P₀-Matrices and the Linear Complementarity Problem

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ABSTRACT

We introduce a new matrix class \( P₀ \), which consists of those matrices \( M \) for which the solution set of the corresponding linear complementarity problem is connected for every \( q \in \mathbb{R}^n \). We consider Lemke's pivotal method from the perspective of piecewise linear homotopies and normal maps and show that Lemke's method processes all matrices in \( P₀ \cap Q₀ \). We further investigate the relationship of the class \( P₀ \) to other known matrix classes and show that column sufficient matrices are a subclass of \( P₀ \), as are \( 2 \times 2 \) \( P₀ \)-matrices.

1. INTRODUCTION

The linear complementarity problem is a classical problem from optimization theory of finding \( x \in \mathbb{R}^n \) with

\[
    z \geq 0, \quad Mz + q \geq 0, \quad z^T (Mz + q) = 0.
\]

Here \( M \in \mathbb{R}^{n \times n} \) and \( q \in \mathbb{R}^n \) are given data, and the resulting problem will be denoted by \( \text{LCP}(q, M) \). We also define the set of feasible points of

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LCP\( (q, M) \) by

\[
\text{FEA}(q, M) := \{ z \mid z \succeq 0, Mz + q \succeq 0 \}.
\]

In this paper, we investigate a new class of matrices, \( P_e \), which is defined by

\[
M \in P_e \iff \text{SOL}(q, M) \text{ is connected for all } q \in \mathbb{R}^n,
\]

where \( \text{SOL}(q, M) \) is the set of solutions of LCP\( (q, M) \). The most widely used algorithm for solving LCP\( (q, M) \) is the pivotal algorithm of Lemke [10]. In [1], it is shown that Lemke’s method processes all matrices \( M \in P_0 \cap Q_0 \), that is, it either finds a solution, or determines that \( \text{FEA}(q, M) = \emptyset \). Here \( P_0 \) is the class of matrices having nonnegative principal minors. The principal result of this paper, given in Section 2, is that if \( M \in P_0 \cap Q_0 \), then Lemke’s method processes LCP\( (q, M) \). Note that \( Q_0 \) is the set of matrices for which feasibility of LCP\( (q, M) \) implies its solvability, that is,

\[
M \in Q_0 \iff [\text{FEA}(q, M) \neq \emptyset \Rightarrow \text{SOL}(q, M) \neq \emptyset].
\]

Before proving this result, let us explain our motivation. An \( n \times n \) matrix is a member of the matrix class \( P \) if all its principal minors are positive. It is well known that an equivalent definition is that \( \text{SOL}(q, M) \) is a singleton for every \( q \in \mathbb{R}^n \). Therefore, a natural extension of the class \( P \) is the class of column sufficient matrices \( S_e \), characterized by

\[
M \in S_e \iff \text{SOL}(q, M) \text{ is convex for all } q \in \mathbb{R}^n.
\]

Although there are other extensions of the class \( S_e \), the most natural geometric extension would seem to be \( P_e \). Note that it is clear that

\[
P \subset S_e \subset P_e.
\]

In order to relate our result to others found in the literature, we explore the class \( P_e \) further in Section 3. It is known [4, Theorems 3.3.4, 3.4.2] that

\[
P \subset S_e \subset P_0 \subset E_0,
\]

where \( E_0 \) is the class of matrices for which \( \text{SOL}(q, M) \) is a singleton for all \( q > 0 \). We know of no geometric properties of LCP\( (q, M) \) that characterize \( P_0 \) or \( E_0 \). In this paper, the geometrically defined class of matrices \( P_e \) is
shown to be closely related to the algebraically defined class $P_0$. The interplay between algebraic and geometric characterizations of matrix classes is of paramount importance to a complete understanding of such classes. We show that within the class of $2 \times 2$ matrices

$$P_0 \subset P_c \subset E_0,$$

and that these inclusions are strict. We conjecture that (2) holds for $n \times n$ matrices and hence that our main result extends the class of matrices that Lemke’s method is known to process. However, in the $2 \times 2$ case, we also show that

$$P_0 \cap Q_0 = P_c \cap Q_0.$$

2. TERMINATION OF LEMKE’S METHOD

Although the basic step of Lemke’s method is a pivot (as in the simplex method for linear programming), the choice of pivot step is fundamentally different and is motivated by a path following or homotopy approach. An equivalent formulation of $\text{LCP}(q, M)$ is to find a zero of the nonsmooth mapping

