3. $A \sim m \times n$, $m \leq n$

(i) let x_0 satisfy $Ax_0 = b$. Then $Ax = b$ if and only if $x \in x_0 + \ker(A)$:

if $Ax = b$, then

$$A(x - x_0) = Ax - Ax_0 = b - b = 0$$

so $x - x_0 \in \ker(A)$ so $x \in x_0 + \ker(A)$.

if $x \in x_0 + \ker(A)$, then

$$0 = A(x - x_0) = Ax - Ax_0 = Ax - b$$

so $Ax = b$.

(ii) QR factorization:

$\cancel{Q_1 \sim n \times (n-m)}$

$Q_1 \sim n \times m$

$Q_2 \sim n \times (n-m)$

$Q = [Q_1 \quad Q_2]$, $Q^T Q = I$

$R \sim m \times m$ upper triangular

$$A^T = [Q_1 \quad Q_2] \begin{bmatrix} R \\ 0 \end{bmatrix}$$

Lemma: $\ker(A^T) = \mathrm{im}(A)^\perp$:

Suppose $u \in \ker(A^T)$. ~~Suppose~~ Then $\forall v$, $(Av)^T u = v^T A^T u = v^T 0 = 0$ so $u \in \mathrm{im}(A)^\perp$.

Suppose $u \in \mathrm{im}(A)^\perp$. Then $\forall v$,

$v^T A^T u = (Av)^T u = 0$, so $A^T u = 0$,

so $u \in \ker(A^T)$.

By this lemma, the columns of Q_2 are an orthonormal basis of $\ker(A^T)$.

(iii) To apply this to

$$\min_x \|Cx - d\|_2^2$$

$$\text{s.t. } Ax = b,$$

assume A is full rank so that R is invertible. Let

$$\boxed{x_0 = Q_1 R^{-T} b.}$$

Then

$$A^T x_0 = R^T Q_1^T Q_1 R^{-T} b = R^T R^{-T} b = b$$

By (ii) we introduce the change of variables

$$\boxed{x = x_0 + Q_2 y}, \quad y \in \mathbb{R}^{n-m}$$

and the problem becomes

$$\min_y \| C Q_2 y + C x_0 - d \|_2^2$$

which we solve as an unconstrained linear least squares problem in y.

$$\boxed{\begin{array}{l}\text{SVD:}\\ CQ_2 = USV^T\end{array}}$$ $U^TU = I$ $V^TV = I$

S $k \times k$ pos def diag

$$\|CQ_2 y + Cx_0 - d\|_2^2$$

$$= y^T V S U^T U S V^T y$$

$$+ 2\left[VSU^T(Cx_0 - d)\right]^T y + \|Cx_0 - d\|^2$$

$$= (V^Ty)^T SS\, V^Ty + 2\left(S\,U^T(Cx_0 - d)\right)^T V^T y$$

$$+ \|U^T(Cx_0 - d)\|^2 + M_0$$

$$= \|SV^Ty + U^T(Cx_0 - d)\|^2 + M_0$$

where M_0 is a constant. Then

$$V^Ty = -S^{-1}U^T(Cx_0 - d)$$

so take

$$\boxed{y = -VS^{-1}U^T(Cx_0 - d)}$$