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Interior-Point Algorithms for Monotone Affine Variational Inequalities[1,2]

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Abstract. Given an $n \times n$ matrix M, a vector q in $\mathbb{R}^n$, a polyhedral convex set $X = \{x \mid Ax \leq b, Bx = d\}$, where A is an $m \times n$ matrix and B is a $p \times n$ matrix, the affine variational inequality problem is to find $x \in X$ such that $(Mx + q)^T(y - x) \geq 0$, for all $y \in X$. If M is positive semidefinite (not necessarily symmetric), the affine variational inequality can be transformed to a generalized complementarity problem, which can be solved in polynomial time using interior-point algorithms due to Kojima et al. We develop interior-point algorithms that exploit the particular structure of the problem, rather than directly reducing the problem to a standard linear complementarity problem.

Key Words. Interior-point algorithms, affine variational inequalities, complementarity problems.

1. Introduction

In this paper, we investigate how interior-point algorithms can be used to solve monotone affine variational inequalities. Given an $n \times n$ matrix M, a vector q in $\mathbb{R}^n$, and a polyhedral convex set

$$X = \{x \mid Ax \leq b, Bx = d\},$$

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where A is an $m \times n$ matrix and B is a $p \times n$ matrix, the affine variational inequality problem, abbreviated as AVI(q, M, X), is to find $x \in X$ such that

$$\text{(AVI)} \quad (Mx + q)^T(y - x) \geq 0, \qquad \text{for all } y \in X.$$

In this paper, we assume that M is positive semidefinite, and we say that AVI(q, M, X) is monotone.

It is well known (see Ref. 1) that AVI(q, M, X) is equivalent to the following complementary problem:

$$\text{(CP)} \quad (s, x, u) \in \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^m_+,$$

$$H(s, x, u) = \begin{bmatrix} 0 & -B & 0 \\ B^T & M & A^T \\ 0 & -A & 0 \end{bmatrix} \begin{bmatrix} s \\ x \\ u \end{bmatrix} + \begin{bmatrix} d \\ q \\ b \end{bmatrix} \in \{0\} \times \{0\} \times \mathbb{R}^m_+,$$

$$(s, x, u)^T H(s, x, u) = 0.$$

Complementarity problems of this form belong to the category of generalized complementarity problems (see Ref. 2). In Section 2, it will be shown that (CP) is essentially equivalent to a standard linear complementarity problem. Therefore, the interior algorithms used by Kojima et al (see Refs. 3, 4) for solving monotone linear complementarity problem can be used to solve (CP). However, the direct reduction of (CP) to a standard LCP is impractical. So, in Section 3, we present a different approach for applying interior algorithms to solve (AVI).

The following is a summary of our notation and the basic concepts employed. Given any matrix C and the index sets α and β, C_α denotes the submatrix formed by those rows of C with indices in α, $C_{\cdot\beta}$ denotes the submatrix formed by those columns of C with indices in β, and $C_{\alpha\beta}$ denotes the submatrix formed by those elements of C with row indices in α and column indices in β. For any vector or matrix, a superscript T indicates the transpose and $\|\cdot\|_p$ denotes their p-norm; see Ref. 5. For any vector v diag(v) is the diagonal matrix whose diagonal elements are the components of v. Finally, for any closed convex set $S \subset \mathbb{R}^n$,

$$\text{rec}\, S := \{d \in \mathbb{R}^n \mid s + \lambda d \in S, \forall s \in S, \forall \lambda \geq 0\}$$

is the recession cone of S, and the set

$$L(S) := \{d \in \mathbb{R}^n \mid s + \mu d \in S, \forall s \in S, \forall \mu \in \mathbb{R}\}$$

is the lineality space of S; see Ref. 6.

