Homework 6

CS 730, Semester II, 2007–08

March 26, 2008

1. The following is a simplified mathematical model for the economical management of electrical power dispatch. There are n power generating plants indexed by $j = 1, \ldots, n$. Plant j is capable of producing any power amount x_j in a fixed interval $l_j \leq x_j \leq u_j$, where $0 \leq l_j < u_j < \infty$, the cost being $\phi_j(x_j) = c_j x_j + \frac{1}{2} b_j x_j^2$ with known coefficients $c_j > 0$ and $b_j > 0$. The plants are all connected to the same transmission network, so that when each produces an amount x_j the sum of these amounts enters the network; but due to transmission losses, which involve electrical interactions, the amount of power actually made available to customers is not this sum but

   $$h(x) = h(x_1, \ldots, x_n) := \sum_{j=1}^{n} x_j - \frac{1}{2} \sum_{j=1,k=1}^{n,n} a_{jk} x_j x_k$$

   for a certain symmetric, positive definite matrix $A \in \mathbb{R}^{n \times n}$ with entries $a_{jk} \geq 0$. It is assumed in the model that the entries a_{jk} are small enough that the partial derivatives $(\partial h / \partial x_j)(x)$ are positive at all vectors $x = (x_1, \ldots, x_n)$ having $l_j \leq x_j \leq u_j$. This ensures that h is an increasing function with respect to each variable over these ranges; in other words, an increase in power at one of the plants always results in an increase in power available from the network. Note that the highest value h can achieve is $h(u) = h(u_1, \ldots, u_n)$, whereas the lowest is $h(l) = h(l_1, \ldots, l_n)$. The exercise revolves around the following problem in these circumstances: for a given load demand d (power to be withdrawn from the network), with $h(l) < d < h(u)$, determine a scheduling vector $x = (x_1, \ldots, x_n)$ that meets this demand as cheaply as possible.

   (a) Express this as a problem (P): $\min_{x \in X} f_0(x)$ subject to one equality or inequality constraint involving a scalar function f_1. Is this quadratic programming? convex programming? Is the corresponding Lagrangian $L(x, y)$ convex in $x \in X$ for each $y \in Y$ (define Y yourself), as well as affine in $y \in Y$ for each $x \in X$?

   (b) Does (P) have at least one optimal solution? At most one optimal solution?

   (c) Show that, by virtue of the assumptions in the model, the standard constraint qualification is satisfied at every feasible solution $\bar{x}$.

   (d) If the Kuhn-Tucker conditions for (P) hold at $\bar{x}$, can you legitimately conclude that $\bar{x}$ is optimal? On the other hand, if they don't hold at $\bar{x}$, might $\bar{x}$ be optimal anyway?

(e) Show that the Kuhn-Tucker conditions come down to relations between the single Lagrange multiplier $\bar{y} = \bar{y}_1$ and the ratio of $\phi_j'(\bar{x}_j)$ to $(\partial h/\partial x_j)(\bar{x}_1, \ldots, \bar{x}_n)$ for $j = 1, \ldots, n$, and moreover that they imply $\bar{y}$ has to be positive.

(f) What must be the units in which $L(x, y)$ and y are measured, considering that the costs $\phi_j(x_j)$ are in dollars, whereas $h(x)$ and the x_j's are in "power units"? In such terms, try to interpret the ratio relations required by the Kuhn-Tucker conditions. What do they tell you about schedule levels $\bar{x}_j$ that are optimal with respect to the given demand d?