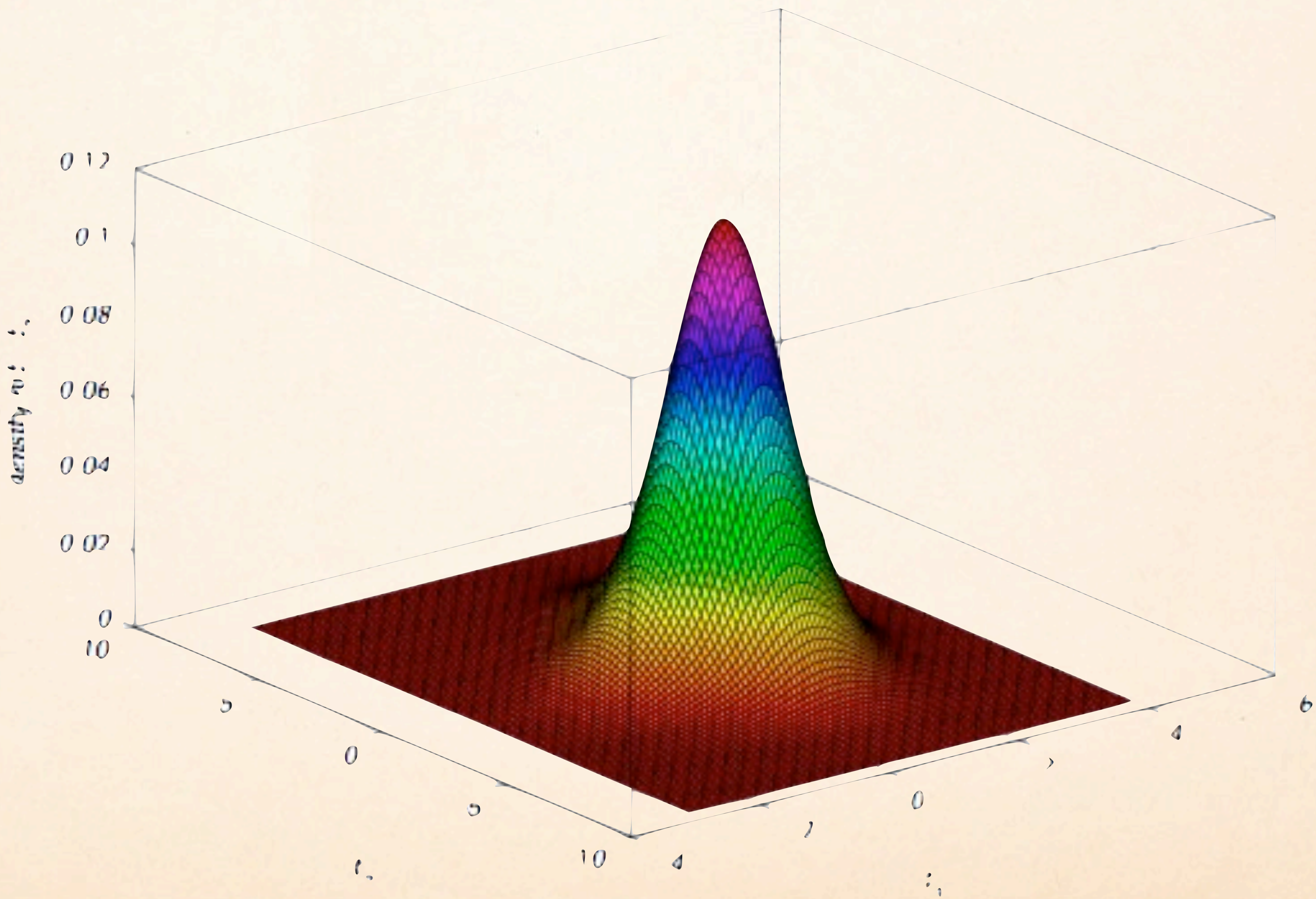


Absolute Error

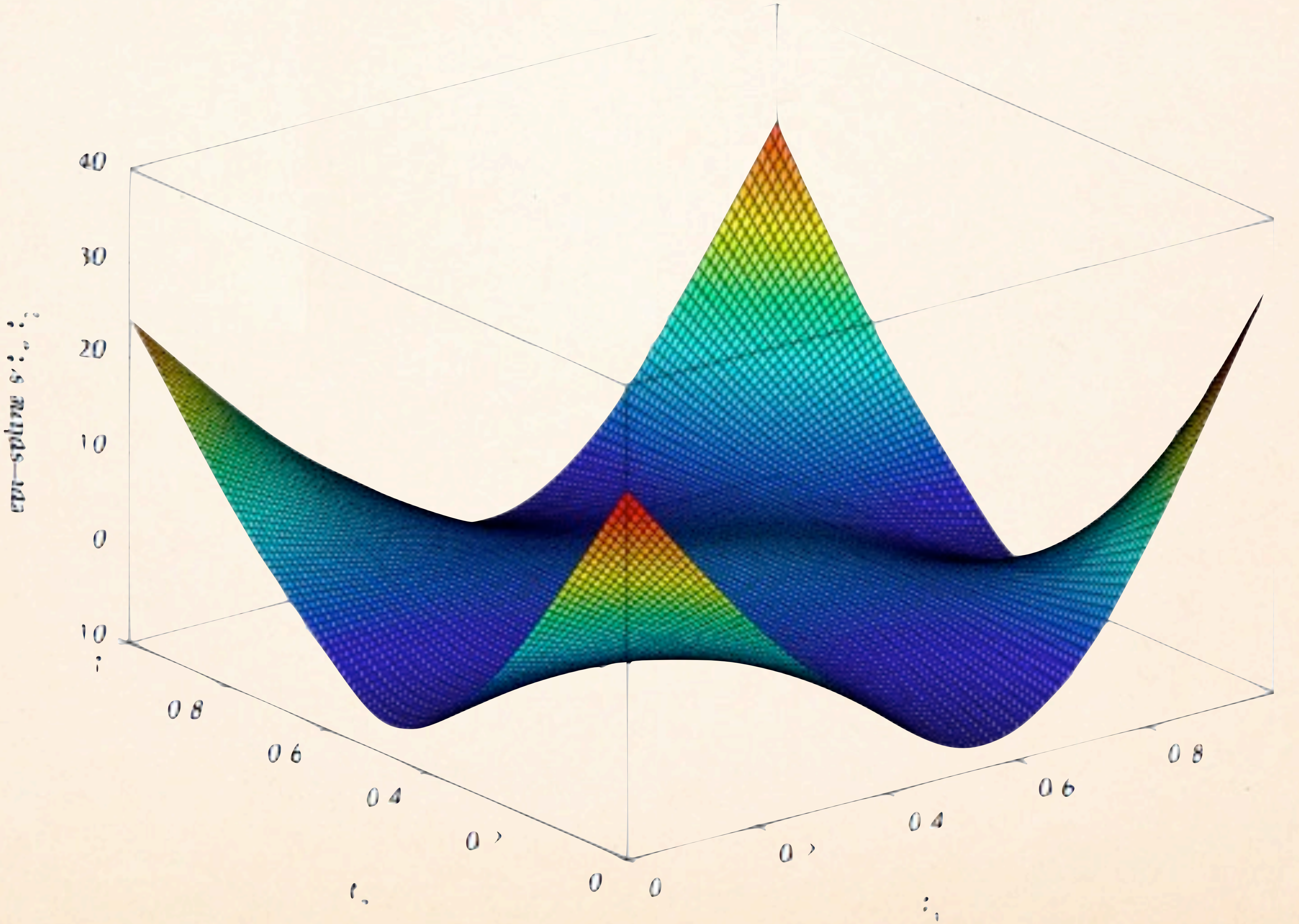


Level curves: true & estimate