2. Generalized Linear Complementarity Problem

We begin with a few basic concepts from the theory of monotone multifunctions. A multifunction T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a subset of $\mathbb{R}^n \times \mathbb{R}^m$. A multifunction T from $\mathbb{R}^n$ to $\mathbb{R}^n$ is said to be monotone if, for each pair $(x_1, y_1), (x_2, y_2) \in T$,

$$(x_1 - x_2)^T(y_1 - y_2) \geq 0.$$

T is said to be maximal monotone if it is not properly contained in any other monotone multifunction. Also, T is said to be affine if T is an affine subset of $\mathbb{R}^n$ to $\mathbb{R}^n$.

For each monotone multifunction T, we can define a complementarity problem of finding (x, y) such that

$$(x, y) \in T, \qquad (x, y) \geq 0, \qquad x^T y = 0. \tag{1}$$

We call (1) the generalized linear complementarity problem when T is affine and maximal monotone.

Theorem 2.1. Problem (CP) is a generalized linear complementarity problem with

$$T = \{(u, v) \in \mathbb{R}^m \times \mathbb{R}^m \mid v = b - Ax,\ Bx = d,\ Mx + A^T u + B^T s + q = 0\}. \tag{2}$$

Proof. Obviously, T is affine. It suffices to show that T is maximal monotone.

For any $(u_i, v_i) \in T$, $i = 1, 2$,

$$\Delta u = u_2 - u_1, \qquad \Delta v = v_2 - v_1,$$

and some appropriate Δs, Δx satisfy the following homogeneous equation:

$$\begin{bmatrix} B^T & A^T & 0 & M \\ 0 & 0 & I & A \\ 0 & 0 & 0 & B \end{bmatrix} \begin{bmatrix} \Delta s \\ \Delta u \\ \Delta v \\ \Delta x \end{bmatrix} = 0. \tag{3}$$

Therefore,

$$\Delta u^T \Delta v = \Delta x^T M^T \Delta x.$$

It follows from the positive semidefiniteness of M that

$$\Delta u^T \Delta v \geq 0, \tag{4}$$

which implies that T is a monotone multifunction.

In showing that T is maximal, we may assume without loss c generality that

$$b = 0, \qquad d = 0, \qquad q = 0.$$

By Minty's theorem, it suffices to show that the range of $I + T$ is $\mathbb{R}^m$. Le $z \in \mathbb{R}^m$ be arbitrary. We show the existence of $(u, v) \in T$ such that $z = u + \iota$ It follows from (2) that this is equivalent to the solvability of the systen

$$u + v = z,$$

$$Bx = 0,$$

$$Ax + v = 0,$$

$$Mx + A^T u + B^T s = 0.$$

Equivalently, the system

$$Bx = 0, \qquad Mx + A^T(Ax + z) + B^T s = 0 \tag{5}$$

must be solvable for (x, s). Let $C \in \mathbb{R}^{n \times n-p}$ be a matrix such that

$$\ker B = \operatorname{im} C.$$

Letting $x = Ct$, (5) reduces to the system

$$(M + A^T A)Ct + A^T z \in \operatorname{im} B^T = \ker C^T,$$

or

$$C^T(M + A^T A)Ct + C^T A^T z = 0.$$

Since z is arbitrary, we must show that

$$\operatorname{im}(C^T A^T) \subset \operatorname{im}(C^T(M + A^T A)C),$$

which is in turn equivalent to the statement

$$\ker(C^T(M^T + A^T A)C) \subset \ker(AC).$$

To prove the last statement, assume that

$$C^T(M^T + A^T A)Cw = 0.$$

Then,

$$w^T C^T(M^T + A^T A)Cw = 0,$$

or

$$w^T C^T(M^T)Cw + \|ACw\|_2^2 = 0.$$

But M is positive semidefinite; hence,

$$w^T C^T (M^T) C w = 0 \quad \text{and} \quad A C w = 0.$$

The claim is proved. □

Now that we know that T is maximal monotone, the following result (Corollary 2.1, Ref. 2) illustrates the connection between (CP) and the class of horizontal LCPs defined in Ref. 7.

