1. The following tasks provide some ways to get acquainted with basic facts about convex sets and their tangent cones, including connections with feasible directions.

(a) Show that for a convex set \( C \) and a point \( \bar{x} \in C \), the vectors giving feasible directions into \( C \) at \( \bar{x} \) are the vectors \( w \) that can be expressed in the form \( w = \lambda(x - \bar{x}) \) for some \( x \in C \) other than \( \bar{x} \) and some \( \lambda > 0 \). (Note that this is a two-way assertion.)

(b) Verify that the set \( C = \{(x_1, x_2, x_3): x_2 \geq 0, |x_3| \leq 1 - x_1^2 - x_2^2\} \) is convex in \( \mathbb{R}^3 \). (Simplify your task by making use of various rules for deriving convex sets from other convex sets or convex functions.) What does this set \( C \) look like? (A picture isn’t required, but one could help you in understanding parts (c) and (d) that follow.)

(c) For the set \( C \) in (b) and the point \( \bar{x} = (1, 0, 0) \), show that if a vector \((w_1, w_2, w_3) \neq (0, 0, 0)\) gives a feasible direction into \( C \) at \( \bar{x} \), then it satisfies \( w_2 \geq 0 \) and \(|w_3| \leq -2w_1\). On the other hand, show that if a vector \((w_1, w_2, w_3) \neq (0, 0, 0)\) satisfies \( w_2 \geq 0 \) and \(|w_3| < -2w_1\), then it gives a feasible direction into \( C \) at \( \bar{x} \). (In each case rely on the definition of “feasible directions” and the constraints specifying \( C \) in (b).) Note: The first argument develops a necessary condition for a feasible direction in this special setting, whereas the second argument develops a sufficient condition.

(d) Derive from the necessary and sufficient conditions on feasible directions in (c) the fact that, for the set \( C \) and point \( \bar{x} \) in question, the tangent cone is given by

\[
T_C(\bar{x}) = \{(w_1, w_2, w_3): w_2 \geq 0, |w_3| \leq -2w_1\}
\]

and is a convex set, moreover polyhedral.
2. This exercise explores the role of Lagrange multipliers in optimality. It concerns the set $C \subset \mathbb{R}^2$ consisting of all $x = (x_1, x_2) \in X = \mathbb{R}^2_+$ that satisfy

\[
0 \geq f_1(x_1, x_2) = x_1 + x_2 - 5, \\
0 \geq f_2(x_1, x_2) = x_1 - x_2 - 1, \\
0 \geq f_3(x_1, x_2) = 2x_1 - x_2 - 4.
\]

(a) Draw a picture of $C$, indicating the tangent and normal cones to $C$ at representative points $\bar{x}$ and giving an algebraic expression in terms of Lagrange multipliers for the vectors $v = (v_1, v_2)$ that belong to the normal cone $N_C(\bar{x})$ in each case. (Hint: Use Theorem 10. There’s no need to be concerned about the standard constraint qualification (CQ) there, because this is a system of linear constraints only; cf. Theorem 12(a).) The description of $N_C(\bar{x})$ yields equations for $v_1$ and $v_2$ in terms of $y_1, y_2, y_3, z_1$, and $z_2$ as parameters (these being the coordinates of the vectors $y$ and $z$ there). Having written down these equations in general, specialize them at each $\bar{x}$ by suppressing any parameters that have to equal 0 and indicating the sign restrictions, if any, on the remaining parameters.

(b) For the problem of minimizing $f_0(x_1, x_2) = (x_1 - 1)^2 - x_2$ over $C$, add to your picture in (a) some indication of representative level sets $f_0(x_1, x_2) = \alpha$. Determine all locally and globally optimal solutions $\bar{x}$ “by hand” through Theorem 9. (Note: this amounts to checking where the Kuhn-Tucker conditions are satisfied; justifiable shortcuts based on the problem’s structure are encouraged.) For each such $\bar{x}$ give the specific parameter values $y_1, y_2, y_3, z_1$, and $z_2$ in part (a) for the normal vector representation of $-\nabla f_0(\bar{x})$.

(c) Solve the minimization problem in (b) in GAMS using a solver for quadratic programs. Note that the Lagrange multipliers are returned as consname.m. Do the latter correspond to the parameter values you calculated in (b)? What about the values of $z$?
3. Consider an optimization problem \((P)\) in standard format in which the functions \(f_i\) are all of class \(C^1\) and \(X = \mathbb{R}^n\). Let \(\bar{x}\) denote any feasible solution.

(a) Show that the standard constraint qualification (CQ) is satisfied at \(\bar{x}\) if the following conditions, comprising the Mangasarian-Fromovitz constraint qualification, hold:

1. the vectors \(\nabla f_i(\bar{x})\) for \(i \in [s + 1, m]\) are linearly independent, and
2. there exists a vector \(w\) such that \(\langle \nabla f_i(\bar{x}), w \rangle\) \(\begin{cases} < 0 & \text{for } i \in [1, s] \text{ active at } \bar{x}, \\ = 0 & \text{for } i \in [s + 1, m]. \end{cases}\)

(Remark: The converse is true as well; these two constraint qualifications are equivalent – for problems with \(X = \mathbb{R}^n\). But you’re not being asked to prove it.)

(b) Let \(s = m\), so that the constraints in \((P)\) are \(f_i(x) \leq 0\) for \(i = 1, \ldots, m\). Suppose the following condition, called the Slater constraint qualification, is satisfied: all the constraint functions \(f_i\) are convex and there is a point \(\tilde{x}\) with \(f_i(\tilde{x}) < 0\) for \(i = 1, \ldots, m\). Show that then the Mangasarian-Fromovitz constraint qualification (in which now only the inequality part counts) is satisfied at \(\bar{x}\) (no matter how \(\bar{x}\) is located relative to \(\tilde{x}\)).

(c) Go back to the set \(C\) and point \(\bar{x}\) in Question 1 (b)-(d) above, viewing this set as given in the manner above by three constraints \(f_i(x) \leq 0\). (The condition involving \(|x_3|\) converts into two of these). Determine the normal cone \(N_C(\bar{x})\), taking care to check somehow that the assumptions behind the formula you are using are fulfilled.