$$x \mapsto Mx_+ + q + x - x_+,$$

where $(x_+) = \max\{x, 0\}$ is the projection of $x$ onto the nonnegative orthant. This map is sometimes referred to as the “normal map” [12]; the earliest known reference is [7]. The equivalence is established by noting that if $z$ solves $\text{LCP}(q, M)$, then $x = z - Mz - q$ is a zero of the normal map, and if $x$ is a zero of the normal map, then $z = x_+$ is a solution of $\text{LCP}(q, M)$. It is easy to see that the normal map is an affine map on each of the orthants of $\mathbb{R}^n$ and is continuous on $\mathbb{R}^n$. The normal mapping is thus an example of a piecewise affine map and is intimately related to the manifold defined by the collection of the faces of the set $\mathbb{R}_+^n$, called the normal manifold [12]. Lemke’s method can be viewed as a clever way of traversing this manifold, as each pivot step corresponds to changing the affine map that currently represents the normal map. In fact, Lemke’s method is an instance of a more general algorithm for solving equations with piecewise linear homotopies due to Eaves [8]. The analysis in this paper uses many of the ideas contained in [5] without further proof. We will also use the fact that the general algorithm applied to $\text{LCP}(q, M)$ is in fact Lemke’s algorithm; this is shown elsewhere [2, 5].
Let $\mathcal{N}$ be the piecewise-linear manifold in $\mathbb{R}^{n+1}$ constructed by forming the Cartesian product of each orthant of $\mathbb{R}^n$ with $\mathbb{R}_+$, the nonnegative half line in $\mathbb{R}$. We abuse notation slightly and let $\mathcal{N}$ represent both the collection of cells of the manifold and the union of this collection. $\mathcal{N}$ is a piecewise linear $(n+1)$-manifold in $\mathbb{R}^{n+1}$, as can easily be verified (see [8, Example 4.3]). Now let $\epsilon > 0$ and consider the piecewise linear map $F: \mathcal{N} \to \mathbb{R}^n$ defined by

$$F(\mathbf{x}, \mu) := M\mathbf{x} + q + \mathbf{x} - x_\mu + \mu \epsilon.$$ 

Clearly any $x$ satisfying $F(x, 0) = 0$ solves LCP$(q, M)$. Let $w(\mu) := -q - \mu \epsilon$, and note that since

$$w(\mu) = -\mu [\epsilon + \mu^{-1}q],$$

(3)

$w(\mu)$ lies interior to the orthant $\mathbb{R}^n_+$ for large positive $\mu$. Therefore $(w(\mu), \mu)$ lies interior to the cell $\mathbb{R}^n_+ \times \mathbb{R}_+$ of $\mathcal{N}$, and so it is a regular point of $\mathcal{N}$ (see the proof of Theorem 2). Further, for such $\mu$ we have $(w(\mu), \mu) = 0$, so that

$$F(\mathbf{w}(\mu), \mu) = -q - \mu \epsilon - (q + \mu \epsilon) = 0.$$ 

Therefore, for some $\mu_0 \geq 0$, $F^{-1}(0)$ contains the ray $\{(w(\mu), \mu) \mid \mu \geq \mu_0\}$. Now the algorithm of [8] is applied to the PL equation $F(x, \mu) = 0$, using a ray start at $(w(\mu_1), \mu_1)$ for some $\mu_1 > \mu_0$ and proceeding in the direction $(-e, -1)$. As the manifold $\mathcal{N}$ is finite, according to [8, Theorem 15.13] the algorithm generates, in finitely many steps, either a point $(x_\mu, \mu_\mu)$ in the boundary of $\mathcal{N}$ with $F(x_\mu, \mu_\mu) = 0$, or a secondary ray in $F^{-1}(0)$ different from the starting ray. In the first case $\mu_\mu = 0$ and, by our earlier remarks, $x_\mu$ solves LCP$(q, M)$. Many of the results pertaining to Lemke’s method processing different classes of matrices just show that secondary ray termination cannot occur, or that ray termination guarantees FEA$(q, M) = \emptyset$.