Theorem 2.2. Let T be an affine multifunction on $\mathbb{R}^m$. T is maximal monotone if and only if there exist matrices $H_1, H_2 \in \mathbb{R}^{m \times m}$ and $a \in \mathbb{R}^m$ such that the pair H_1, H_2 is column monotone, that is, $H_1 + H_2 = I$, $H_1^T H_2$ is positive semidefinite, and

$$T = \{(u, v) \mid H_1 u - H_2 v = a\}.$$

When T is represented as in Theorem 2.2, (CP) is equivalent to the following horizontal LCP (see Ref. 7):

$$H_1 u - H_2 v = a, \qquad u, v \geq 0, \qquad u^T v = 0. \tag{6}$$

The pair H_1, H_2 is column monotone due to the maximality of T; therefore, Theorem 7 of Ref. 7 applies.

Theorem 2.3. Given H_1, H_2 column monotone, then for any $a \in \mathbb{R}^m$, (6) is equivalent to LCP($C^{-1}D$, $C^{-1}a$), where C and D are column representatives (see Ref. 7) of H_1, H_2 and $C^{-1}D$ is positive semidefinite.

Since (CP) is equivalent to a standard monotone LCP, interior-point algorithms (e.g., the path-following algorithm in Ref. 3, the potential-reduction algorithm in Ref. 4, and the infeasible-path-following algorithm in Ref. 8) can be applied to provide polynomial algorithms for (CP) and hence for (AVI). However, the construction carried out to arrive at the standard LCP is generally impractical. In the next section, we show that the path-following and the potential-reduction-algorithms can be carried out without specifically reducing (CP) to a monotone LCP.

3. Interior-Point Algorithms

Section 2 shows that (CP) is equivalent to a standard LCP. However, directly reducing (CP) to a standard LCP using the method outlined in the last section will not provide a practical algorithm. We now show how to exploit the structure of Problem (CP) in applying the path-following and potential-reduction algorithms.

We assume that all the elements of the matrix

$$Q = \begin{bmatrix} M & q \\ A & b \\ B & d \end{bmatrix}$$

are integers. The size of Problem (AVI) is defined by

$$L = 1 + \log(m + n + p)^2 + \left\lceil \sum_{i=1}^{m+n+p} \sum_{j=1}^{n+1} \log(1 + |q_{ij}|) \right\rceil,$$

where q_{ij}'s are elements of the matrix Q.

3.1. Path-Following Method. To solve (CP) using a path-following method, we begin with an initial point (s^0, u^0, v^0, x^0) which is close to the central path, that is, a point in the set

$$S^\alpha := \{(s, u, v, x) \in S \mid u, v > 0, \|UVe - \zeta e\|_2 \leq \alpha\zeta, \text{ where } \zeta = (1/m)u^T v\}, \quad (7)$$

where S is defined by (16).

At each step, the Newton step for the nonlinear equation

$$F(s, u, v, x, \mu) = (UV - \mu e, Mx + q + B^T s + A^T u, v + Ax - b, Bx - d) = 0 \quad (8)$$

is used to compute a new point in S^α such that ζ is reduced from the previous value by a constant factor. The algorithm temrinates when ζ is sufficiently small.

Given a point $(s^0, u^0, v^0, x^0) \in S^\alpha$, here is the algorithm.

Algorithm 1.

Step 1. Choose $0 < \alpha \leq 1/10$; let $\delta = \alpha/(1 - \alpha)$, and let $k = 0$.

Step 2. If $u^{kT} v^k < 2^{-4L}$, then stop.

Step 3. Let

$$\zeta = u^{kT} v^k / m,$$

$$\mu = (1 - \delta/m^{1/2})\zeta,$$

$$(s, u, v, x) = (s^k, u^k, v^k, x^k).$$

Step 4. Compute $(\Delta s, \Delta u, \Delta v, \Delta x)$ by constructing a Newton step for the nonlinear equation (8), that is, solve

$$\begin{bmatrix} 0 & V & U & 0 \\ B^T & A^T & 0 & M \\ 0 & 0 & I & A \\ 0 & 0 & 0 & B \end{bmatrix} \begin{bmatrix} \Delta s \\ \Delta u \\ \Delta v \\ \Delta x \end{bmatrix} = \begin{bmatrix} UVe - \mu e \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{9}$$

and set

$$(s^{k+1}, u^{k+1}, v^{k+1}, x^{k+1}) = (s, u, v, x) - (\Delta s, \Delta u, \Delta v, \Delta x).$$

Step 5. Set $k = k + 1$, and go to Step 2.

