Homework 7 (Solutions)

Problem 1

a) $[L, U, P] = lu(A')$

$\Rightarrow PA^T = LU$

$\Rightarrow AP^T = U^T L^T$, where $A \in \mathbb{R}^{m\times n}$

$U \in \mathbb{R}^{m\times m}$ (upper triangular matrix)

$L \in \mathbb{R}^{n\times m}$ (lower triangular matrix)

$r(A) = r(A^T) = m = r(U)$

$\Rightarrow U$ is an invertible matrix.

Let $R = L^T(1:m, 1:m)$

$S = L^T(1:m, m+1:n)$

$\therefore L^T = [R \;\; S]$

$\therefore AP^T = U^T L^T = U^T [R \;\; S] = [U^T R \;\; U^T S] = [B \;\; N]$

Here U^T is lower triangular ~~matrix~~ invertible matrix & R is upper triangular matrix with all diagonals having 1 (hence invertible).

Thus,

$B = U^T R$,

where U^T & R are invertible lower & upper triangular factors of matrix ~~the~~ B that satisfies

$AP^T = [B \;\; N]$

b) $Y = \begin{bmatrix} B^{-1} \\ 0 \end{bmatrix}$ $\quad Z = \begin{bmatrix} -B^{-1}N \\ I \end{bmatrix}$

As B is invertible.

$\therefore r(Y) = r(B) = m$

Also I is invertible.

$\therefore r(Z) = r(I) = n-m.$

Let $X = [Y \;\; Z]$

We need to show that $r(X) = n$

Let there be $\lambda \in \mathbb{R}^{n \times 1}$ such that

$X\lambda = 0$

$$\Rightarrow \begin{bmatrix} B^{-1} & -B^{-1}N \\ 0 & I \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 0 \qquad \begin{array}{l} \lambda_1 \in \mathbb{R}^{m \times 1} \\ \lambda_2 \in \mathbb{R}^{n-m \times 1} \end{array}$$

$$\Rightarrow \begin{bmatrix} B^{-1}\lambda_1 + (-B^{-1}N)\lambda_2 \\ \lambda_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 0$$

$\therefore r(X) = r([Y \;\; Z]) = n$

Hence $[Y \;\; Z]$ form basis of whole space.

$r(\mathrm{Ker}(A)) = n-m$

$r(P^T Z) = r(Z) = r(I) = n-m = r(\mathrm{Ker}(A))$

Also.

$$A(P^T Z) = (AP^T)Z$$

$$= \begin{bmatrix} B & N \end{bmatrix} \begin{bmatrix} -B^{-1}N \\ I \end{bmatrix}$$

$$= -BB^{-1}N + NI$$

$$= -N + N = 0$$

Hence,

$P^T Z$ forms basis for the kernel of A

Proble 2

a) There are three cases of which I will show the proof of third one.

<u>Case I</u> $x \in C, y \in C$.

<u>Case II</u> $x \in cl(C), x \notin C, y \in C$

<u>Case III</u> $x \in cl(C), y \in cl(C), x \notin C, y \notin C$.

In this case x & y are limit points of C.

$\therefore$ There exist a sequence $\{x_k\}$ such that $\lim_{k \to \infty} x_k = x$ & a sequence $\{y_k\}$ such that $\lim_{k \to \infty} y_k = y$, where $x_k, y_k \in C$.

$\therefore$ for $\alpha \in [0,1]$

$\alpha x_k + (1-\alpha) y_k \in C$

$\Rightarrow \lim_{k \to \infty} (\alpha x_k + (1-\alpha) y_k) = \alpha x + (1-\alpha) y$ is a limit point C.

Thus $\alpha x + (1-\alpha) y \in cl(C)$.

Hence $cl(C)$ is convex.

b) Two cases.

Case I : $\bar{x} \notin cl(C)$

Let $P_{cl(C)}(\bar{x})$ be the projection of $\bar{x}$ onto $cl(C)$.

$\therefore$ By projection theorem.

$$\left(\bar{x} - P_{cl(C)}(\bar{x})\right)^T \left(x - P_{cl(C)}(\bar{x})\right) \leq 0 \qquad \forall x \in cl(C)$$

$$\Rightarrow \left(\bar{x} - P_{cl(C)}(\bar{x})\right)^T \left(x - \bar{x} + \bar{x} - P_{cl(C)}(\bar{x})\right) \leq 0$$

$$\Rightarrow \underbrace{\|\bar{x} - P_{cl(C)}(\bar{x})\|_2^2}_{\geq 0} + \left(\bar{x} - P_{cl(C)}(\bar{x})\right)^T (x - \bar{x}) \leq 0$$

