Final Exam

CS 525, Semester I, 2006-2007

Tuesday December 19, 2006
2 hours (starting 5:05)

All questions carry equal credit. No calculators allowed. Be sure to quote any results you use accurately.

1. Solve the following problem:

\[
\begin{align*}
\text{max} & \quad \min \{x_1 - 2x_2, 4 - 3x_1 - x_2\} \\
\text{subject to} & \quad x_1 \leq 2 + x_2 \\
& \quad x_1 \geq 1
\end{align*}
\]

Is the solution unique? Justify.

2. Consider

\[
z(t) = \min \quad 3x_1 + 2x_2
\]
subject to \(x_1 + x_2 \geq 7 + t\)
subject to \(x_1 \leq 5 - t\)
subject to \(x_1 \geq 0, x_2 \leq 0\)

(a) Find a value of \(t\) for which \(z(t)\) is finite.
(b) Find \(z(t)\) for all \(t \in \mathbb{R}\).
(c) What properties does the function \(z(t)\) have?
3. Show that precisely one of (I) and (II) has a solution:

\[(I) \quad Cx < d\]
\[(II) \quad C^Tv = 0, d^Tv \leq 0, 0 \neq v \geq 0\]

You may want to do this via the following steps:

(a) Show that both systems cannot have a solution.
(b) Suppose (II) has no solution. Construct a linear program that under this assumption will be solvable.
(c) Construct the dual of the above linear program.
(d) Quote and use appropriate theory to generate a system that has solution, and then show this implies (I) has a solution.

4. Solve

\[
\begin{align*}
\text{min} & \quad \frac{1}{2}x_1^2 - 2x_1x_2 + 2x_2^2 + 4x_2 \\
\text{subject to} & \quad x_1 + 2x_2 \leq 3 \\
& \quad 2x_1 - x_2 \geq 4 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

(a) What if the constraint \(2x_1 - x_2 \geq 4\) is replaced by \(2x_1 - x_2 = 4\)?
(b) What if the objective is replaced by

\[
\frac{1}{2}x_1^2 - 2x_1x_2 + 3x_2^2 + 4x_2
\]

(Note the only change is to the coefficient of \(x_2^2\)).
(c) With the modified objective, is the solution unique?

Be sure to quote any theorems that you appeal to carefully.