There are two crucial issues concerning the validity of the algorithm; one is the solvability of (9), and the other is the justification that each new iterate stays in S^α and that ζ is reduced. In fact, by the analysis from the last section, (9) is equivalent to

$$H_1 \Delta u - H_2 \Delta v = 0, \tag{10a}$$

$$V\Delta u + U\Delta v = UVe - \mu e. \tag{10b}$$

Hence, $(\Delta u, \Delta v)$ is uniquely solvable from (9) by the maximality of T; see Theorem 2.1, Ref. 2. Furthermore, in view of Theorem 2.3, the step computed from (9) is the same as the interior step used by Kojima et al. in Ref. 3 for LCP$(C^{-1}D, C^{-1}a)$. Therefore, we have the following theorem.

Theorem 3.1. Let $(s, u, v, x) \in S$ with $u, v > 0$ satisfy

$$\|UVe - \zeta e\|_2 \leq \alpha\zeta, \qquad \text{with } \zeta = (1/m)u^T v,$$

for $\alpha \in (0, 1/10)$. Let

$$\mu = (1 - \delta/m^{1/2})\zeta.$$

Suppose that $(\Delta s, \Delta u, \Delta v, \Delta x)$ is a solution of (9) and that

$$(\bar{s}, \bar{u}, \bar{v}, \bar{x}) = (s, u, v, x) - (\Delta s, \Delta u, \Delta v, \Delta x).$$

Then, $(\bar{u}, \bar{v}) > 0$, and

$$\|\bar{U}\bar{V}e - \bar{\zeta}e\|_2 \leq \alpha\bar{\zeta},$$

$$\bar{\zeta} = (1/m)\bar{u}^T\bar{v} \leq (1 - \delta/6m^{1/2})\zeta.$$

As a result of this theorem, Algorithm 1 stops in $O(m^{1/2}L)$ iterations, each of which requires $O((m+n+p)^3)$ operations to compute a new point. Therefore, the number of arithmetic operations needed for finding a point $\{(s^k, u^k, v^k, x^k)\}$ such that $u^{kT}v^k < 2^{-4L}$ is no more than $O(m^{1/2}(m+n+p)^3 L)$. Furthermore, an exact solution of AVI(q, M, X) can be constructed from such a point in no more than $O((m+n+p)^3)$ arithmetic operations using a technique similar to that of Ref. 3.

3.2. Potential-Reduction Method. Potential-reduction algorithms stai with a point in

$$S^0 := \{(s, u, v, x) \in S \mid u, v > 0\},$$

where S is defined by (16),

such that $f(u, v)$ does not exceed $O(m^{1/2}L)$, where the potential function i defined by

$$f(u, v) = \sqrt{m} \log u^T v - \sum_{i=1}^{m} \log(u_i v_i) - m \log m, \quad \text{for } (s, u, v, x) \in S^0. \quad (11$$

The algorithm is as follows.

Algorithm 2.

Step 1. Choose $(s^0, u^0, v^0, x^0) \in S^0$, such that $f(u, v)$ does not excee $O(m^{1/2}L)$, and let $k = 0$.

Step 2. Let $(s, u, v, x) = (s^k, u^k, v^k, x^k)$. If $f(u^k, v^k) < -4m^{1/2}L$, the stop.

