Homework 8 Solutions

December 20, 2011

3. A given rectangle with dimensions $2x$ and $2y$ is contained in some closed unit disc if and only if $x^2 + y^2 \leq 1$. This is intuitively obvious, but a rigorous proof can be given using the triangle inequality. So it suffices to show that $f(x, y) = xy$ attains a maximum relative to the upper-right quarter of the closed unit disc D at $(x, y) = (1, 1)$ and at no other point (besides $(-1, -1)$). Since f is continuous and D is compact, the maximum is attained. The interior first order necessary condition $df = 0$ implies that the only interior maximum must be at the origin, which is clearly not a maximizer. For the boundary we introduce polar coordinates:

$$f(\cos(\theta), \sin(\theta)) = \cos(\theta)\sin(\theta) = 0.5\sin(2\theta)$$

which attains its maximum value at $\theta = (k + 1/4)\pi$, $k \in \mathbb{Z}$, i.e. at $(x, y) = (1, 1)$ and $(-1, -1)$.

4. Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $f : \mathbb{R}^m \to \mathbb{R}$ be convex. Consider the problem

$$\min_x f(Ax + b) \tag{1}$$

with optimal value p^*. Define the Lagrangian

$$L(x, y) = f(Ax + b)$$

for $x \in \mathbb{R}^n$, $y \in \mathbb{R}^0 = \{0\}$. Then the dual problem is

$$\max_y g(y) \tag{2}$$

with dual objective

$$g(y) = \inf_x L(x, y) = \inf_x f(Ax + b) = p^*$$

5. Reformulate (1) as

$$\min_{x,z}\{f(z) : z = Ax\} \tag{3}$$

Define the Lagrangian

$$L(x,z,y) = f(z) - (z - Ax - b)^T y$$

for $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $y \in \mathbb{R}^m$. Then the dual objective is

$$\begin{array}{rcl} g(y) & = & \inf_{x,z} L(x,z,y) \\ & = & b^T y + \inf_z \left(f(z) - y^T z\right) + \inf_x \left(A^T y\right)^T x \\ & = & \left\{ \begin{array}{ll} b^T y - f^*(y) & \text{if } A^T y = 0 \\ -\infty & \text{else} \end{array} \right. \end{array}$$

Then the dual problem is

$$\max_y\{b^T y - f^*(y) : A^T y = 0\} \tag{4}$$

6. Consider the convex function $f : \mathbb{R}^m \to \overline{\mathbb{R}}$ given by

$$f(z) = \log\left(\sum_i \exp z_i\right)$$

The domain of f is $\mathbb{R}^m$. The conjugate $f^* : \mathbb{R}^m \to \overline{\mathbb{R}}$ is the convex function

$$f^*(y) = \sup_z \left(z^T y - f(z)\right) = \left\{ \begin{array}{ll} \sum_i y_i \log y_i & \text{if } y \geq 0 \text{ and } 1^T y = 1 \\ +\infty & \text{else} \end{array} \right. \tag{5}$$

with the convention that $0 \log 0 = \lim_{t \to 0^+} t \log t = 0$. To see this, first suppose $y_{i_0} < 0$. Then, with $z_i = 0$ for $i \neq i_0$ and $z_{i_0} = t$,

$$y^T z - f(z) = y_{i_0} t - \log\left(m - 1 + \exp t\right)$$

which is unbounded above as $t \to -\infty$. And with $z = 1t$,

$$y^T z - f(z) = \left(1^T y - 1\right) t - \log m$$

which is unbounded above as $t \to \pm\infty$ if $1^T y \neq 1$. Now if $y > 0$ and $1^T y = 1$, then the first-order necesary and sufficient condition for the supremum in (5) is

$$0 = y_i - \frac{\exp z_i}{\sum_{i'} \exp z_{i'}}$$

with solution set

$$\{z : z_i = \log y_i + t, t \in \mathbb{R}\}$$

on which

$$y^T z - f(z) = \sum_i y_i \log y_i$$

Finally, if $y \geq 0$ and $1^T y = 1$ and some components of y are 0, then (5) holds by a limiting argument since the conjugate of a convex function is lower semicontinuous and convex.

Now using (5) in (4) we see that the primal problem

$$\min_x \log \left(\sum_i \exp \left(a_i^T x + b_i \right) \right)$$

has a dual

$$\max_y \{ b^T y - \sum_i y_i \log y_i : y \geq 0, 1^T y = 1, A^T y = 0 \}$$