

$$\frac{|b_1 - b_2|}{\|a\|}$$

10 pts each question,  
each part

# Homework 1

100 pts total

CS 726, Semester I, 2011-12

September 9, 2011

First of all read Appendix A of the Nocedal-Wright book. This material covers most of the prerequisites of this course. Only four of the questions below will actually be graded for credit - I choose which ones!

1. What is the distance between two parallel hyperplanes  $\{x \mid a^T x = b_1\}$  and  $\{x \mid a^T x = b_2\}$ ?
2. Which of the following sets  $S$  are polyhedra? If possible, express  $S$  in the form  $S = \{x \mid Ax \leq b, Fx = g\}$ .
  - (a)  $S = \{y_1 a_1 + y_2 a_2 \mid -1 \leq y_1 \leq 1, -1 \leq y_2 \leq 1\}$ , where  $a_1, a_2 \in \mathbf{R}^n$ . ✓
  - (b)  $S = \{x \in \mathbf{R}^n \mid x \geq 0, 1^T x = 1, \sum_{i=1}^n x_i a_i = b_1, \sum_{i=1}^n x_i a_i^2 = b_2\}$ , where  $a_1, \dots, a_n \in \mathbf{R}$  and  $b_1, b_2 \in \mathbf{R}$ . ✓
  - (c)  $S = \{x \in \mathbf{R}^n \mid x \geq 0, x^T y \leq 1 \forall y \text{ with } \|y\|_2 = 1\}$ . ✗
  - (d)  $S = \{x \in \mathbf{R}^n \mid x \geq 0, x^T y \leq 1 \forall y \text{ with } \sum_{i=1}^n |y_i| = 1\}$ . ✓
3. For each value of the scalar  $\beta$ , find the set of all stationary points  $\{x \mid \nabla f(x) = 0\}$  of the following function of variables  $x_1$  and  $x_2$ :

$$f(x) = x_1^2 + x_2^2 + \beta x_1 x_2 + x_1 + 2x_2$$

Which of these stationary points are global minima?

4. Suppose  $f: \mathbf{R} \rightarrow \mathbf{R}$  is strictly increasing and convex on its domain  $(a, b)$ . Let  $g$  denote its inverse, i.e. the function with domain  $(f(a), f(b))$  and  $g(f(x)) = x$  for  $a < x < b$ . What can you say about convexity or concavity of  $g$ ? *concave*
5. Suppose  $\text{dom } f$  is an open set. Prove that a twice differentiable function  $f$  is convex if and only if its domain is convex and  $\nabla^2 f(x)$  is positive semidefinite for all  $x \in \text{dom } f$ . Hint: First consider the case  $f: \mathbf{R} \rightarrow \mathbf{R}$ . You can use the first-order condition for convexity (which was proved in class).
6. In each of the following questions fully justify your answers using the optimality conditions:

1.)

$$\min \frac{1}{2} \|x - y\|^2$$

$$\text{s.t.} \quad ax = b_1 \\ \quad \quad \quad ax = b_2$$

$$a^T (x - y) = b_1 - b_2$$

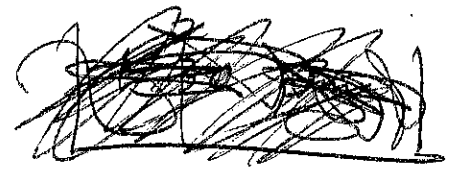
$$\min \frac{1}{2} \|u\|^2$$

$$\text{s.t.} \quad a^T u = b_1 - b_2$$

$$u = (b_1 - b_2) \frac{a}{\|a\|^2}$$

~~u~~

$$\|u\| = \frac{|b_1 - b_2|}{\|a\|}$$



2. (a), (b), (d) are polyhedral

~~(c) is not~~

(c) is not polyhedral

2. a.  $a_1, a_2 \in \mathbb{R}^n$

$$S = \{y_1 a_1 + y_2 a_2 : -1 \leq y_1 \leq 1, -1 \leq y_2 \leq 1\}$$

$S \subseteq L = \{y_1 a_1 + y_2 a_2\}$ , so  $S$  is a set of dimension  $\dim L$ . If  $\dim L = 0$ ,

then  $S = \mathbf{0}$ . ~~It is a point centered at 0.~~ If  $\dim L = 1$ , then  $S$  is a line segment centered at 0.