Step 3. Let

$$w = (\sqrt{u_1 v_1}, \sqrt{u_2, v_2}, \ldots, \sqrt{u_m v_m}),$$

$$W = \operatorname{diag} w,$$

$$z = W^{-1}e - ((n + \sqrt{n})/\|w\|_2^2)w.$$

Step 4. Compute $(\Delta s, \Delta u, \Delta v, \Delta x)$ by constructing a Newton step fo the function F, that is, solve

$$\begin{bmatrix} B^T & A^T & 0 & M \\ 0 & 0 & I & A \\ 0 & 0 & 0 & B \end{bmatrix} \begin{bmatrix} \Delta s \\ \Delta u \\ \Delta v \\ \Delta x \end{bmatrix} = 0, \quad (12a$$

$$W^{-1}(U\Delta v + V\Delta u) = z/\|z\|_2, \quad (12b$$

and set

$$(s^{k+1}, u^{k+1}, v^{k+1}, x^{k+1}) = (s, u, v, x) - (\Delta s, \Delta u, \Delta v, \Delta x).$$

Step 5. Set $k = k + 1$, and go to Step 2.

We notice that the system (12) is equivalent to

$$H_1 \Delta u - H_2 \Delta v = 0, \quad (13a$$

$$U\Delta v + V\Delta u = Wz/\|z\|_2. \quad (13b$$

A reference to Theorem 2.3 and Theorem 2.2, Ref. 4 leads to the following result.

Theorem 3.2. $(\Delta u, \Delta v)$ is uniquely determined by (12), and at each iteration we have

$$f(u^{k+1}, v^{k+1}) < f(u^k, v^k) - 0.2.$$

Similar to the case of Algorithm 1, Theorem 3.2 guarantees that the number of arithmetic operations needed by the potential-reduction algorithm for finding a solution of AVI(q, M, X) is bounded by $O(m^{1/2}(m+n+p)^3L)$.

4. Implementation Issue

Although $(\Delta u, \Delta v)$ can be uniquely determined from the system (10) or (13), in practice we solve (9) or (12). The task of computing $(\Delta u, \Delta v)$ can be significantly simplified if the solution to each of these systems is unique. Our next lemma shows that the assumption

$$\operatorname{rank}\begin{bmatrix} 0 & -B \\ B^T & M \\ 0 & -A \end{bmatrix} = n + p \tag{14}$$

guarantees unique solutions of (9) and (12).

Lemma 4.1. Suppose that the condition (14) holds. Then, for any positive diagonal matrices D_1, D_2, and $r \in \mathbb{R}^m$, the equation

$$\begin{bmatrix} 0 & D_1 & D_2 & 0 \\ B^T & A^T & 0 & M \\ 0 & 0 & I & A \\ 0 & 0 & 0 & B \end{bmatrix} \begin{bmatrix} \Delta s \\ \Delta u \\ \Delta v \\ \Delta x \end{bmatrix} = \begin{bmatrix} r \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

has a unique solution.

Proof. It suffices to show that the homogeneous system

$$\begin{bmatrix} 0 & D_1 & D_2 & 0 \\ B^T & A^T & 0 & M \\ 0 & 0 & I & A \\ 0 & 0 & 0 & B \end{bmatrix} \begin{bmatrix} \Delta s \\ \Delta u \\ \Delta v \\ \Delta x \end{bmatrix} = 0 \tag{15}$$

has a unique solution.

Suppose that $(\Delta s, \Delta u, \Delta v, \Delta x)$ is a solution. Then,

$$D_1 \Delta u + D_2 \Delta v = 0,$$

hence

$$D \Delta u + D^{-1} \Delta v = 0,$$

where

$$D = (D_1 D_2^{-1})^{1/2}.$$

Therefore,

$$\|D\Delta u\|_2^2 + 2(D\Delta u)^T(D^{-1}\Delta v) + \|D^{-1}\Delta v\|_2^2 = 0.$$