NORMAL, 20 SAMPLES DIAGONAL COVARIANCE

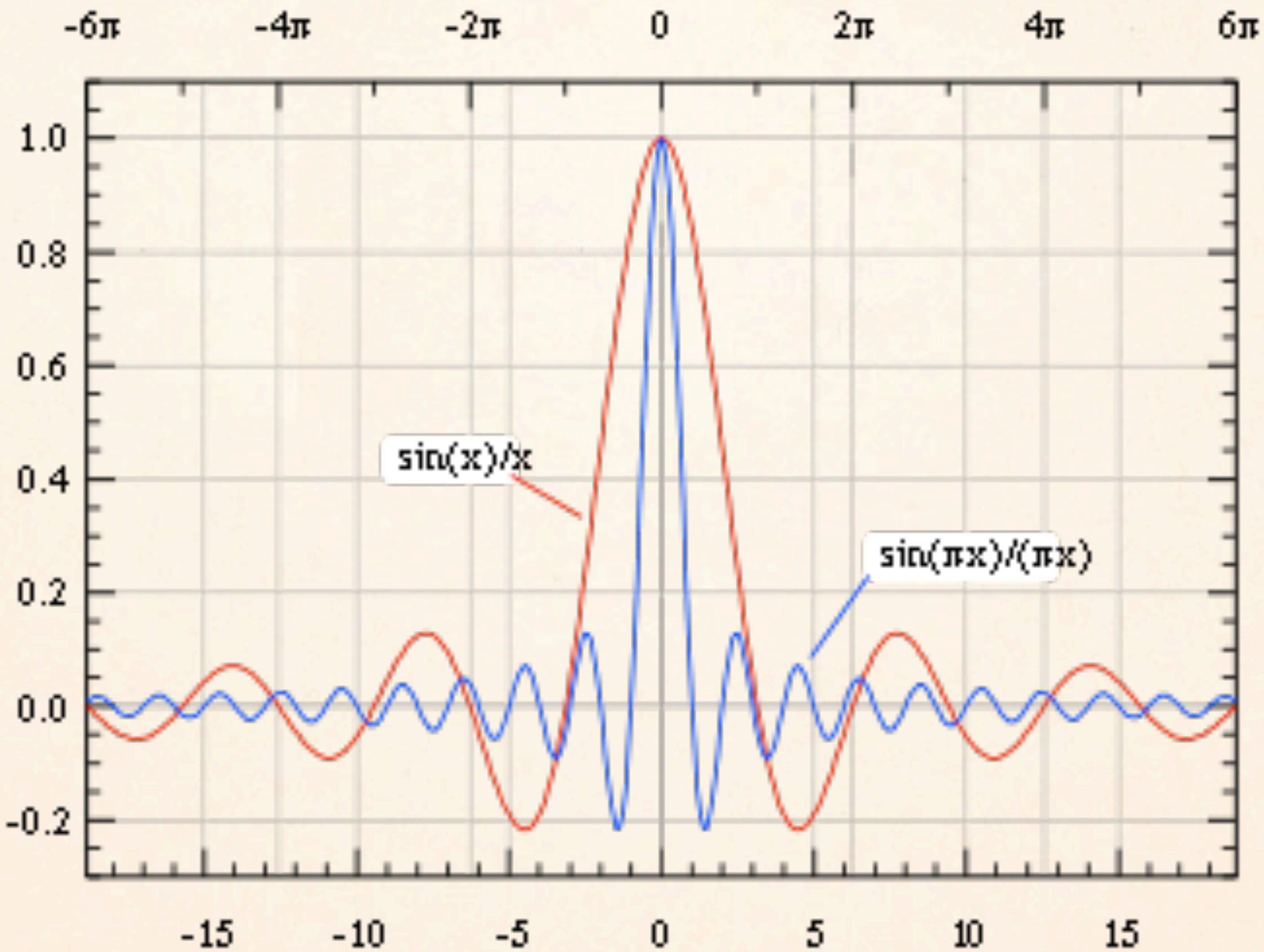


... THE -(EPI-SPLINE FUNCTION)



