Nonlinear Programming Algorithms
Handout

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1 BFGS

Theorem 1 Suppose $B_k$ is positive definite and

$$y_k^T s_k = (g_{k+1} - g_k)^T s_k > 0.$$  

Then $B_{k+1}^{BFGS}$ is positive definite.

Proof $B_k = R^T R$ by assumption with $R$ invertible upper triangular. Let $z \neq 0$ and define $a = Rz$, $b = Rs_k$.

$$z^T B_{k+1}^{BFGS} z = z^T \left( B_k + \frac{y_k y_k^T}{y_k^T s_k} - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} \right) z$$

$$= a^T a + \frac{(z^T y_k)^2}{y_k^T s_k} - \frac{(a^T b)^2}{b^T b}$$

$$= \frac{(z^T y_k)^2}{y_k^T s_k} + \left\{ \|a\|^2 \|b\|^2 - (a^T b)^2 \right\} / \|b\|^2 \geq 0$$

The latter term is nonnegative by Cauchy Schwartz. If it is strictly positive, we are done. Otherwise it is zero, in which case $a$ is proportional to $b$. Invertibility of $R$ guarantees that $z = \mu s_k$ for some $\mu \neq 0$. Then

$$z^T B_{k+1}^{BFGS} z = \mu^2 y_k^T s_k > 0$$

\[ \square \]

Lemma 2 Under Wolfe linesearch test:

$$\phi'(\alpha) \geq c_2 \phi'(0)$$

where $c_2 < 1$ and $p_k$ satisfies $B_k p_k = -g_k$ for some positive definite $B_k$, it follows that

$$y_k^T s_k > 0$$
Proof
\[ g_k^T p_k = -p_k^T B_k p_k < 0 \] (1)
since \( B_k \) is positive definite. Since \( \phi(\alpha) = f(x_k + \alpha p_k) \) the linesearch condition is equivalent to
\[ g_{k+1}^T p_k \geq c_2 g_k^T p_k \]
Thus
\[ y_k^T s_k = \alpha_k (g_{k+1} - g_k)^T p_k = \alpha_k \left\{ (g_{k+1} - c_2 g_k)^T p_k - (1 - c_2) g_k^T p_k \right\} \]
The first term is nonnegative, and the second term is strictly negative by (1) leading to the required result.

Note that if we do exact linesearch along \( p_k \), then we still get \( y_k^T p_k > 0 \).

Proof The first order optimality conditions guarantee that \( g_{k+1}^T p_k = 0 \). Thus
\[ y_k^T s_k = \alpha_k (g_{k+1} - g_k)^T p_k = -\alpha_k g_k^T p_k > 0 \]
from the positive definiteness of \( B_k \) (see (1)).