Notice that

$$(D\Delta u)^T(D^{-1}\Delta v) = \Delta u^T \Delta v \geq 0$$

as a result of (4), so we have

$$\|D\Delta u\|_2 = 0, \qquad \|D^{-1}\Delta v\|_2 = 0.$$

It follows that

$$\Delta u = 0, \qquad \Delta v = 0.$$

Consequently,

$$\Delta s = 0, \qquad \Delta x = 0,$$

since

$$\operatorname{rank}\begin{bmatrix} 0 & -B \\ B^T & M \\ 0 & -A \end{bmatrix} = n + p.$$

We now turn to the general case and in the rest of this section devel(a technique to reduce a problem in the form (CP) to a smaller proble satisfying (14). Define the feasible set of (CP) by

$$S := \{(s, u, v, x) \mid u, v \geq 0, v = Ax - b, Bx - d = 0,$$
$$Mx + A^T u + B^T s + q = 0\}. \qquad (1$$

The lineality space of S is

$$L(S) = \{(s, 0, 0, x) \mid B^T s + Mx = 0, -Ax = 0, -Bx = 0\}.$$

So, $L(S) = \{0\}$ if and only if (14) holds.

For convenience of notation, define

$$Q = \begin{bmatrix} 0 & -B \\ B^T & M \end{bmatrix}, \qquad C = [0, \quad A].$$

Problem (CP) can be reformulated as

$$\text{(CP}'\text{)} \quad (z, u) \in \mathbb{R}^{p+n} \times \mathbb{R}^m_+,$$

$$H(z, u) = \begin{bmatrix} Q & C^T \\ -C & 0 \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix} + \begin{bmatrix} q' \\ b \end{bmatrix} \in \{0\} \times \mathbb{R}^m_+,$$

$$(z, u)^T H(z, u) = 0,$$

where

$$z = \begin{bmatrix} s \\ x \end{bmatrix}, \qquad q' = \begin{bmatrix} d \\ q \end{bmatrix}.$$

Suppose $L(S) \neq \{0\}$. Then, the columns of the matrix $\begin{bmatrix} Q \\ -C \end{bmatrix}$ are linearly dependent. There exists index sets α and β such that

$$\begin{bmatrix} Q \\ -C \end{bmatrix} = \begin{bmatrix} Q_{\cdot\alpha} & Q_{\cdot\beta} \\ -C_{\cdot\alpha} & -C_{\cdot\beta} \end{bmatrix}$$

and $\begin{bmatrix} Q_{\cdot\alpha} \\ -C_{\cdot\alpha} \end{bmatrix}$ is a maximum subset of linearly independent columns of the matrix $\begin{bmatrix} Q \\ -C \end{bmatrix}$. Thus,

$$\begin{bmatrix} Q_{\cdot\beta} \\ -C_{\cdot\beta} \end{bmatrix} = \begin{bmatrix} Q_{\cdot\alpha} \\ -C_{\cdot\alpha} \end{bmatrix} P, \tag{17}$$

for some $|\alpha| \times |\beta|$ matrix P.

Lemma 4.2. Let α, β, and P be as in (17), $\beta \neq \varnothing$. If (CP′) is solvable, then there exists a solution $(\bar{z}, \bar{u})$ such that $\bar{z}_\beta = 0$.

Proof. Let $(\bar{z}, \bar{u}) = (\bar{z}_\alpha, \bar{z}_\beta, \bar{u})$ be a solution of (CP′). Then, it is clear that $(\bar{z}_\alpha + P\bar{z}_\beta, 0, \bar{u})$ is the desired solution. □

In order to complete our reduction, we will need a particular property of positive-semidefinite matrices. The following lemma and its corollaries indicate precisely the form of this property that we need. Notice that M is not assumed symmetric.

Lemma 4.3. See Ref. 9, Page 13. Let M be a positive-semidefini matrix, and assume that

$$M=\begin{bmatrix} 0 & u^T \\ 0 & M' \end{bmatrix}.$$

Then, $u=0$.

Corollary 4.1. Let M be an $n\times n$ positive-semidefinite matrix, and l

$$\gamma\subset\{1,2,\dots,n\}.$$

Assume that $M_{\cdot\gamma}=0$. Then, $M_{\gamma\cdot}=0$.