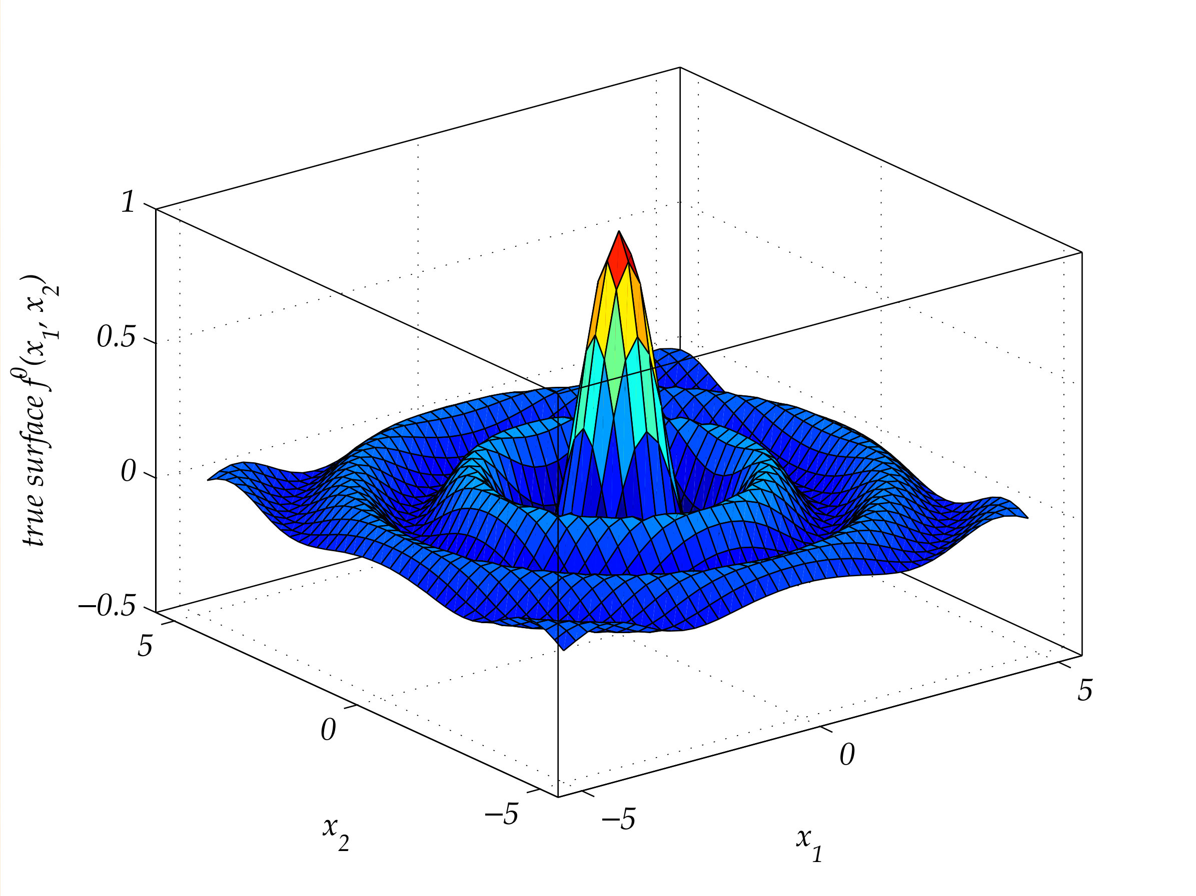
SPLINES & EPI-SPLINES

DIRICHLET (SINC-)FUNCTION

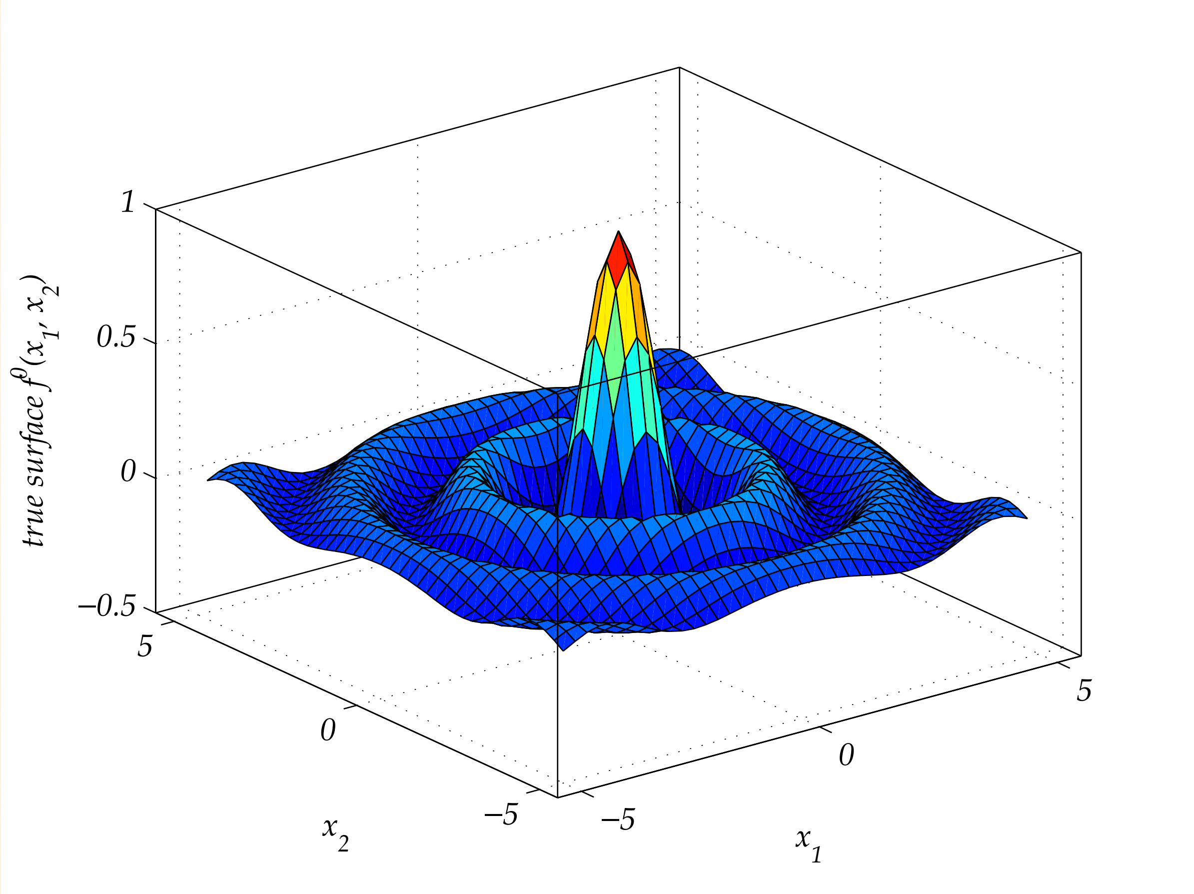


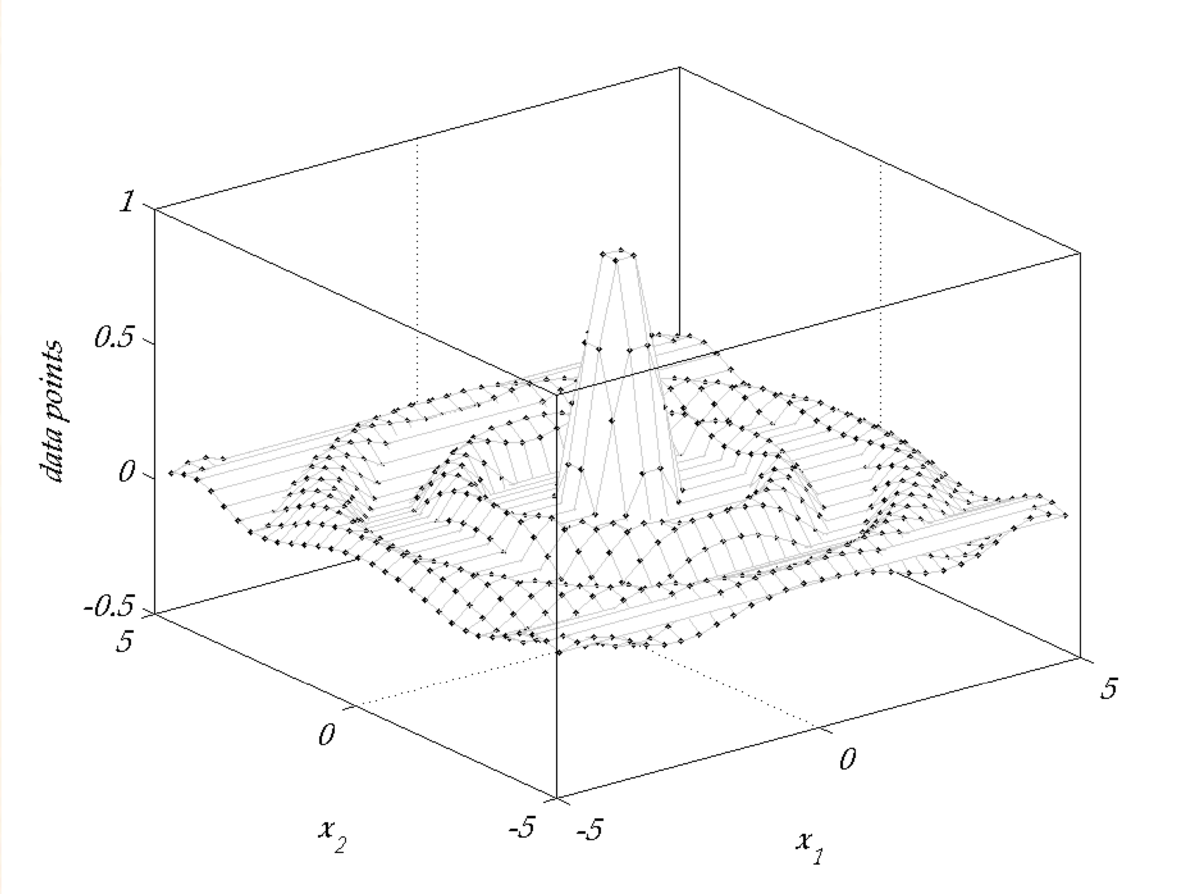
$$f(x) = \sin(\pi x / 2) / \pi x \text{ for } x \neq 0, \quad = 1 \text{ for } x = 0$$

