Final Exam

CS 525, Semester II, 2012-2013

Monday May 13, 2013
2 hours (starting 2:45)

All questions carry equal credit. No calculators allowed. Be sure to quote any results you use accurately and justify steps clearly. No problem needs more than 3 pivots.

1. Consider the problem

$$\max_x \quad 7x_1 + 11x_2 + 7x_3 - x_1^2 - x_1x_2 - x_2x_3 - \frac{1}{2}x_2^2 - x_3^2$$
subject to
$$Cx \leq d, \ x \geq 0$$

where
$$C = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

(a) Reformulate the problem so that it is in standard form

$$\min_x \quad f(x) = \frac{1}{2}x^TQx + p^Tx$$
subject to
$$Ax \geq b, \ x \geq 0$$

(b) Show that $Q$ is positive semidefinite. Is $f(x)$ convex or concave?

(c) Write down the KKT conditions for this problem explicitly.

(d) Prove that $\bar{x}^T = (1, 1, 1)$ solves the problem. [Hint: you do not have to use Lemke’s method]

(e) Evaluate $\| (\bar{x}^T, u^T) \|_\infty$ where $u$ is the multiplier vector on the $Cx \leq d$ constraints.

2. Consider the following problem:

$$\min_{x_1, x_2, x_3} \quad \max(2x_1 - 3x_2 + x_3 + 2, -x_1 + x_2 + 2x_3 + 5)$$
subject to
$$x_1 - x_2 + x_3 = 4, \quad x_1, x_2, x_3 \geq 0.$$

(a) Formulate this problem as a linear program.

(b) Solve this problem.

(c) Find the dual of the LP formulation in (a).
(d) Is the solution to the primal problem unique? If not, is the solution set bounded or unbounded?

3. Suppose a manufacturer makes two products (call them “1” and “2”) that both require the use of a raw material (call it “3”). Suppose that two units of raw material are needed to make each unit of product 1, and two units of raw material are also needed for each unit of product 2. The manufacturer can charge $3 for each unit of product 1 and $4 for each unit of product 2, and the raw material costs $t per unit (where t is a parameter). Because of equipment constraints, no more than 8 units of product 1 can be manufactured. In addition, there are 10 units of labor available, and each unit of product 1 requires 2 units of labor while each unit of product 2 requires 1 unit. Finally, it makes sense to consider only nonnegative quantities of both products and the raw material.

(a) Write down a linear program, parametrized by the raw material cost t, that the manufacturer can solve to maximize his profit (defined as the difference between total revenues on the two products and total raw material costs).

(b) Solve this linear program for all values \( t \in [0, +\infty) \) and interpret the results. In particular, indicate how much profit is made for each value of t. (Hint: This is easier if you eliminate the variable that quantifies the amount of raw material purchased.)

4. We aim to show that \( X \cap Y = \emptyset \iff Z \neq \emptyset \), using the following steps. Let \( X = \{ x \mid 0 \neq Bx \geq 0 \} \), \( Y = \{ x \mid Cx \geq 0 \} \) and \( Z = \{ (u, v) \mid B^T u + C^T v = 0, u > 0, v \geq 0 \} \).

(A system is a “system of linear equalities and inequalities, some of which may be strict - as in Farkas Lemma).

(a) Construct a system II that has a solution if and only if \( Z \neq \emptyset \).

(b) Construct a system I that has solution if and only if \( X \cap Y \neq \emptyset \). You may want to introduce a variable \( w = Bx \) and you could have \( w \neq 0 \) as part of I.

(c) Show that both systems I and II cannot have a solution.

(d) Suppose (I) has no solution. Construct a linear program that under this assumption will be solvable.

(e) Construct the dual of the linear program you formed in (b).

(f) Quote and use appropriate theory to generate a system that has solution, and then show this implies (II) has a solution.