Such results are plentiful, and Lemke’s algorithm is known to process many classes of matrices; see, for example [3, 4, 11]. There are, in fact, two large but distinct classes that contain most of these classes of matrices, namely L-matrices [6] and the class $P_0 \cap Q_0$ [1]. This paper is concerned with extending the algebraically defined class $P_0 \cap Q_0$ using geometric ideas. To this purpose, we introduce the class $P_c$. In the remainder of this section we will show that Lemke’s method processes matrices from $P_c \cap Q_0$. We shall explore the class $P_c$ more fully in the following section.
The set

\[ K(M) := \{ q \in \mathbb{R}^n \mid \text{SOL}(q, M) \neq \emptyset \} \]

is the set of all right hand side vectors for which \( \text{LCP}(q, M) \) is solvable. This set is intimately related to the class \( Q_0 \), as the following theorem shows.

**Theorem 1** [6]. For an \( n \times n \) matrix \( M \), the following are equivalent:

1. \( M \in Q_0 \).
2. \( K(M) \) is convex
3. \( K(M) = \text{pos}(I, -M) \).

Here \( \text{pos}(I, -M) \) represents the cone generated by the columns of the matrix \( (I, -M) \) and the origin.

Note that \( \text{pos}(I, -M) \) is a polyhedral convex cone. Our main result is summarized in the following theorem. The two key geometric facts that we use in the proof are

1. the connectedness of \( \text{SOL}(q, M) \) for all \( q \),
2. the convexity of \( K(M) \).

**Theorem 2.** Suppose \( M \) is in \( P_\ast \cap Q_0 \). Then Lemke's algorithm terminates at a solution of \( \text{LCP}(q, M) \) or determines that \( \text{FEA}(q, M) = \emptyset \). Furthermore, the parameter \( \mu \) in Lemke's algorithm is nonincreasing.

**Proof.** Since \( 0 \) may not be a regular value of \( F \), we use the pivotal algorithm from [8] which generates a solution of the original problem by solving the perturbed system

\[ F(x, \mu) = -[\epsilon], \]

where \( [\epsilon] = (\epsilon, \epsilon^2, \ldots, \epsilon^n)^T \), with \( \epsilon > 0 \).

Let \( w(\mu) := -q - \mu e = -\mu [e + \mu^{-1}q] \), so that \( w(\mu) \) lies interior to the orthant \( \mathbb{R}_+^n \) for large positive \( \mu \). Therefore \( (w(\mu) - [\epsilon], \mu) \) lies interior to the cell \( \mathbb{R}_+^n \times \mathbb{R}_+^m \) of \( \mathcal{N} \) for \( \mu \) sufficiently large and \( \epsilon \) sufficiently small. It is a regular point of \( \mathcal{N} \), since \( F(\mathbb{R}_+^n \times \mathbb{R}_+^m) \) has a nonempty interior. Further, for large \( \mu \) we have \( (w(\mu) - [\epsilon])_+ = 0 \), so that

\[ F(w(\mu) - [\epsilon], \mu) = M(w(\mu) - [\epsilon])_+ + q + w(\mu) - [\epsilon] - (w(\mu) - [\epsilon])_+ + \mu e = -[\epsilon]. \]
Hence, $F^{-1}(-\epsilon)$ contains the ray \((w(\mu) - \epsilon, \mu) | \mu \geqslant \mu_0\) for some $\mu_0 > 0$.

Now the algorithm of [8] is applied to the PL equation $F(x, \mu) = -\epsilon$, using a ray start at \((w(\mu_1), \mu_1)\) for some $\mu_1 > \mu_0$ and proceeding in the direction $(-e, -1)$.

Since $-\epsilon \in F(\mathcal{N})$ for all sufficiently small $\epsilon$, it follows from [8, Lemma 14.2], that $-\epsilon$ is a regular value of $F$ for each small positive $\epsilon$. It then follows by [8, Theorem 9.1] that for such $\epsilon$, $F^{-1}(-\epsilon)$ is a 1-manifold near $\mathcal{N}$. This means that $F^{-1}(-\epsilon)$ is closed in $\mathcal{N}$ and its boundary is its intersection with the boundary of $\mathcal{N}$. It is subdivided by $\sigma \cap F^{-1}(-\epsilon)$, where $\sigma$ is an $n$-cell of $\mathcal{N}$. Furthermore, we have $(w(\mu) - \epsilon, \mu) \in F^{-1}(-\epsilon)$ for sufficiently large $\mu$.