SPLINES & EPI-SPLINES

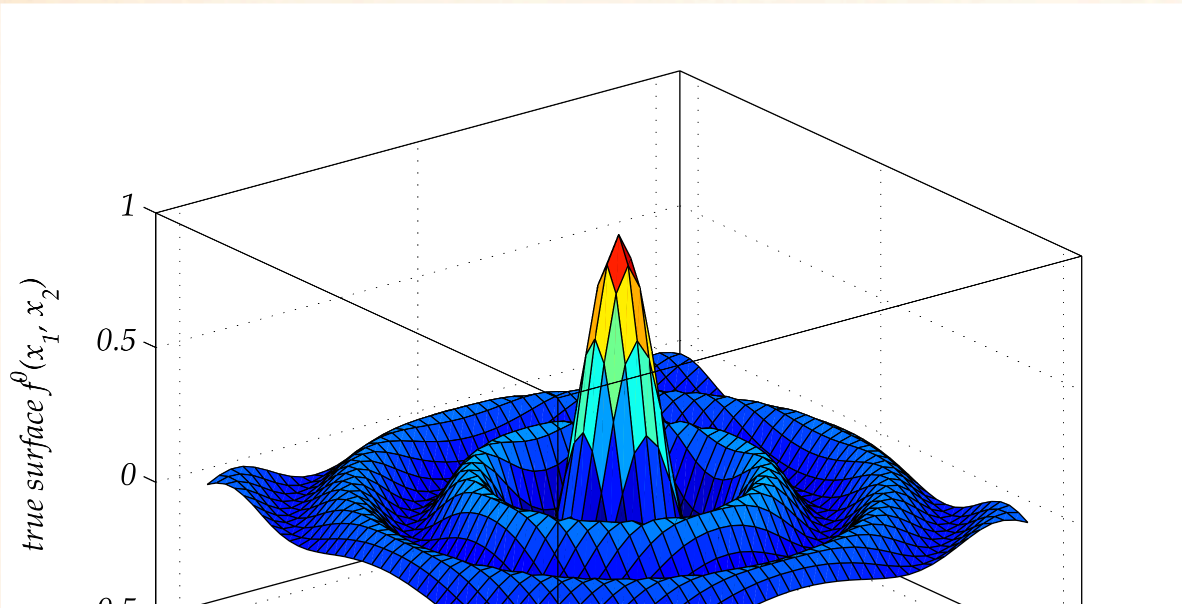


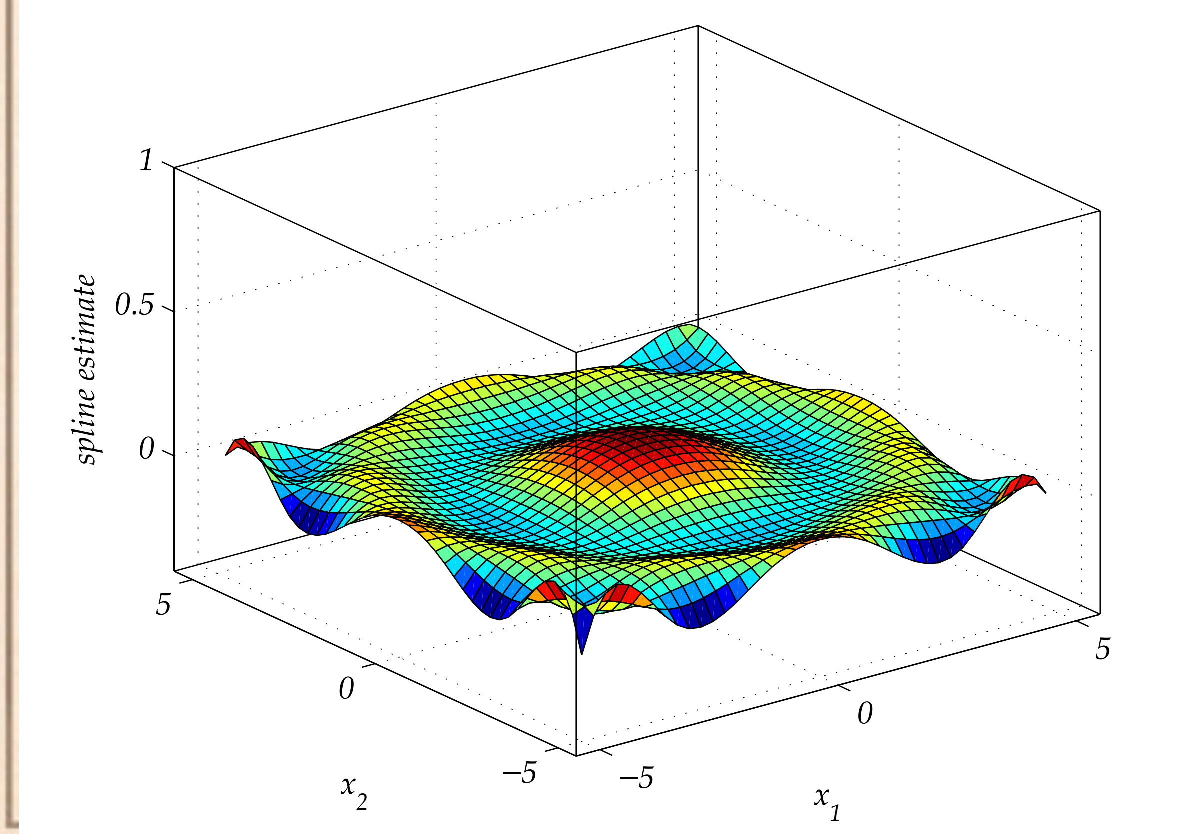
SPLINES & EPI-SPLINES

