Theorem 8

$$\min f_0(x) \ s.t.\ x \in X, F(x) \in K$$

using a penalty function $\beta(x)$ which is zero if $F(x) \in K$ and positive otherwise.

$$x_i \in \arg\min_{x \in X} f_0(x) + \gamma_i \beta(x)$$

Let $C = \{x \in X \colon F(x) \in K\}$ and suppose $\{\gamma_i\} \to \infty$.

Suppose $C \neq \emptyset$, X is closed, and f_0 and β are lowersemicontinuous on X. Each accumulation point $\bar{x}$ of $\{x_i\}$ solves

$$\min_{x \in C} f_0(x)$$

Furthermore,

$$\lim_{j \to \infty} \gamma_{i_j} \beta(x_{i_j}) = 0$$

for $x_{i_j} \to \bar{x}$.

Proof Since X is closed, $\bar{x} \in X$. Since C is nonempty, (iv) implies $\lim_{i \to \infty} \beta(x_i) = 0$. By lower semicontinuity of β

$$0 \leq \beta(\bar{x}) \leq \lim_{i \to \infty} \beta(x_i) = 0$$

so that $\beta(\bar{x}) = 0$. Hence $\bar{x} \in C$. Similarly, the lower semicontinuity of f_0 implies

$$f_0(\bar{x}) \leq \lim_{i \to \infty} f_0(x_i) \leq \inf_{x \in C} f_0(x)$$

the last inequality from (i). Hence $f_0(\bar{x}) = \min_{x \in C} f_0(x)$.

Suppose $\{x_{i_j}\} \to \bar{x}$. Then

$$0 \geq P_{i_j}(x_{i_j}) - P_{i_j}(\bar{x}) = f_0(x_{i_j}) + \gamma_{i_j} \beta(x_{i_j}) - f_0(\bar{x})$$

Rearranging:

$$f_0(\bar{x}) - f_0(x_{i_j}) \geq \gamma_{i_j} \beta(x_{i_j}) \geq 0$$

Taking limits gives the required result. $\square$