Now, assume that the algorithm generates a sequence of points $(x_1, \mu_1), (x_2, \mu_2), \ldots, (x_k, \mu_k)$ with $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_k$ and either terminates at step $k$ with a ray different from the starting one or generates a point $(x_{k+1}, \mu_{k+1})$ with $\mu_{k+1} > \mu_k$. Let $\mathcal{W}(\epsilon)$ be the set of chords traversed by the algorithm up to this point. Then, due to the ray start, $\mathcal{W}(\epsilon)$ cannot be PA homeomorphic to a circle, and therefore it is homeomorphic to an interval [8, Lemma 5.1].

Upon ray termination, $\mu$ is nondecreasing on the terminating ray. Thus, the set

$$\Xi = \{ \mu | (x, \mu) \in \mathcal{W}(\epsilon) \}$$

admits a minimum $\overline{\mu} = \inf\{ \mu \in \Xi | 0 \geqslant \mu \}$, which is achieved on $(x_j, \mu_j)$ for some $1 \leqslant j \leqslant k$. Let

$$S = \{ x | (x, \overline{\mu}) \in \mathcal{W}(\epsilon) \};$$

then $F(x, \overline{\mu}) = -\epsilon$ for $x \in S$. Hence

$$S \subset \text{SOL}(q + \epsilon + \overline{\mu}e, M).$$

But $\text{SOL}(q + \epsilon + \overline{\mu}e, M)$ cannot contain any other point $z_1$ such that $(z_1, \overline{\mu}) \notin \mathcal{W}(\epsilon)$; otherwise, by our hypothesis on the connectedness of the solution set, there is a continuous path $z: [0, 1] \to \text{SOL}(q + \epsilon + \overline{\mu}e, M)$ with $z(1) = z_1$ and $z(0) = z_0$ for any $z_0 \in S$. Thus

$$\{(z(t), \overline{\mu}) | 0 \leqslant t \leqslant 1\} \subset F^{-1}(-\epsilon).$$
But this contradicts the fact that $F^{-1}([-\epsilon])$ is a 1-manifold, since $(z_0, \mu)$ contains a neighborhood not homeomorphic to an interval (see Figure 1).

Thus $S = \text{SOL}(q + [\epsilon] + \mu e, M)$ is a connected set. It is either a single point, or the union of finite number of consecutive chords in $W(\epsilon)$. In particular, $S$ is closed.

We now show that if $q \in K(M)$, then $\mu = 0$. The cone $K(M)$ contains the positive orthant in its interior, and it is convex because $M \in Q_0$. Since $q + [\epsilon] \in K(M)$ and $q + [\epsilon] + \mu e \in K(M)$, it follows that

$$q + [\epsilon] + \lambda \mu e \in K(M)$$

for every $\lambda \in [0, \mu]$. Hence

$$\text{SOL}(q + [\epsilon] + \mu e, M) \neq \emptyset$$

for all $\lambda \in [0, \mu]$. Consider a strictly increasing sequence $\{\lambda_j | j = 1, 2, \ldots\}$ with $\lambda_1 < \mu$ and $\lim_{j \to \infty} \lambda_j = \mu$. Assume that $x(\lambda_j) \in \text{SOL}(q + [\epsilon] + \mu_j e, M)$. Then $x(\lambda_j), \mu_j) \in F^{-1}([-\epsilon])$; hence each $x(\lambda_j), \mu_j) \in F^{-1}([-\epsilon])$ is contained in a 1-chord of $F^{-1}([-\epsilon])$. Since the 1-manifold

\[ W \]

\[ S \]

\[ z_1 \]

**Fig. 1.** The path connecting $z_1$ to $S$ forms a branch of $W$. 
$F^{-1}(-[\epsilon])$ is finite, there exists a chord $l$ (which is a line segment) such that $(x(\mu_j), \mu_j) \in l$ for infinitely many $j$, and without loss of generality we can assume that $(x(\mu_j), \mu_j) \in l$ for all $j$. Therefore $l$ contains the set

$$\{(x(\mu), \mu) \in F^{-1}(-[\epsilon]) \mid \bar{\mu} - \delta \leq \mu < \bar{\mu}\}$$

for some $\delta > 0$. Thus $l$ contains a point $(\omega(\bar{\mu}), \bar{\mu})$ with $\omega(\bar{\mu}) \in S$.

On the other hand, by definition of $\bar{\mu}$,

$$(\omega(\mu), \mu) \notin W(\epsilon)$$

for any $\mu < \bar{\mu}$. Hence $l$ is not a subset of $W(\epsilon)$, and $l$ forms a branch from $S \times \{\bar{\mu}\}$ (see Figure 2). This is in contradiction to the fact that $F^{-1}(-[\epsilon])$ is a 1-manifold.