If  $\dim L = 2$  then  $S$  is a parallelogram centered at 0. In this case, let

$$v_1 = a_1 - \text{proj}_{a_2} a_1 = a_1 - \frac{a_1^T a_2}{a_2^T a_2} a_2$$

$$v_2 = a_2 - \text{proj}_{a_1} a_2 = a_2 - \frac{a_2^T a_1}{a_1^T a_1} a_1$$

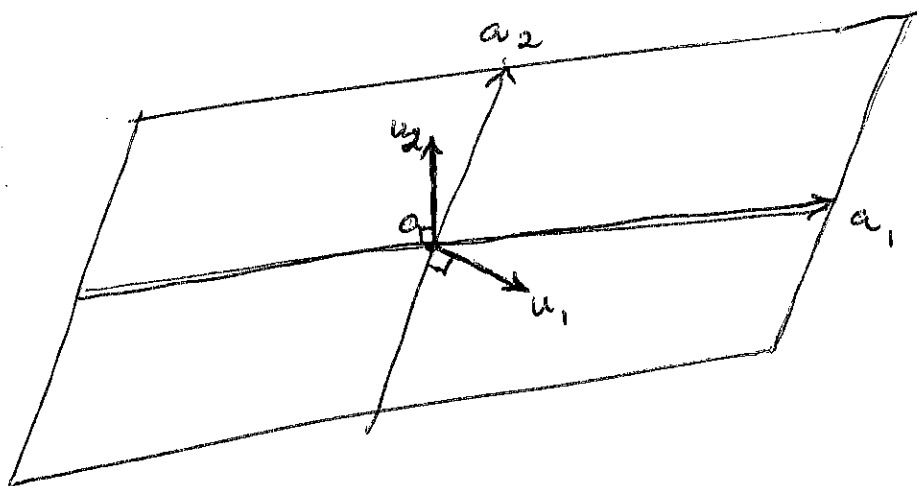
$$u_1 = \frac{v_1}{\|v_1\|}$$

$$u_2 = \frac{v_2}{\|v_2\|}$$

$$b_1 = u_1^T a_1$$

$$b_2 = u_2^T a_2$$

Here is a drawing of  $S$  in the plane of  $L$ :



Note that  $u_1^T a_2 = 0$ ,  $u_2^T a_1 = 0$ . Finally, let  $u_3, \dots, u_n$  be an orthonormal basis of  $L^\perp$ .

Then

$$S = \left\{ x : \begin{array}{l} -b_1 \leq u_1^T x \leq b_1, \\ -b_2 \leq u_2^T x \leq b_2, \\ u_3^T x = 0, \\ \vdots \\ u_n^T x = 0 \end{array} \right\}$$

2.  ~~$\{x: Ax \leq b, Fx = g\}$~~

$$b. \{x: Ax \leq b, Fx = g\}$$

$$A = [-1 \dots -1]$$

$$b = 0$$

$$F = \begin{bmatrix} 1 & \dots & 1 \\ a_1 & \dots & a_n \\ a_1^2 & \dots & a_n^2 \end{bmatrix}$$

$$g = \begin{bmatrix} 1 \\ b_1 \\ b_2 \end{bmatrix}$$

$$c. \text{ No. } S = \{x: \|x\|_2 \leq 1\} \cap \{x \geq 0\}$$

$$d. S = \{x: 0 \leq x_i \leq 1\}$$

$$A = \begin{bmatrix} -I \\ I \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$3. f(x) = x_1^2 + x_2^2 + \beta x_1 x_2 + x_1 + 2x_2$$

$$\nabla f(x) = \begin{bmatrix} 2 & \beta \\ \beta & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