Proof. Apply the previous lemma to each index of γ. ∎

Corollary 4.2. Let M be an $n\times n$ positive-semidefinite matrix, and l γ,α,β be a partition of $\{1,2,\dots,n\}$, so that

$$M=[M_{\cdot\gamma}, M_{\cdot\alpha}, M_{\cdot\beta}].$$

Assume that

$$M_{\cdot\gamma}=M_{\cdot\alpha}P,$$

for some $|\alpha|\times|\gamma|$ matrix P. Then,

$$M_{\gamma\cdot}=P^TM_{\alpha\cdot}.$$

Proof. Observe that

$$\begin{bmatrix} I & -P^T & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} M \begin{bmatrix} I & 0 & 0 \\ -P & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & -P^T & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & M_{\gamma\alpha} & M_{\gamma\beta} \\ 0 & M_{\alpha\alpha} & M_{\alpha\beta} \\ 0 & M_{\beta\alpha} & M_{\beta\beta} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & M_{\gamma\alpha}-P^TM_{\alpha\alpha} & M_{\gamma\beta}-P^TM_{\alpha\beta} \\ 0 & M_{\alpha\alpha} & M_{\alpha\beta} \\ 0 & M_{\beta\alpha} & M_{\beta\beta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & M_{\alpha\alpha} & M_{\alpha\beta} \\ 0 & M_{\beta\alpha} & M_{\beta\beta} \end{bmatrix},$$

where the last equality follows from Corollary 4.1. It now follows that

$$M=\begin{bmatrix} I & P^T & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & M_{\alpha\alpha} & M_{\alpha\beta} \\ 0 & M_{\beta\alpha} & M_{\beta\beta} \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ P & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} P^TM_{\alpha\alpha}P & P^TM_{\alpha\alpha} & P^TM_{\alpha\beta} \\ M_{\alpha\alpha}P & M_{\alpha\alpha} & M_{\alpha\beta} \\ M_{\alpha\beta}P & M_{\alpha\beta} & M_{\beta\beta} \end{bmatrix},$$

so that

$$\begin{aligned} M_\gamma &= [P^T M_{\alpha\alpha} P, P^T M_{\alpha\alpha}, P^T M_{\alpha\beta}] \\ &= P^T [M_{\alpha\alpha} P, M_{\alpha\alpha}, M_{\alpha\beta}] \\ &= P^T M_\alpha . \end{aligned}$$

□

Corollary 4.2 allows our final reduction to remove the lineality. Problem (CP″) satisfies (14).

Theorem 4.1. Let (α, β) be defined by (17). Define Problem (CP″) by

$$\text{(CP}''\text{)} \quad (w, u) \in \mathbb{R}^{p+n-|\beta|} \times \mathbb{R}^m_+,$$

$$\tilde{H}(w, u) = \begin{bmatrix} Q_{\alpha\alpha} & (C^T)_{\alpha\cdot} \\ -C_{\cdot\alpha} & 0 \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} + \begin{bmatrix} q'_\alpha \\ b \end{bmatrix} \in \{0\} \times \mathbb{R}^m_+,$$

$$(w, u)^T \tilde{H}(w, u) = 0.$$

Then, (z, u) is a solution of (CP′) with $z_\beta = 0$ if and only if (z_α, u) is a solution of (CP″). Furthermore, a solution x of AVI(q, M, X) is immediate from the definition of (CP′) and $z = (s, x)$.

Proof. If (z, u) is a solution of (CP′) with $z_\beta = 0$, then it is easily verified that (z_α, u) is a solution of (CP″).