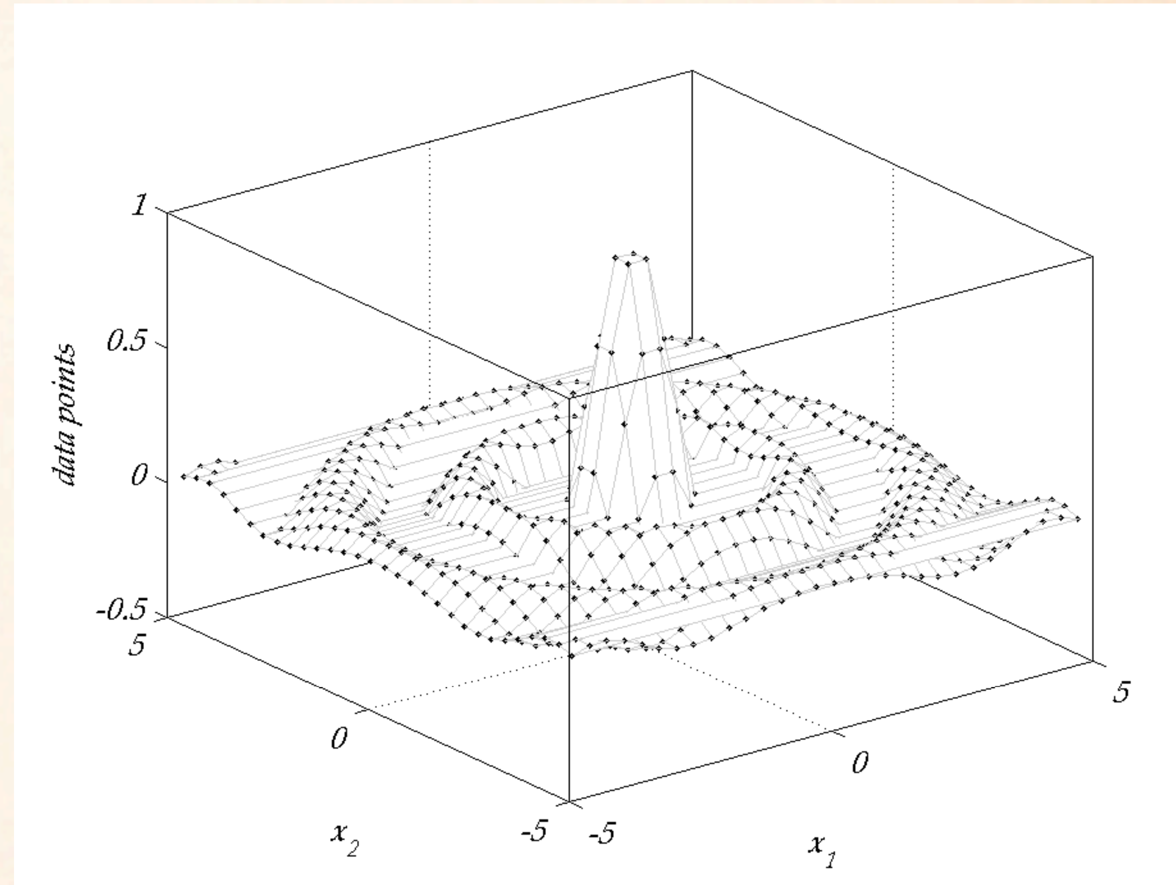




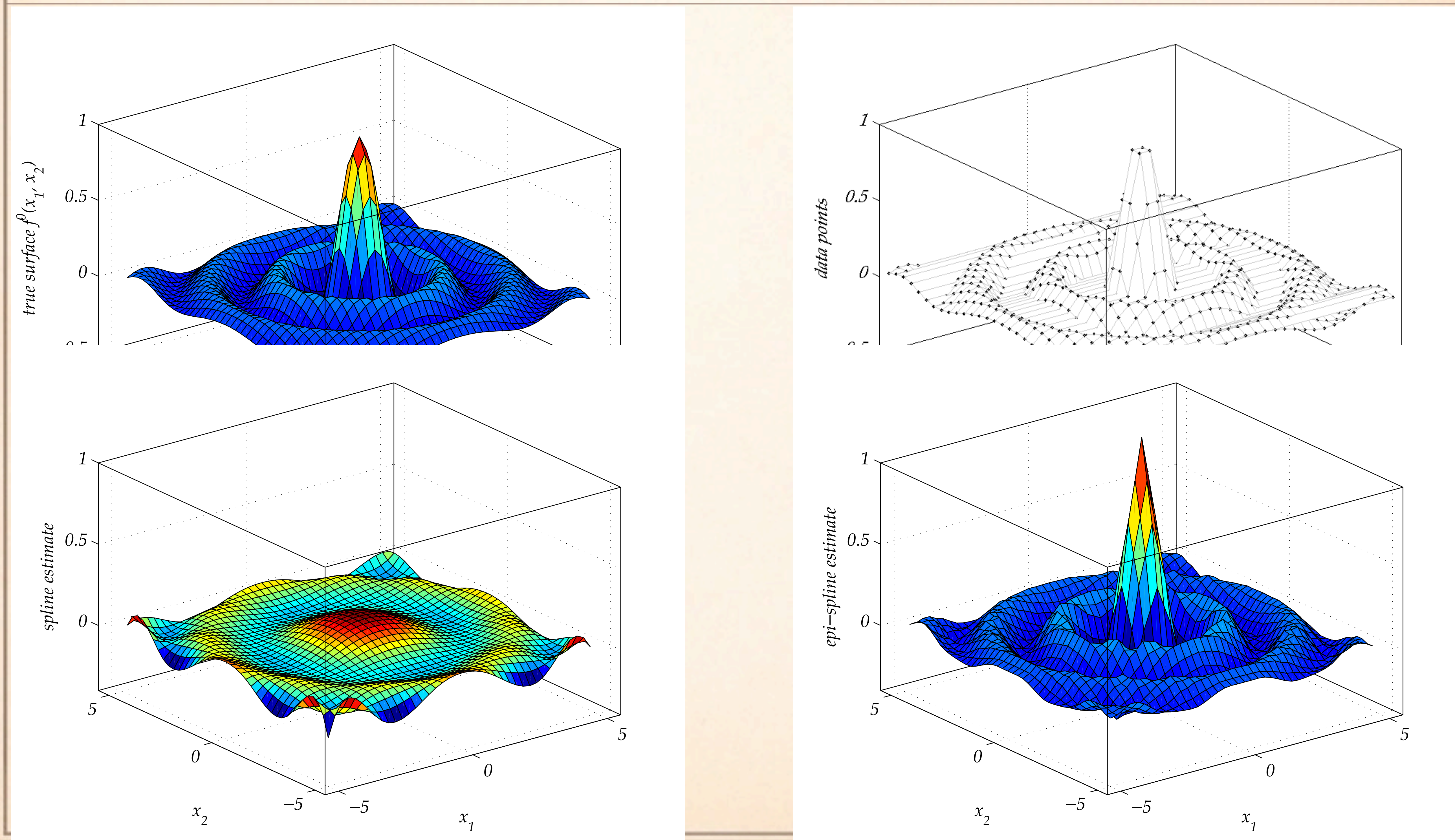
SPLINES & EPI-SPLINES



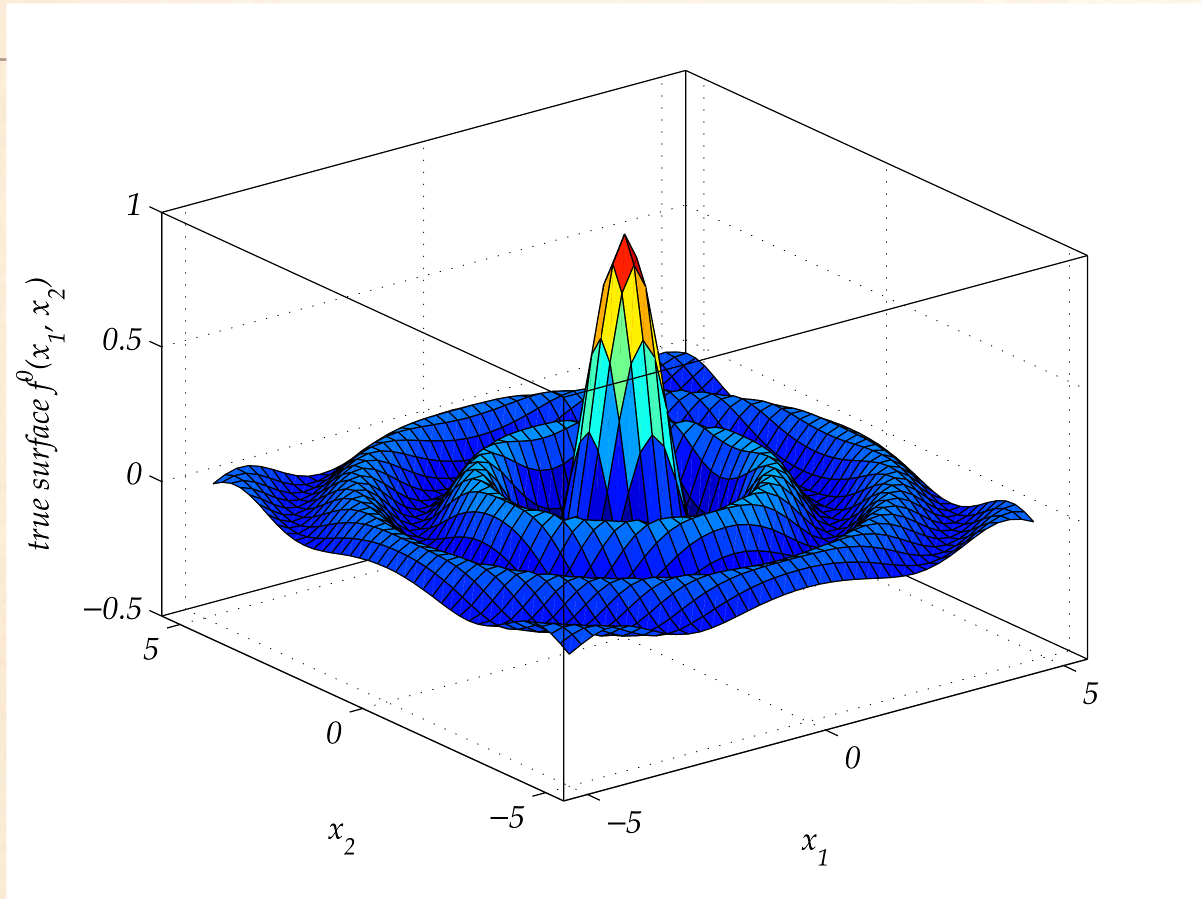