~~$$\{x: \nabla f(x) = 0\} = \{ \}$$~~

$$0 = \nabla f(x) \iff$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{1}{4 - \beta^2} \begin{bmatrix} 2 & -\beta \\ -\beta & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{\beta^2 - 4} \begin{bmatrix} 2 - 2\beta \\ 4 - \beta \end{bmatrix}, \quad \beta^2 \neq 4$$

or, if  $\beta^2 = 4$ , no such  $x$ .

$$\{x: \nabla f(x) = 0\} = \begin{cases} \left\{ \left( \frac{2-2\beta}{\beta^2-4}, \frac{4-\beta}{\beta^2-4} \right) \right\}, & \beta^2 \neq 4 \\ \{ \}, & \beta^2 = 4 \end{cases}$$

~~the~~ stat. pt. is a global min if and only if  $4 > \beta^2$

$$6. a. \nabla f = \begin{bmatrix} 4x(x^2 - 4) \\ 2y \end{bmatrix}$$

stat pts :  $(0,0), (-2,0), (2,0)$

$$\nabla^2 f = \begin{bmatrix} 12x^2 - 16 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{cases} \begin{bmatrix} -16 & 0 \\ 0 & 2 \end{bmatrix} & \text{at } (0,0) \\ \begin{bmatrix} 32 & 0 \\ 0 & 2 \end{bmatrix} & \text{at } (-2,0), (2,0) \end{cases}$$

$(0,0)$  is a stat. pt that is  
neither a local max nor  
a local min

$(-2,0), (2,0)$  are local minima

$f$  is bdd below by a quadratic

$$x^2 + y^2 + K \text{ for some constant } K,$$

so ~~it attains~~ it ~~is~~ attains a global minimum.

$f$  has the same value at  $(-2, 0)$  and  $(2, 0)$ , so both are global minimizers.

$$6.b. \quad f = \frac{1}{2}x^2 + x \cos y$$

$$\nabla f = \begin{bmatrix} x + \cos y \\ -x \sin y \end{bmatrix}$$

$$\text{stat pts: } -x = \cos y$$

$$\sin y \cos y = 0$$

$$y = \pi/2$$

$$y \in 2\pi\mathbb{Z} \Rightarrow x = -1$$

$$y \in 2\pi\mathbb{Z} + \pi \Rightarrow x = 1$$

$$y \in \pi\mathbb{Z} + \frac{\pi}{2} \Rightarrow x = 0$$

$$\nabla^2 f = \begin{bmatrix} 1 & -\sin y \\ -\sin y & -x \cos y \end{bmatrix} \quad \det = -(x \cos y + \sin^2 y)$$

$$\det > 0 \quad \text{iff} \quad y \in 2\pi\mathbb{Z} \quad \text{or} \quad y \in 2\pi\mathbb{Z} + \pi$$

$$\det = -1 \quad \text{iff} \quad \underline{\quad} y \in 2\pi\mathbb{Z} + \frac{3\pi}{2}$$

local min :

$$\{(-1, 2\pi k) : k \in \mathbb{Z}\}$$

$$\cup \{(1, (2k+1)\pi) : k \in \mathbb{Z}\}$$

$$\cancel{\cup \{(0, (2k + \frac{3}{2})\pi) : k \in \mathbb{Z}\}}$$

b.c.  $f = (y - x^2)^2 - x^2$

$$\nabla f = \begin{bmatrix} -4x(y - x^2) - 2x \\ 2(y - x^2) \end{bmatrix} = -2 \begin{bmatrix} x(2y - 2x^2 + 1) \\ x^2 - y \end{bmatrix}$$

~~$\nabla^2 f$~~

stat pt. :  $x = 0, y = 0$  or

$$x \neq 0, y = x^2, 1 = 0 \Rightarrow \Leftarrow$$

$$(x^*, y^*) = \boxed{(0, 0)}$$

$$\nabla^2 f \Big|_{(x^*, y^*)} = -2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

indefinite so

$(x^*, y^*)$  is neither a local min  
nor a local max.