$$\Rightarrow \left(\bar{x} - P_{cl(C)}(\bar{x})\right)^T (x - \bar{x}) \leq 0 \qquad \forall x \in cl(C)$$

$$\Rightarrow b^T (x - \bar{x}) \leq 0, \quad b = \bar{x} - P_{cl(C)}(\bar{x}) \neq 0, \quad \forall x \in cl(C)$$

$$\Rightarrow b^T x \leq b^T \bar{x}$$

$$\Rightarrow -b^T \bar{x} \leq -b^T x$$

$$\Rightarrow \underline{a^T \bar{x} \leq a^T x} \qquad \forall x \in cl(C)$$

Case II $\bar{x} \in cl(C)$, $\bar{x} \notin int(C)$

$\bar{x}$ is a limit point.

There exist $\{x_k\} \notin cl(C)$ such that

$$\lim_{k \to \infty} x_k = \bar{x}$$

Let $$a_k = \frac{x_k - P_{cl(C)}(x_k)}{\| x_k - P_{cl(C)}(x_k) \|_2}$$

Then $\|a_k\|_2 = 1$ & $\{a_k\}$ is bounded.

From previous case we have

$$b_k^T(x - x_k) \leq 0 \quad \forall x \in cl(C)$$

As $\{b_k\}$ is bounded, let $\lim_{k\to\infty} b_k = b$

$$\therefore \lim_{k\to\infty} b_k^T(x - x_k) = b^T(x - \bar{x}) \leq 0 \quad \forall x \in cl(C)$$

$$\therefore b^T x \leq b^T \bar{x}$$

$$\Rightarrow -b^T\bar{x} \leq -b^T x$$

$$\Rightarrow a^T\bar{x} \leq a^T x \quad \square$$

Problem 3

a) C_1 & C_2 are convex.

$$C : \{x \mid x = x_1 - x_2,\ x_1 \in C_1, x_2 \in C_2\}$$

Let $x \in C$, $y \in C$, $x_1, y_1 \in C_1$, $x_2, y_2 \in C_2$

such that

$$x = x_1 - x_2, \quad y = y_1 - y_2.$$

For any $\alpha \in [0,1]$

$$\alpha x + (1-\alpha) y = \alpha(x_1 - x_2) + (1-\alpha)(y_1 - y_2)$$

$$= \underbrace{\{\alpha x_1 + (1-\alpha) y_1\}}_{\in C_1} - \underbrace{\{\alpha x_2 + (1-\alpha) y_2\}}_{\in C_2}$$

$$= z_1 - z_2 \in C$$

Hence $C_1 - C_2 = C$ is convex.

b) We know that C_1 & C_2 are convex & disjoint

$\therefore$ $C_1 \cap C_2 = \phi$

$\therefore$ $0 \notin C_1 - C_2$.

Hence. there exist a hyperplane passing through 0 such that $C = C_1 - C_2$ is contained in one of its closed half space. From previous problem.

$\exists a \neq 0$ such that.

$$a^T (0) \leq a^T x \qquad \forall x \in C$$

$$\Rightarrow \quad 0 \leq a^T (x_1 - x_2) \qquad \forall x_1 \in C_1,\ x_2 \in C_2$$

$$\Rightarrow \quad a^T x_2 \leq a^T x_1 \qquad \forall x_1 \in C_1,\ \forall x_2 \in C_2$$

$$\Rightarrow \quad -a^T x_1 \leq -a^T x_2 \qquad ''$$

$$\Rightarrow \quad b^T x_1 \leq b^T x_2 \qquad ''.$$

Problem 4.

a) Feasible set $= D = \{x \mid x \in [2,4]\}$

Optimal point $x = 2$

Optimal value $f(2) = 5$

b) $L(x, \lambda) = (1+\lambda) x^2 - 6\lambda x + 8\lambda + 1$

$$g(\lambda) = \begin{cases} \dfrac{-\lambda^2 + 9\lambda + 1}{1+\lambda} & \lambda > -1 \\ -\infty & \lambda \leq -1 \end{cases}$$

c) Dual problem.

$$\max_{\lambda} \; g(\lambda)$$

s.t. $\lambda \geq 0$.

$$g''(\lambda) = -\frac{18}{(1+\lambda)^3} < 0$$ ~~heng~~ hence $g(\lambda)$ is concave.

Dual optimal point $= \lambda^* = 2$

" " sol^n $= g(\lambda^*) = 5 = f(x^*) = p^*$

hence strong duality holds.

d)
$$p^*(u) = \begin{cases} 1 & u \geq 8 \\ u - 6\sqrt{1+u} + 1 & -1 \leq u < 8 \\ \text{UND} & \text{else} \end{cases}$$

$$\frac{d\,p^*(0)}{du} = -2 = -\lambda^*.$$