Equivalence of polyhedrality and finite generation for convex sets.

Definition. *A subset S of* $\mathbf{R}^n$ *is a* **finitely generated convex set** *if there are points* $y_1, \ldots, y_K$ *and* $z_1, \ldots, z_L$ *in* $\mathbf{R}^n$ *such that*

$$S = \left\{ \sum_{k=1}^{K} \lambda_k y_k + \sum_{l=1}^{L} \zeta_l z_l \mid \lambda_k \geq 0,\ \zeta_k \geq 0,\ \sum_{k=1}^{K} \lambda_k = 1 \right\}.$$

Theorem. *A polyhedral convex set in* $\mathbf{R}^n$ *is finitely generated.*

Proof: If C is empty it is ~~polyhedral~~ finitely generated, so assume that C is a nonempty polyhedral convex set, and represent it as $\{ x \mid Dx \leq d \}$ for appropriate D and d. Define a cone Q in $\mathbf{R}^{n+1}$ by

$$Q := \left\{ \lambda \begin{pmatrix} c \\ -1 \end{pmatrix} \mid c \in C, \lambda > 0 \right\} \cup \left\{ \begin{pmatrix} v \\ 0 \end{pmatrix} \mid Dv \leq 0 \right\}.$$

~~We will show that~~ Claim

$$Q = \left\{ \begin{pmatrix} s \\ \sigma \end{pmatrix} \mid \begin{pmatrix} D & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s \\ \sigma \end{pmatrix} \leq 0 \right\}. \tag{1}$$

Pf To see this, note first that if $\begin{pmatrix} s \\ \sigma \end{pmatrix} \in Q$ then either $\sigma < 0$ or $\sigma = 0$. If $\sigma < 0$ then

$$\begin{pmatrix} s \\ \sigma \end{pmatrix} = (-\sigma) \begin{pmatrix} -s/\sigma \\ -1 \end{pmatrix},$$

and $D(-s/\sigma) \leq d$. Then $Ds + d\sigma \leq 0$ (remember that $(-\sigma) > 0$!), and of course $\sigma \leq 0$. On the other hand, if $\sigma = 0$ then the definition of Q implies that $Ds \leq 0$ so we have $Ds + d(0) \leq 0$ and $0 \leq 0$. This proves the inclusion "$\subset$".

To prove "$\supset$", suppose that $Ds + d\sigma \leq 0$ and $\sigma \leq 0$. Either $\sigma < 0$ or $\sigma = 0$. In the former case we have $D(-s/\sigma) \leq d$, so $-s/\sigma \in C$ and then

$$\begin{pmatrix} s \\ \sigma \end{pmatrix} = (-\sigma) \begin{pmatrix} -s/\sigma \\ -1 \end{pmatrix} \in Q.$$

In the latter case we have $Ds \leq 0$, so

$$\begin{pmatrix} s \\ \sigma \end{pmatrix} = \begin{pmatrix} s \\ 0 \end{pmatrix} \in Q.$$

$\square$

This proves (1), and therefore we know that the set Q is actually a polyhedral convex cone and, therefore, also finitely generated. Each element of Q has a non-positive last component, so we can write the generators of Q as

$$\begin{pmatrix} y_1 \\ -1 \end{pmatrix}, \ldots, \begin{pmatrix} y_K \\ -1 \end{pmatrix}, \begin{pmatrix} z_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} z_l \\ 0 \end{pmatrix},$$

where not necessarily both kinds of generators need be present.

Now the points $c \in C$ are exactly those c such that $(c,-1)^* \in Q$, and therefore c belongs to C if and only if there are $\lambda_1, \ldots, \lambda_K$ and $\zeta_1, \ldots, \zeta_L$, all non-negative, with $c = \sum_{k=1}^K \lambda_k y_k + \sum_{l=1}^L \zeta_l z_l$ and with $-1 = \sum_{k=1}^K \lambda_k(-1) + \sum_{l=1}^L \zeta_l(0)$: that is, with $\sum_{k=1}^K \lambda_k = 1$. ∎

Corollary. *Let $C = \{x \mid Dx \le d\}$ be the set described in the last theorem. If C is bounded then in fact $C = \operatorname{conv}\{y_1, \ldots, y_K\}$; also, D has the property that $Dv \le 0$ only for $v = 0$.*

Proof: If any z_k is present in the representation of C then z_k must be zero (otherwise C could not be bounded), so $(z_k, 0)^*$ can be deleted from the representation without changing C. This proves the first assertion. Now if $Dv \le 0$ then $(v,0)* \in Q$; however, since no $(z_k,0)^*$ are included among the generators of Q, we must have $v = 0$. ∎

Theorem. *A finitely generated convex set is polyhedral.*

Proof: Let $C \subset \mathbf{R}^n$ be such a set. If $C = \emptyset$ then this is obvious, so assume $C \neq \emptyset$, and represent C as

$$C = \left\{ \sum_{k=1}^K \lambda_k y_k + \sum_{l=1}^L \zeta_l z_l \mid \lambda_k \ge 0,\ \zeta_l \ge 0,\ \sum_{k=1}^K \lambda_k = 1 \right\}.$$

$\{Eu \mid u \geq 0\}$

Let Q be the cone generated by the pairs $(y_1,-1)^*, \ldots, (y_K,-1)^*, (z_1,0)^*, \ldots, (z_L,0)^*$. Then Q° is the intersection of the halfspaces of the form $\{w \in \mathbf{R}^{n+1} \mid \langle g, w\rangle \le 0\}$, where g is one of the above generators. So Q° is polyhedral, hence finitely generated by what we have already proved. Then Q $(= Q^{\circ\circ})$ is polyhedral. Represent Q as

as above (2.1.7)

$$Q = \left\{ \begin{pmatrix} x \\ \xi \end{pmatrix} \mid (D \quad d) \begin{pmatrix} x \\ \xi \end{pmatrix} \le 0 \right\}.$$

Now $x \in C$ if and only if $(x,-1)^* \in Q$: that is, if and only if $Dx + d(-1) \le 0$. Hence $C = \{x \mid Dx \le d\}$, and this is a polyhedral convex set. ∎