6-d.  $f = \sin x + \sin y + \sin(x+y)$

$$\nabla f = \begin{bmatrix} \cos x + \cos(x+y) \\ \cos y + \cos(x+y) \end{bmatrix}$$

$$\nabla f = 0 : \quad \cos x - \cos y = 0$$

$$0 < x, y < 2\pi \quad \Rightarrow$$

$$x \neq y = 2\pi \quad y = x \quad \text{or} \quad y = 2\pi - x$$

$$y = x$$

$$0 = \cos x + \cos 2x$$

$$= \cos x + 2\cos^2 x - 1$$

$$\equiv \cancel{2} \left( \cancel{\cos^2 x} + \frac{1}{2} \cos x - \frac{1}{2} \right)$$

$$\cos x = \frac{-1 \pm \sqrt{1+8}}{4} = -\frac{1}{4} \pm \frac{3}{4} = \left\{ -1, \frac{1}{2} \right\}$$

$$(x, y) = (\pi, \pi) \text{ or } \left(\frac{\pi}{3}, \frac{\pi}{3}\right) \text{ or } \left(\frac{5\pi}{3}, \frac{5\pi}{3}\right)$$

$$y = 2\pi - x:$$

$$0 = \cos x + \cos(x + 2\pi - x)$$

$$= \cos x + \cos 2\pi$$

$$= \cos x + 1$$

$$x = \pi, y = \pi$$

Altogether stat pts are

$$\left\{ \left(\frac{\pi}{3}, \frac{\pi}{3}\right), (\pi, \pi), \left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) \right\}$$

$$\nabla^2 f = \begin{bmatrix} -\sin x - \sin(x+y) & -\sin(x+y) \\ -\sin(x+y) & -\sin y - \sin(x+y) \end{bmatrix}$$

at  $(\pi, \pi)$ ,  $\nabla^2 = 0$ , so 2nd-deriv test gives no information

~~at  $(\frac{\pi}{3}, \frac{\pi}{3})$~~

at all the stat pts,  $y=x$  so we have

$$\nabla^2 f = -\sin x \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \sin 2x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

~~$= -\sin x \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$~~

$$= e \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$C = \begin{cases} -\frac{\sqrt{3}}{2}, & x = \frac{\pi}{3} \\ 0, & x = \frac{\pi}{2} \\ \frac{\sqrt{3}}{2}, & x = \frac{2\pi}{3} \end{cases}$$

so  $(\frac{\pi}{3}, \frac{\pi}{3})$  is a local max

$(\frac{5\pi}{3}, \frac{5\pi}{3})$  is a local min

For  $(\pi, \pi)$ , consider the restriction

$$g = f|_{y=x} : \mathbb{R} \rightarrow \mathbb{R}$$

$$g(x) = f(x, x) = 2\sin x + \sin 2x$$

~~$$g'(x) = 2\cos x + 2\cos 2x$$~~

~~$$g''(x) = -2\sin x - 4\sin 2x$$~~

Note  ~~$g(\pi) = 0$ ,  $g(\pi - \epsilon)$~~

$k$	$g^{(k)}(x)$	$g^{(k)}(\pi)$
0	$g(x) = 2\sin x + \sin 2x$	0
1	$g'(x) = 2\cos x + 2\cos 2x$	0
2	$g''(x) = -2\sin x - 4\sin 2x$	0
3	$-2\cos x - 8\cos 2x$	-6

~~It is analytic~~

The first nonzero derivative of  $g$

at  $\pi$  is  $g^{(3)}(\pi) = -6 > 0$ , so

$g(\pi) = 0$ ,  $g(\pi - \varepsilon) > 0$ ,  $g(\pi + \varepsilon) < 0$

for  $\varepsilon > 0$  sufficiently small.

Hence  $(\pi, \pi)$  is neither a

local minimum nor a local

maximum for  $f$ .