Proposition 11 *Tangents versus normals under regularity: When a closed set C is regular at $\bar{x}$, the geometric relationship between tangents and normals is beautifully symmetric:*

$$N_C(\bar{x}) = \{v \colon v \cdot w \le 0 \textit{ for all } w \in T_C(\bar{x})\},$$
$$T_C(\bar{x}) = \{w \colon v \cdot w \le 0 \textit{ for all } v \in N_C(\bar{x})\}.$$

Proof The first equation follows from the definition of regularity of C at $\bar{x}$. It yields the '$\subset$' half of the second equation. To show that $\supset$ holds in that equation as well, we fix any vector $\bar{w} \notin T_C(\bar{x})$ and aim at demonstrating the existence of a vector $\bar{v} \in N_C(\bar{x})$ such that $\bar{v} \cdot \bar{w} > 0$.

Replacing C by its intersection with some closed ball around $\bar{x}$ if necessary (which involves no loss of generality, since the generation of normal vectors depends only on a neighborhood of $\bar{x}$), we can suppose that C is compact. Let B stand for some closed ball around $\bar{w}$ that doesn't meet $T_C(\bar{x})$ (as exists because $T_C(\bar{x})$ is closed). The definition of $T_C(\bar{x})$, in conjunction with having $T_C(\bar{x}) \cap B = \emptyset$, implies the existence of an $\epsilon > 0$ such that the compact, convex set $S = \{\bar{x} + \tau w \colon w \in B, \tau \in [0, \epsilon]\}$ meets C only at $\bar{x}$. Consider any sequence $\epsilon^\nu \searrow 0$ with $\epsilon^\nu < \epsilon$ along with the compact, convex sets $S^\nu = \{\bar{x} + \tau w \colon w \in B, \tau \in [\epsilon^\nu, \epsilon]\}$, which are disjoint from C.

The function $h(x, u) = \frac{1}{2} \|x - u\|^2$ attains its minimum over the compact set $C \times S^\nu$ at some point (x^ν, u^ν). In particular, x minimizes $h(x, u^\nu)$ with respect to $x \in C$, so by Theorem 9 the vector $-\nabla_x h(x^\nu, u^\nu) = u^\nu - x^\nu$ belongs to $N_C(x^\nu)$. Likewise, the vector $\nabla_u h(x^\nu, u^\nu) = x^\nu - u^\nu$ belongs to $N_{S^\nu}(u^\nu)$. Necessarily $x^\nu \neq u^\nu$ because $C \cap S^\nu = \emptyset$, but $x^\nu \to \bar{x}$ and $u^\nu \to \bar{x}$ because the sets S^ν increase to S (the closure of their union), and $C \cap S = \{\bar{x}\}$.

Let $v^\nu = (u^\nu - x^\nu)/\|u^\nu - x^\nu\|$, so $\|v^\nu\| = 1$, $v^\nu \in N_C(x^\nu)$, $-v^\nu \in N_{S^\nu}(u^\nu)$. The sequence of vectors v^ν being bounded, it has a cluster point $\bar{v}$, $\|\bar{v}\| = 1$; without loss of generality (by passing to a subsequence if necessary) we can suppose for simplicity that $v^\nu \to \bar{v}$. Along with the fact that $v^\nu \in N_C(x^\nu)$ and $x^\nu \to \bar{x}$, this implies that $\bar{v} \in N_C(\bar{x})$. Because $-v^\nu \in N_{S^\nu}(x^\nu)$ and S^ν is convex, we also have $-v^\nu \cdot [u - u^\nu] \le 0$ for all $u \in S^\nu$. Since S^ν increases to S while $u^\nu \to \bar{x}$, we obtain in the limit that $-\bar{v} \cdot [u - \bar{x}] \le 0$ for all $u \in S$. Recalling the construction of S, we note that among the vectors $u \in S$ are all vectors of the form $\bar{x} + \epsilon w$ with $w \in B$. Further, B is the closed ball of a certain radius $\delta > 0$ around $\bar{w}$, so its elements w have the form $\bar{w} + \delta z$ with $\|z\| \le 1$. Plugging these expressions into the limiting inequality that was obtained, we get $-\bar{v} \cdot \epsilon[\bar{w} + \delta z] \le 0$ for all z with $\|z\| \le 1$. In particular we can take $z = -\bar{v}$ (since $\|\bar{v}\| = 1$) and see that $-\bar{v} \cdot \bar{w} + \delta \|\bar{v}\|^2 \leq 0$. This reveals that $\bar{v} \cdot \bar{w} \geq \delta$, hence $\bar{v} \cdot \bar{w} > 0$ as desired. $\square$