So if $q \in K(M)$, the algorithm terminates at a point in the boundary, that is, a solution of $F(x, 0) = -[\epsilon]$.

If the algorithm also terminates in the boundary when $q \notin K(M)$, this leads immediately to a contradiction. Thus in this case, ray termination must occur.

![Figure 2](image-url)  
**Fig. 2.** The chord $l$ forms a branch of $W$. 
In practice the algorithm does not actually use a positive $\epsilon$, but only maintains the information necessary to compute $W(\epsilon)$ for all small positive $\epsilon$, employing the lexicographic ordering to resolve possible ambiguities when $\epsilon = 0$. Therefore after finitely many steps it will actually have computed $x_0$ with $M_{R_+^q}(x_0) + q = 0$, or prove that $\text{FEA}(q, M) = \emptyset$.

3. RELATIONSHIP TO OTHER CLASSES

The aim of this section is to explore the relationship of $P_\epsilon$ to other known classes of matrices. In particular, we consider its relationship to column sufficient matrices, $P_0$ and $E_0$.

We first show how column sufficient matrices are related to $P_0$, and also to $P_\epsilon$. A matrix $M$ is said to be column sufficient if, given $z \in \mathbb{R}^n$,

$$z_i(Mz)_i \leq 0 \quad \text{for all } i \quad \Rightarrow \quad z_i(Mz)_i = 0 \quad \text{for all } i.$$

The class of such matrices is denoted as $S_\epsilon$, and it is shown in [4, Proposition 3.5.8] that this definition is equivalent to the one given in the introduction. $M$ is row sufficient if its transpose is column sufficient, and $M$ is sufficient if it is both column and row sufficient.

Clearly, $P \subset S_\epsilon \subset P_\epsilon$, since the solutions sets are a singleton, convex, and connected, respectively. The following corollary, which also follows from [5, p. 239], is now immediate.

**Corollary 3.** Suppose $M \in S_\epsilon \cap Q_0$. Then Lemke’s algorithm terminates at a solution of $\text{LCP}(q, M)$ or determines that $\text{FEA}(q, M) = \emptyset$.

**Proof.** Since $M$ is column sufficient, $\text{SOL}(q, M)$ is convex, and is hence connected for all $q$. The corollary now follows from Theorem 2.

It is also known that Lemke’s method processes row sufficient matrices, since these are contained in $P_0 \cap Q_0$ [4, 3.5.3 and 3.5.5].

The following example shows that $P_\epsilon$ is not a subclass of $P_0$. The matrix

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
does not belong to \( P_0 \). But for any \( q \in \mathbb{R}^n \), we have

\[
\text{SOL}(q, M) = \begin{cases}
(0,0) & \text{if } q_1 > 0, q_2 > 0 \\
((0,y) \mid y \geq 0) & \text{if } q_1 > 0, q_2 = 0, \\
\emptyset & \text{if } q_1 > 0, q_2 < 0, \\
((x,0) \mid x \geq 0) & \text{if } q_1 = 0, q_2 > 0, \\
((x,0) \mid x \geq 0) \cup ((0,y) \mid y \geq 0) & \text{if } q_1 = 0, q_2 = 0, \\
((x,0) \mid x \geq -q_2) & \text{if } q_1 = 0, q_2 < 0, \\
\emptyset & \text{if } q_1 < 0, q_2 > 0, \\
((0,y) \mid y \geq -q_1) & \text{if } q_1 < 0, q_2 = 0, \\
\{(q_2,-q_1)\} & \text{if } q_1 < 0, q_2 < 0.
\end{cases}
\]

We see that \( \text{SOL}(q, M) \) is connected for all \( q \) and hence \( M \in P_\ast \). Clearly \( M \notin Q_0 \).

Note also that \( E_0 \) is not contained in \( P_\ast \). The following example proves this fact:

\[
M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad q = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.
\]

Here, \( M \in E_0 \), but the solution set for the given \( q \) is

\[
\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix} \right\},
\]

which is not connected.

Now that \( P_0 \) does not contain \( P_\ast \), does \( P_\ast \) contain \( P_0 \)? When an \( n \times n \) matrix \( M \) is in \( P_0 \), we can prove that for all \( q \) except those in a set \( \kappa(M) \) of measure zero, the solution set is connected.

