

Homework 7

CS 726, Semester I, 2010–11

December 5, 2011

The computational part of this assignment should be submitted electronically using the instructions on the course web page. The assignment name is hwk7 and you should hand in exactly 1 file with the following name:

TNewtonEq.m

The remaining parts can be handwritten (or submitted in hwk7.txt) and should be submitted at class time on December 7.

1. Consider the problem

$$\min f(x) = \sum_{i=1}^n [(1 - x_{2i-1})^2 + 10 * (x_{2*i} - x_{2i-1}^2)^2] + \frac{\gamma}{2} (\sum_{i=1}^{2n} x_i - 1)^2 \text{ subject to } Ax = b$$

with $\gamma = 100$, where $A \in \mathbf{R}^{m \times n}$ has rank m , $b \in \mathbf{R}^m$ is given and $m < n$.

- (a) Explain carefully how to use the *lu* routine from Matlab to construct invertible lower and upper triangular factors of a matrix B that satisfies

$$AP^T = [B \ N]$$

Here P is a permutation matrix.

- (b) Show that the matrices

$$Y = \begin{bmatrix} B^{-1} \\ 0 \end{bmatrix} \text{ and } Z = \begin{bmatrix} -B^{-1}N \\ I \end{bmatrix}$$

together form a basis for the whole space and the columns of $P^T Z$ form a basis for the kernel of A respectively.

- (c) Adapt your TNewton code to solve the above problem using the data found in hwk7.m.

`function [inform,x] = TNewtonEq(fun,x,A,b,nparams)`

Note that instead of forming the matrices Y and Z explicitly, you should use the result in (a) and b to construct products $Y\lambda$ and $Z\mu$ efficiently using forwards and backwards substitution. Be sure to indicate in your code by comments where this manipulation is carried out.

2. Show that if C is convex, then so is $\text{cl}(C)$ (the closure of C). Use this fact and the Projection Theorem to prove the following Supporting Hyperplane Theorem: Let C be a nonempty convex set and \bar{x} be a point not in the interior of C . There exists a hyperplane that passes through \bar{x} and contains C in one of its closed halfspaces, i.e. there exists $a \neq 0$ such that

$$a^T \bar{x} \leq a^T x, \forall x \in C$$

3. Show that $C = C_1 - C_2 = \{x \mid x = c_1 - c_2, c_i \in C_i\}$ is convex if C_1 and C_2 are convex. Use this fact and the above result to prove the Separating Hyperplane Theorem: Let C_1 and C_2 be nonempty, disjoint, convex sets. There exists a hyperplane that separates them, i.e. there exists $a \neq 0$ such that

$$a^T x_1 \leq a^T x_2, \forall x_1 \in C_1, x_2 \in C_2$$

(Note that it is possible to prove even stronger results, namely that if instead of the sets being disjoint, their *relative interiors* are disjoint (i.e. their interiors with respect to their affine hulls), then there exists a proper separating hyperplane (that is the hyperplane does not contain the union of C_1 and C_2).

Furthermore, if $0 \notin \text{cl}(C_1 - C_2)$, then the separation is strong, that is the hyperplane separates $C_1 + \epsilon B$ from $C_2 + \epsilon B$ for some $\epsilon > 0$.)

4. Consider the optimization problem

$$\min x^2 + 1 \text{ s.t. } (x - 2)(x - 4) \leq 0,$$

with variable $x \in \mathbf{R}$.

- (a) Give the feasible set, the optimal value, and the optimal solution.
 (b) Plot the objective $x^2 + 1$ versus x . On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ versus x for a few positive values of λ . Verify the lower bound property

$$p^* \geq \inf_x L(x, \lambda) \text{ for } \lambda \geq 0.$$

Derive and sketch the Lagrange dual function g .

- (c) State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution λ^* . Does strong duality hold?
 (d) Let $p^*(u)$ denote the optimal value of the problem

$$\min x^2 + 1 \text{ s.t. } (x - 2)(x - 4) \leq u,$$

as a function of the parameter u . Plot $p^*(u)$. Verify that $dp^*(0)/du = -\lambda^*$.