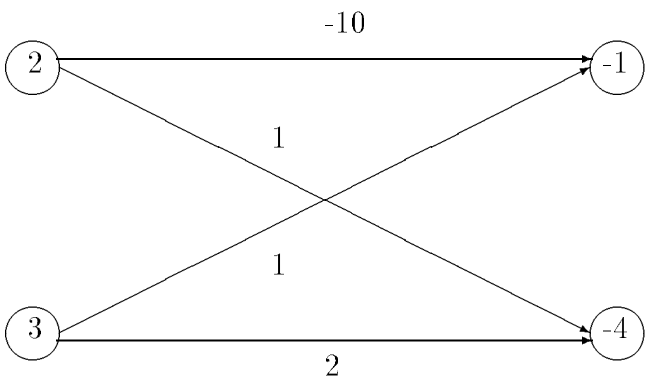
Representative exam questions

CS 719, Semester II, 1995-96

Friday May 3, 1996

1.



(a) Formulate the "big M" problem corresponding to the min cost network problem above, where divergences are shown at the nodes and unit costs are shown on the arcs. Describe how you compute the **numerical** value of "big M" for this problem. Do one pivot of the network simplex method on the big M problem, indicating the primal and dual values before and after the pivot (start with the "all artificial" solution).

(b) Given the **orientation** of the arcs in the network above, what conclusion may be drawn regarding the **existence** of an optimal solution of the corresponding big M problem? Explain your answer.

(c) Given the result of part (b), what additional property (observable from the original data of this problem) relates the big M problem to the original problem? Explain your answer.

2. Let A be an integral matrix of full row rank. Show that all basis submatrices of A are unimodular iff for all integer vectors b

$$\{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\} = \operatorname{conv}\{x \in \mathbb{R}^n : Ax = b,\ x \geq 0, x \text{ integer}\}$$

where conv(S) denotes the convex hull of the set S.

Hint: *To show that a matrix B is unimodular it suffices to show that $B^{-1}t$ is integer for all vectors t.*

3. Let G be a directed graph with specified edge lengths $d_{ij} > 0$ for the m edges (i, j). You may assume that d_{ij} is defined for all $i, j = 1, \cdots, n$ with $i \neq j$.

   (a) Formulate the problem of determining the shortest directed path from node 1 to node n as a network flow problems of the form:

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0, \end{array}$$

   where A is a node-arc incidence matrix. (Be sure to explain the correspondence between an appropriate optimal solution x^* and a directed path.)

   (b) What is the relationship between an appropriate set of optimal dual variables for the network flow problem in part (a) and path lengths for the original digraph? (Be precise.)

4. Consider the constraints $Ax = s,\ b \leq x \leq c$, where A is an $m \times n$ node-arc incidence matrix with corresponding flow vector x and s, b, c are constant vectors. Assume that these constraints have a feasible solution. Let $\bar{x}$ be a vector such that $b \leq \bar{x} \leq c$, but $g := s - A\bar{x} \neq 0$.

   (a) Prove that g has at least one positive component and at least one negative component.

(b) Choose j such that $g_j > 0$ and define $T := \{(j,k) \mid \bar{x}_{jk} < c_{jk}\} \bigcup \{(k,j) \mid \bar{x}_{kj} > b_{kj}\}$. Prove that by adjusting $\bar{x}$ on T only, we may obtain a new flow vector x' such that $b \leq x' \leq c$ and $A_j x' = s_j$, where A_j is the the jth row of A.

5. Consider a network flow problem with n nodes:

$$\min_x \quad c^T x \qquad \text{s.t.} \quad Ax = b, \quad x \geq 0,$$

where c is a vector all of whose components are 1 in absolute value.

Given a primal feasible solution defined on a spanning tree T, let π represent the corresponding node potentials.

a) If π_n=0, derive a bound for π_1 in terms of the problem data, and state necessary and sufficient conditions under which this bound will be attained.

b) If no assumptions are made on the value of any π_i, can a bound still be derived for π_1 from the complementarity conditions for this problem? Explain your answer.