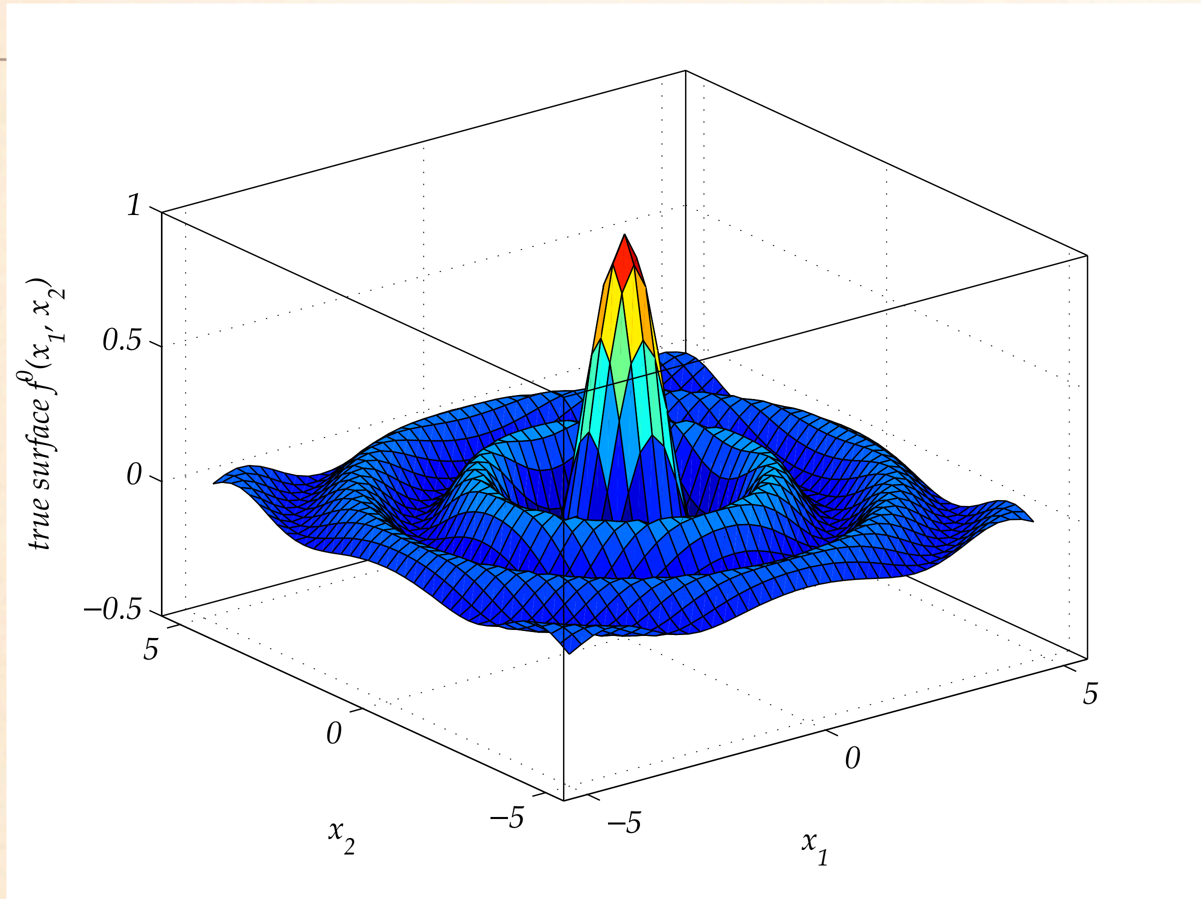
SPLINES & EPI-SPLINES

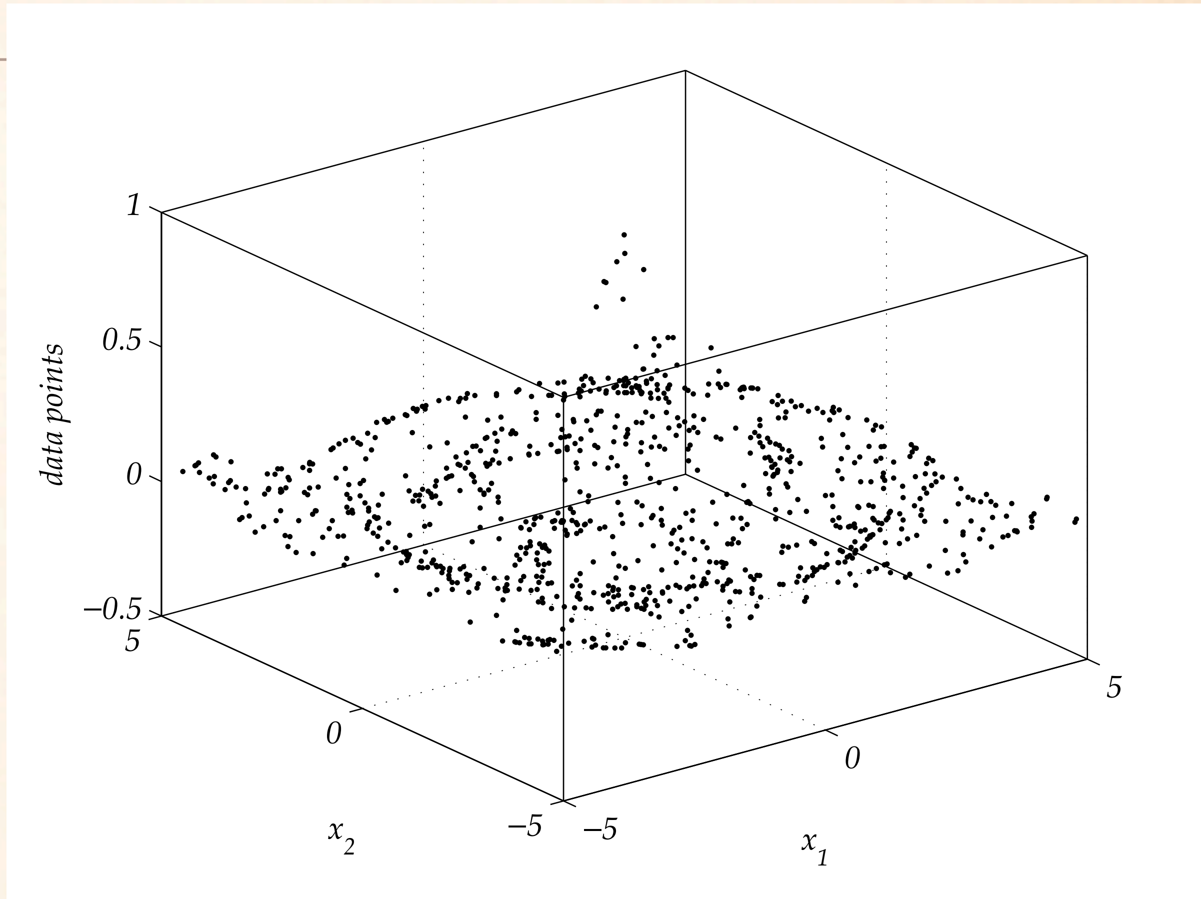


SPLINES & EPI-SPLINES 2