Conversely, if (z_α, u) is a solution of (CP″), then

$$Q_{\alpha\alpha} z_\alpha + (C^T)_{\alpha\cdot} u + q'_\alpha = 0, \tag{18}$$

$$-C_{\cdot\alpha} z_\alpha + b \in \mathbb{R}^m_+, \tag{19}$$

$$u^T(-C_{\cdot\alpha} z_\alpha + b) = 0. \tag{20}$$

Moreover, since the matrix $\begin{bmatrix} Q & C^T \\ -C & 0 \end{bmatrix}$ is positive semidefinite, we can apply Corollary 4.2 to (17), resulting in

$$[Q_{\beta\alpha}, Q_{\beta\beta}, (C^T)_{\beta\cdot}] = P^T [Q_{\alpha\alpha}, Q_{\alpha\beta}, (C^T)_{\alpha\cdot}].$$

Also, taking into account (18), we have

$$\begin{aligned} &[Q_{\beta\alpha}, Q_{\beta\beta}, (C^T)_{\beta\cdot}] \begin{bmatrix} z_\alpha \\ 0 \\ u \end{bmatrix} + q'_\beta \\ &= P^T [Q_{\alpha\alpha}, Q_{\alpha\beta}, (C^T)_{\alpha\cdot}] \begin{bmatrix} z_\alpha \\ 0 \\ u \end{bmatrix} + q'_\beta \\ &= q'_\beta - P^T q'_\alpha . \end{aligned}$$

If $q'_\beta - P^T q'_\alpha \neq 0$, then the system

$$\begin{bmatrix} Q_{\alpha\alpha} & Q_{\alpha\beta} & (C^T)_{\alpha\cdot} \\ Q_{\beta\alpha} & Q_{\beta\beta} & (C^T)_{\beta\cdot} \end{bmatrix} \begin{bmatrix} z_\alpha \\ 0 \\ u \end{bmatrix} + \begin{bmatrix} q'_\alpha \\ q'_\beta \end{bmatrix} = 0$$

is inconsistent, a contradiction to the solvability of (CP′) and Lemma 4.2. Hence,

$$q'_\beta - P^T q'_\alpha = 0.$$

Let $z_0 = (z_\alpha, 0)$. Then,

$$H(z_0, u) = \begin{bmatrix} 0 \\ 0 \\ -C_{\cdot\alpha} z_\alpha + b \end{bmatrix} \in \{0\} \times \mathbb{R}^m_+$$

follows from (18), (19). We also have

$$(z_0, u^T) H(z_0, u) = 0,$$

by reference to (20). □

By definition, we can write

$$Q_{\alpha\alpha} = \begin{bmatrix} 0 & -\bar{B} \\ \bar{B}^T & \bar{M} \end{bmatrix}, \qquad C_{\cdot\alpha} = [0, \bar{A}],$$

for appropriate submatrices $\bar{A}$, $\bar{B}$, $\bar{M}$ of A, B, M respectively. Note that $\bar{M}$ is positive semidefinite and the matrix

$$\begin{bmatrix} 0 & -\bar{B} \\ \bar{B}^T & \bar{M} \\ 0 & -\bar{A} \end{bmatrix}$$

has full column rank. Therefore, (CP″) is equivalent to AVI($\bar{q}, \bar{M}, \bar{X}$) where

$$\bar{X} = \{y \mid \bar{A}y \leq \bar{b}, \bar{B}y = \bar{d}\}$$

and $\bar{q}, \bar{b}, \bar{d}$ are vectors which consist of appropriate components of q, b, d respectively.

The procedure of reducing AVI(q, M, X) to AVI($\bar{q}, \bar{M}, \bar{X}$) can be easily carried out by using Gaussian elimination and deleting appropriate matrix rows and columns. A solution of AVI($\bar{q}, \bar{M}, \bar{X}$) is found by solving (CP″). A solution of (CP), and hence a solution of AVI(q, M, X), can then be constructed from that of (CP″) by applying Theorem 4.1. Therefore these operations do not increase the order of complexity.

5. Conclusions

The monotone affine variational inequality has been shown to be essentially equivalent to a linear complementarity problem. Interior algorithms for LCP can be applied to provide polynomial algorithms for monotone affine variational inequalities. It has also been shown how to adapt path-following and potential-reduction algorithms for solving such variational inequalities without actually reducing them to standard LCPs.

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