**Theorem 4.** Let \( M \in P_0 \), and \( \kappa(M) \) denote the union of the facets of all the complementary cones of \( M \). If \( q \notin \kappa(M) \) or \( q > 0 \), the number of solutions of \( \text{LCP}(q, M) \) is zero or one, and hence the solution set is connected.

**Proof.** Since \( P_0 \subset E_0 \) [Equation (1)], \( \text{SOL}(q, M) \) is a singleton for all \( q > 0 \) and hence connected.
According to a result in [9, Theorem 2], originally due to Cottle and Gau, 
\text{SOL}(q, M)\) contains either 0, 1, or infinitely many points whenever \(M \in \mathcal{P}_0\). 
However, by [4, Theorem 6.1.8], \(q \in \kappa(M)\) implies that the local degree of 
\(q\) relative to \(M\) is well defined, which implies that \(\text{SOL}(q, M)\) is finite. Thus, 
\(\text{SOL}(q, M)\) has 0 or 1 elements for all \(q \in \mathbb{R}^n\) except those that belong to a 
fine union of polyhedral convex cones of dimension less than \(n\).

The question whether \(\text{SOL}(q, M)\) is connected when it has infinitely 
many elements remains open. However, in the \(2 \times 2\) case we can show the 
following results.

**Theorem 5.** Suppose \(M \in \mathbb{R}^{2 \times 2}\). Then

1. \(P_0 \subset P_c\),
2. \(P_c \subset E_0\),
3. \(P_0 \cap Q_0 = P_c \cap Q_0\).

**Proof.** The following proofs assume that

\[
M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},
\]

and essentially consider all cases. Some details are omitted.

1: From Theorem 4, we only need to consider \(q \in \kappa(M)\), that is (without 
loss of generality) \(q = \begin{bmatrix} \lambda \\ 0 \end{bmatrix}\) or \(q = -\lambda \begin{bmatrix} a \\ c \end{bmatrix}\)

for \(\lambda \geq 0\). Furthermore, since \(M \in \mathcal{P}_0\), we have \(a \geq 0\), \(d \geq 0\), and \(ad \geq bc\).

When \(\lambda = 0\), using the set valued inverse operator

\[
A^{-1}(S) := \{ x \mid Ax \in S \},
\]

we see that \(\text{SOL}(0, M)\) is given by

\[
\left[ M^{-1}(0) \cap \mathbb{R}^2_+ \right] \cup \left[ M^{-1}(0 \times \mathbb{R}^+ \cap \mathbb{R}^+ \times 0 \right]
\]

\[
\cup \left[ M^{-1}(\mathbb{R}^+ \times 0) \cap 0 \times \mathbb{R}^+ \right].
\]
Each of the three sets of the union is polyhedral and contains the origin; hence SOL\( (q, M) \) is connected.

For \( \lambda > 0 \) and \( q = (\lambda, 0)^T \),

\[
\text{SOL}(q, M) = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup (0, 0),
\]

where

\[
\mathcal{A}_1 := M^{-1}(-q) \cap \mathbb{R}_+^2,
\]

\[
\mathcal{A}_2 := M_1^{-1}(\lambda) \cap M_2^{-1}(\mathbb{R}_+) \cap \mathbb{R}_+ \times 0,
\]

\[
\mathcal{A}_3 := M_1^{-1}(\lambda a + \mathbb{R}_+) \cap M_2^{-1}(\mathbb{R}_+) \cap 0 \times \mathbb{R}_+,
\]

\[
\mathcal{A}_4 := M_1^{-1}(\lambda a + \mathbb{R}_+) \cap M_2^{-1}(\mathbb{R}_+) \cap (0, 0).
\]

Note that \( \mathcal{A}_2 = \emptyset \). Furthermore, \( (0, 0) \in \mathcal{A}_3 \). It remains to show that \( \mathcal{A}_1 \) and \( \mathcal{A}_3 \) have nontrivial intersection if \( \mathcal{A}_1 \) is nonempty. If \( \mathcal{A}_1 \) is nonempty, then it is easy to show that it has a point in common with \( \mathcal{A}_3 \) by considering the cases when \( M \) is invertible and when \( M \) is not invertible.