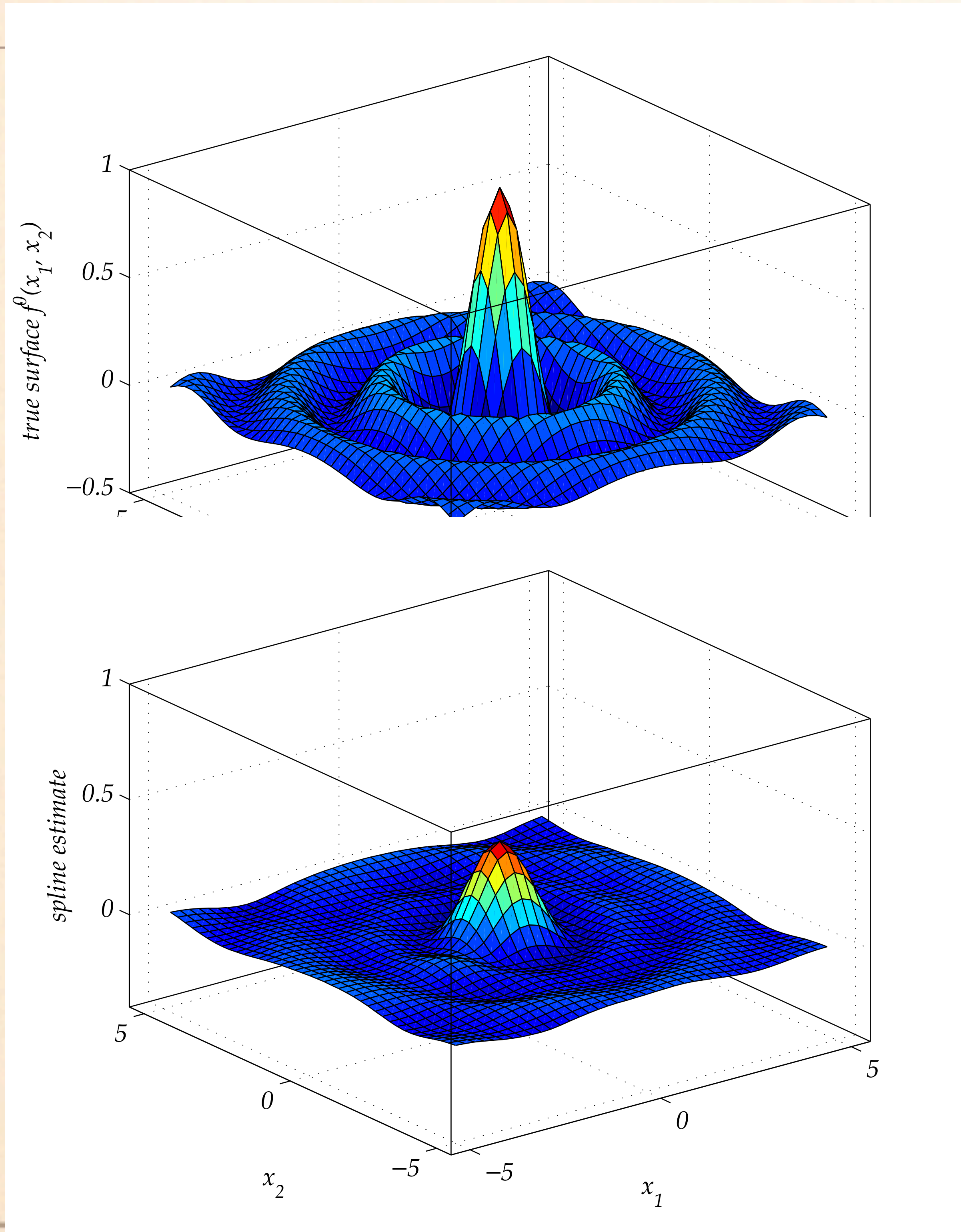


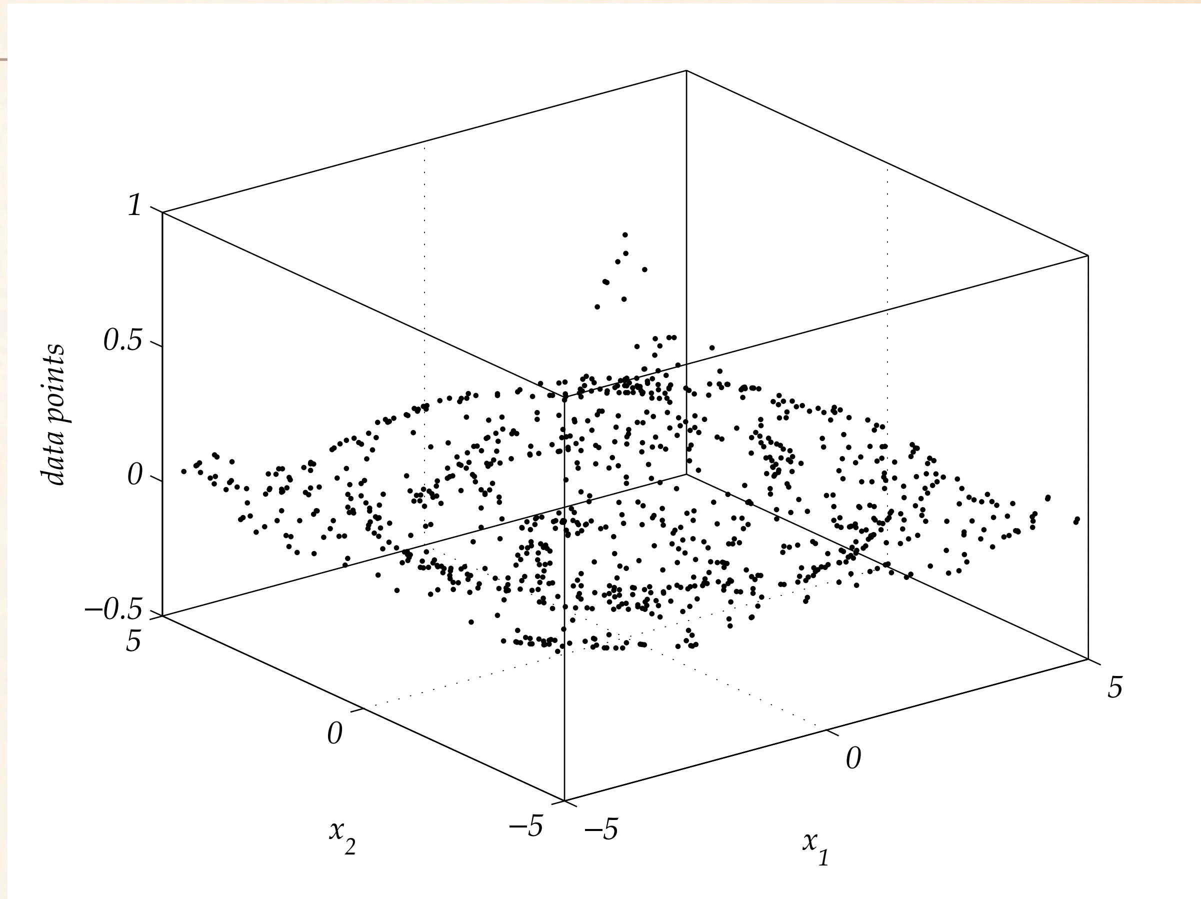
SPLINES & EPI-SPLINES 2



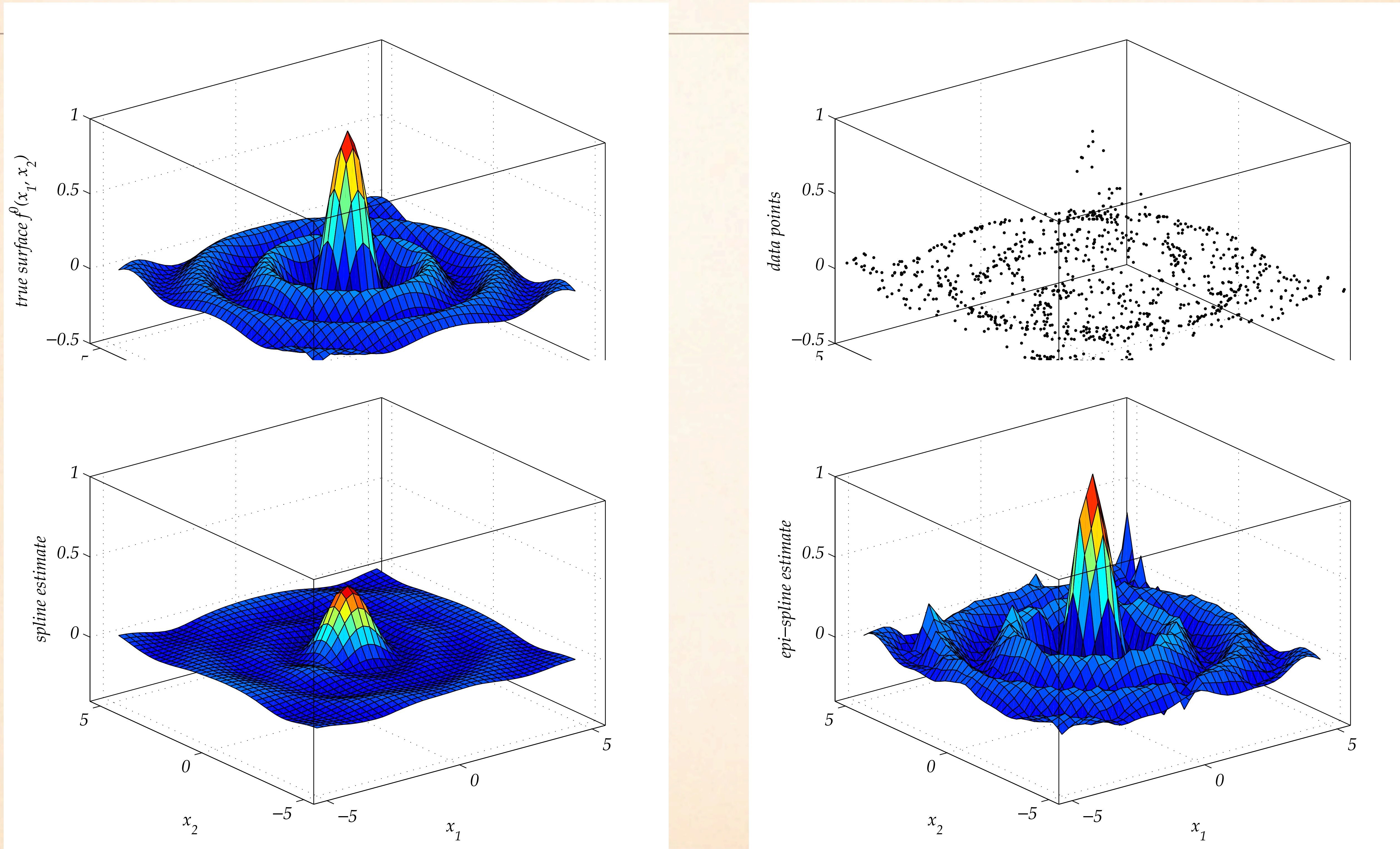


