5 Lagrange Multipliers

Given a nonempty set $C$, the polar cone of $C$ is defined by

$$C^\circ = \{ y : y \cdot x \leq 0, \ \forall x \in C \}.$$  

Note that the polar cone $C^\circ$ is always closed and convex since it is the intersection of closed halfspaces.

Given a nonempty subset $X \in \mathbb{R}^m$, we define the cone generated by $X$, denoted by $\text{cone}(X)$ as the set of all nonnegative combinations of points of $X$. It is easy to prove that $\text{cone}(X)$ is convex, but it is not necessarily closed. If the set $X$ has a finite number of members (say $n$), we say the cone is finitely generated. If we arrange the elements of $X$ as the columns of a matrix $A$, it is clear that

$$\text{cone} X = A(\mathbb{R}^n_+).$$

**Proposition 5** (a) For any nonempty set

$$C^\circ = (\text{cl} \ C)^\circ = (\text{conv} \ C)^\circ = (\text{cone} \ C)^\circ.$$

(b) For any nonempty cone $C$, we have

$$(C^\circ)^\circ = \text{cl}(\text{conv} \ C).$$

In particular, if $C$ is closed and convex, we have $(C^\circ)^\circ = C$.

**Proof** (a) We show the first equivalence, the others follow similarly. Since $C \subseteq \text{cl} \ C$ it follows that $(\text{cl} \ C)^\circ \subseteq C^\circ$. Conversely, if $y \in C^\circ$ then $y \cdot x^\nu \leq 0$ for all sequences $\{x^\nu\} \subseteq C$, so that $y \cdot x \leq 0$ for all $x \in \text{cl} \ C$. Hence $y \in (\text{cl} \ C)^\circ$, implying that $C^\circ \subseteq (\text{cl} \ C)^\circ$.

(b) We first show the result for the case where $C$ is closed and convex. Indeed, for any $x \in C$, we have $x \cdot y \leq 0$ for all $y \in C^\circ$, which implies that $x \in (C^\circ)^\circ$. Hence $C \subseteq (C^\circ)^\circ$.

To prove the reverse inclusion, choose any $z \in (C^\circ)^\circ$. The problem

$$\min_{x \in C} \|x - z\|^2$$

has a solution since for any $c \in C$

$$\min_{x \in C} \|x - z\|^2 = \min_{x \in C, \|x - c\| \leq \|z - c\|} \|x - z\|^2$$
and the latter optimization is a continuous function over a compact set. By the minimum principle, the solution \( \hat{z} \) satisfies
\[
(z - \hat{z}) \cdot (x - \hat{z}) \leq 0, \; \forall x \in C.
\]
Taking \( x = 0 \in C \) (since \( C \) is a closed cone) and \( x = 2\hat{z} \in C \) (again by \( C \) a cone), it follows that
\[
(z - \hat{z}) \cdot \hat{z} = 0
\]
and hence that
\[
(z - \hat{z}) \cdot x \leq 0, \; \forall x \in C,
\]
implying that \( z - \hat{z} \in C^\circ \). Now
\[
\|z - \hat{z}\|^2 = (z - \hat{z}) \cdot z - (z - \hat{z}) \cdot \hat{z} = (z - \hat{z}) \cdot z \leq 0
\]
the second equality following from above, and the last inequality following from the fact that \( z \in (C^\circ)^\circ \). It thus follows that \( z = \hat{z} \) and hence \( z \in C \), so that \( (C^\circ)^\circ \subseteq C \).

Thus, the last statement of the proposition is proven. To prove the result for the case where \( C \) is not closed and convex, we note from above that
\[
((\mathrm{cl}(\mathrm{conv} \, C))^\circ)^\circ = \mathrm{cl}(\mathrm{conv} \, C).\]
However, from part (a),
\[
C^\circ = (\mathrm{conv} \, C)^\circ = (\mathrm{cl}(\mathrm{conv} \, C))^\circ.
\]
Combining these last two relations gives the required conclusion. \( \square \)

**Lemma 6** \( A(\mathbb{R}^n_+) \) is closed, so every finitely generated cone is closed and convex.

**Proof** Suppose \( \{y^\nu\} \subseteq A(\mathbb{R}^n_+) \) and \( y^\nu \to \bar{y} \). The linear program
\[
\min_{x,y} \|y - \bar{y}\|_\infty \text{ such that } Ax = y, x \geq 0
\]
has feasible points whose objective converges to zero, and which is bounded below by zero. Thus it has an optimal solution \( (\bar{x}, \bar{y}) \). Since \( A\bar{x} = \bar{y} \) and \( \bar{x} \geq 0 \), it follows that \( \bar{y} \in A(\mathbb{R}^n_+) \). \( \square \)
Lemma 7 \( A(\mathbb{R}^n_+) = \{ u: u \cdot A \leq 0 \}^\circ \); this is equivalently stated as the polar of a polyhedral cone is finitely generated.