Now consider the case \( \lambda > 0 \) and \( q = -\lambda(a, c)^T \). We may assume without loss of generality that either \( a \) or \( c \) is nonzero. Then

\[
\text{SOL}(q, M) = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4,
\]

where

\[
\mathcal{A}_1 := M^{-1}(-q) \cap \mathbb{R}_+^2,
\]

\[
\mathcal{A}_2 := M_1^{-1}(\lambda a) \cap M_2^{-1}(\lambda c + \mathbb{R}_+) \cap \mathbb{R}_+ \times 0,
\]

\[
\mathcal{A}_3 := M_1^{-1}(\lambda a + \mathbb{R}_+) \cap M_2^{-1}(\mathbb{R}_+) \cap 0 \times \mathbb{R}_+,
\]

\[
\mathcal{A}_4 := M_1^{-1}(\lambda a + \mathbb{R}_+) \cap M_2^{-1}(\mathbb{R}_+) \cap (0, 0).
\]

If \( \mathcal{A}_4 \neq \emptyset \) then \( a < 0 \); hence \( a = 0 \) and thus \( (0, 0) \in \mathcal{A}_2 \). If \( \mathcal{A}_3 \neq \emptyset \), let \( x \in \mathcal{A}_3 \). If \( x_2 = 0 \), then \( a = c = 0 \), which is a contradiction. Thus \( x_2 > 0 \), and it then follows that \( bc = ad \). It is easy to see that this implies \( x \in \mathcal{A}_1 \). The proof is completed by noting that \( (\lambda, 0) \in \mathcal{A}_1 \cup \mathcal{A}_2 \).

2: It is easy to see that the \( 2 \times 2 \) matrix \( M \in E_0 \) if and only if \( a \geq 0 \), \( d \geq 0 \), and either \( ad \geq bc \), \( b \geq 0 \), or \( c \geq 0 \). Thus suppose that \( M \in P_c \) but \( M \notin E_0 \). Then \( M \notin P_0 \), and one of the above inequalities must be violated. It is easy to see that if \( a < 0 \) or \( d < 0 \), then for \( q > 0 \), SOL\( (q, M) \) is not
connected, which is a contradiction. To complete the proof, we derive a contradiction in the case where \( a \geq 0, \ d \geq 0, \ b < 0, \ c < 0, \) and \( bc > ad. \) Again, let \( q > 0. \) Note that is \( x_1 = 0, \) then \( x_2 = 0, \) and conversely. Since \( M \) is invertible, it now follows that the only solutions are \((0,0)\) and \(-M^{-1}q.\) Note that \(-M^{-1}q > 0,\) contradicting the connectedness of the solution set.

3. From the above, it is known that \( P_0 \cap Q_0 \subset P_{\varepsilon} \cap Q_{\varepsilon}. \) We now show the reverse inclusion. Let \( M \in P_{\varepsilon} \cap Q_{\varepsilon}. \) First note that from the above it follows that \( M \in E_0 \cap Q_0 \) and hence that \( a \geq 0, \ d \geq 0, \) and either \( ad > bc, \) \( b > 0, \) or \( c \geq 0. \) Suppose that \( bc > ad, \) so that \( b \geq 0 \) or \( c > 0. \) and thus both are strictly positive. It now follows that \( K(M) = \mathbb{R}^2. \) Taking \( q_1 > 0, q_2 < 0 \) implies that \( d > 0; \) similarly \( a > 0. \) Now let \( q < 0. \) It then follows from the connectedness of \( SOL(q, M) \) that exactly one of the following must hold:

(a) \( aq_2 - cq_1 > 0, \)
(b) \( dq_1 - bq_2 \geq 0, \)
(c) \( aq_2 - cq_1 > 0 \) and \( dq_1 - bq_2 > 0. \)

A contradiction now follows by letting \( q_1 = -a, q_2 = -c. \)

Note that \( E_0 \cap Q_0 \) is strictly bigger than \( P_{\varepsilon} \cap Q_{\varepsilon}, \) as the example given above shows. In fact,

\[
M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}
\]

is in \( E_0 \cap Q_0. \)

4. CONCLUSIONS

This paper has introduced a new class of matrices \( P_{\varepsilon} \) and exhibited some of its properties. Some outstanding questions remain, which include determining an effective test for inclusion in the class \( P_{\varepsilon}. \) An effective test of this sort will allow the conjecture relating \( P_0 \) and \( P_{\varepsilon} \) to be verified or proven false and establish whether the solution set of \( LCP(q, M) \) is in fact connected when \( M \in P_0 \cap Q_0. \) Essentially, a key open question is to establish Theorem 5 or exhibit counterexamples in the general \( n \times n \) case.

REFERENCES


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