SPLINES & EPI-SPLINES 2

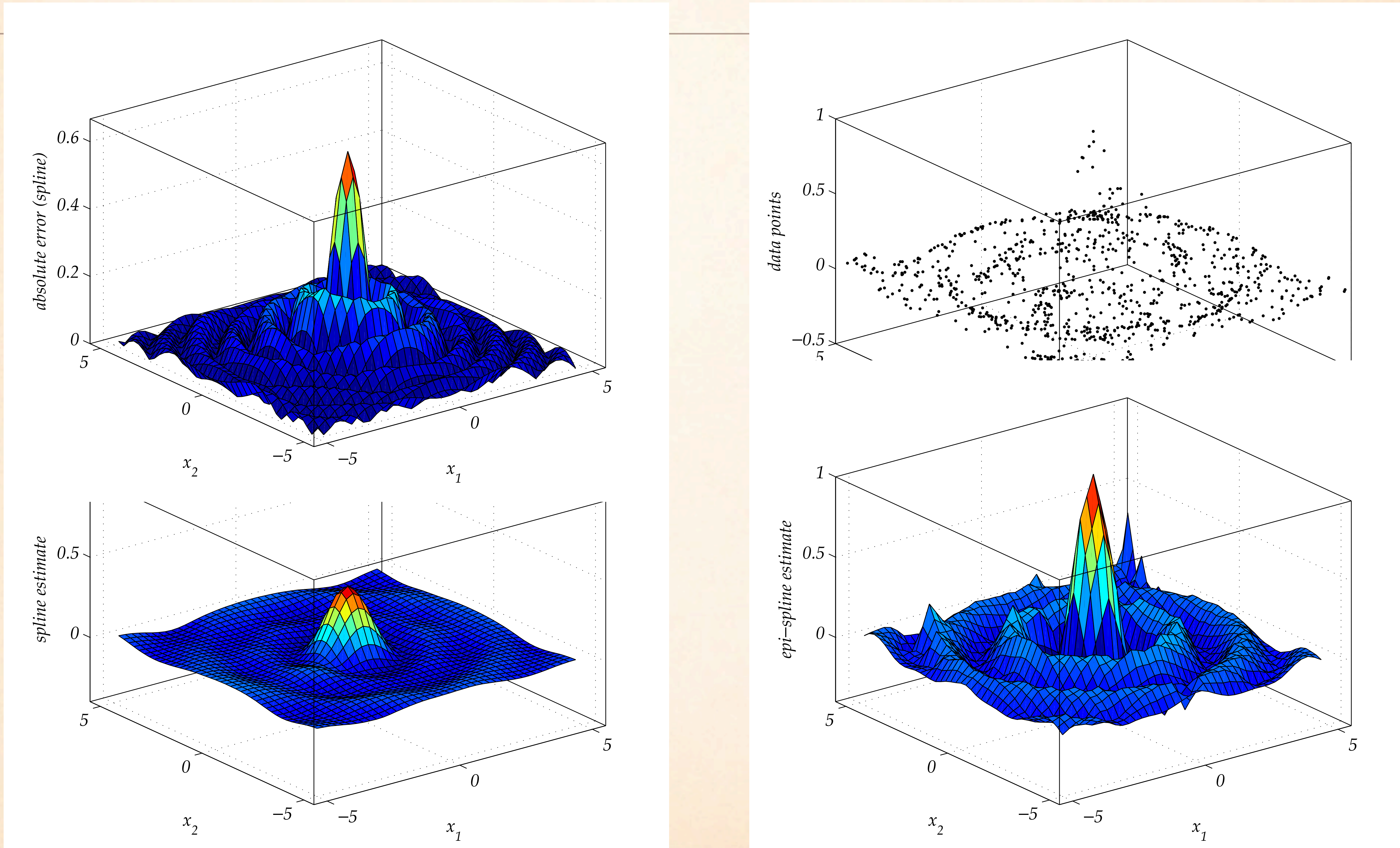




SPLINES & EPI-SPLINES 2



SPLINES & EPI-SPLINES 2



SPLINES & EPI-SPLINES 2