Proof It is clear from the definition that
\[
A(\mathbb{R}^n_+)^\circ = \{ u: u \cdot A \leq 0 \}.
\]
Therefore
\[
\{ u: u \cdot A \leq 0 \}^\circ = (A(\mathbb{R}^n_+)^\circ) = \text{cl conv } A(\mathbb{R}^n_+).
\]
But \( A(\mathbb{R}^n_+) \) is convex and as we have just seen in Lemma 6, closed. \( \square \)

Lemma 8 Let \( e = (1, \ldots, 1) \) and
\[
S := \{ v: Av = 0, e \cdot v = 1, v \geq 0 \}.
\]
\( \) Then \( x \in S \) if and only if \( x \) can be represented as a convex combination of basic feasible solutions of
\[
\begin{bmatrix} A \\ e \end{bmatrix} v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v \geq 0.
\]
Note that there are a finite number of basic feasible solutions.

Proof The if statement is trivial. For the only if proof, suppose that \( x \in S \). Thus \( x \) satisfies
\[
\begin{bmatrix} A \\ e \end{bmatrix} v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]
and if \( x \) is a basic feasible solution, we are done. Suppose it is not. Then by a standard theorem of linear programming, there is a basic feasible solution \( y \) whose support is strictly contained in \( \text{supp}(x) \). Since \( e \cdot x = e \cdot y \), the strict containment implies there is some \( j \in \text{supp}(y) \) with \( x_j < y_j \). Let
\[
w(\mu) := x + \mu(x - y)
\]
so that for small \( \mu > 0 \), \( w(\mu) \geq 0 \), and there is at least one component \( j \) of \( w(\mu) \) that is decreasing as \( \mu \) increases. Thus, there is a \( \mu^* \) such that \( w(\mu^*) \geq 0 \) and \( \text{supp } w(\mu^*) \) is strictly contained in \( \text{supp}(x) \). Since
\[
x = \frac{1}{1 + \mu^*}w(\mu^*) + \frac{\mu^*}{1 + \mu^*}y
\]
the required result follows by repeating the above argument to \( w(\mu^*) \) instead of \( x \). \( \square \)
Lemma 9  A polyhedral cone in $\mathbb{R}^n$ is finitely generated.

Proof  If the cone $K$ is empty, it is certainly finitely generated. If not, it can be written as $K = \{x: Dx \geq 0\}$ for some matrix $D$, and suppose $x \in K$.

Consider the set $L := \text{im}(D) \cap \mathbb{R}^m$. Since $\text{im}(D)$ is a subspace, it can be represented as $\ker(A)$ for some matrix $A$. Furthermore, any point in this set is a nonnegative multiple of a point in $S$, the set defined in Lemma 8. Thus, every point in $L$ is a positive multiple of the finite number of basic feasible solutions identified in Lemma 8; thus $L$ is finitely generated. Suppose these generators are $y^1, \ldots, y^k$, then there exist $x^1, \ldots, x^k$ with $y^i = Dx^i$ for each $i$. Then

$$K = \ker D + [x^1 \cdots x^k](\mathbb{R}_+^k)$$

which is a finitely generated cone. To see this last statement, it is clear that the right hand side is contained in $K$. However,

$$x \in K \implies y = Dx \text{ and } y \in L$$

$$\implies y = Dx \text{ and } y = \sum_{i=1}^k \mu_i y^i, \mu_i \geq 0$$

$$\implies y = Dx \text{ and } y = D \sum_{i=1}^k \mu_i x^i, \mu_i \geq 0$$

Thus $D(x - \sum_{i=1}^k \mu_i x^i) = 0$, so $x \in \ker D + [x^1 \cdots x^k](\mathbb{R}_+^k)$ as required.

Proposition 10  The class of polyhedral cones and the class of finitely generated cones are the same.

Proof  Suppose that $K$ is a finitely generated cone. Then $K = A(\mathbb{R}_+^k)$ for some $A$. Hence $K^\circ = \{u: u \cdot A \leq 0\}$. This is evidently polyhedral, and hence from Lemma 9 finitely generated. Thus its polar $(K^\circ)^\circ$ is also polyhedral; but this is $K$ since $K$ is closed and convex. Hence $K$ is polyhedral.

The reverse implication was already shown in Lemma 9.

Theorem 7 (Minkowski-Weyl)  The class of polyhedral sets and the class of finitely generated sets are the same.

See separate sheet.