Games, Paths and Complementarity

CDGO 2007

Michael C. Ferris
University of Wisconsin, Computer Sciences
ferris@cs.wisc.edu

Nash Equilibria

Given a collection of players $i \in \mathcal{I}$
with decision variables $x_i \in \mathbf{R}^{n_i}$
x^* is a Nash Equilibrium if

$$x_i^* \in \arg\min_{x_i \in X_i} f_i(x_i, x_{-i}^*), \forall i \in \mathcal{I}$$

x_{-i} are the decisions of other players.

2-person game

Player I plays i with prob p_i

Player II plays j with prob q_j

Player I loss matrix A_{ij}

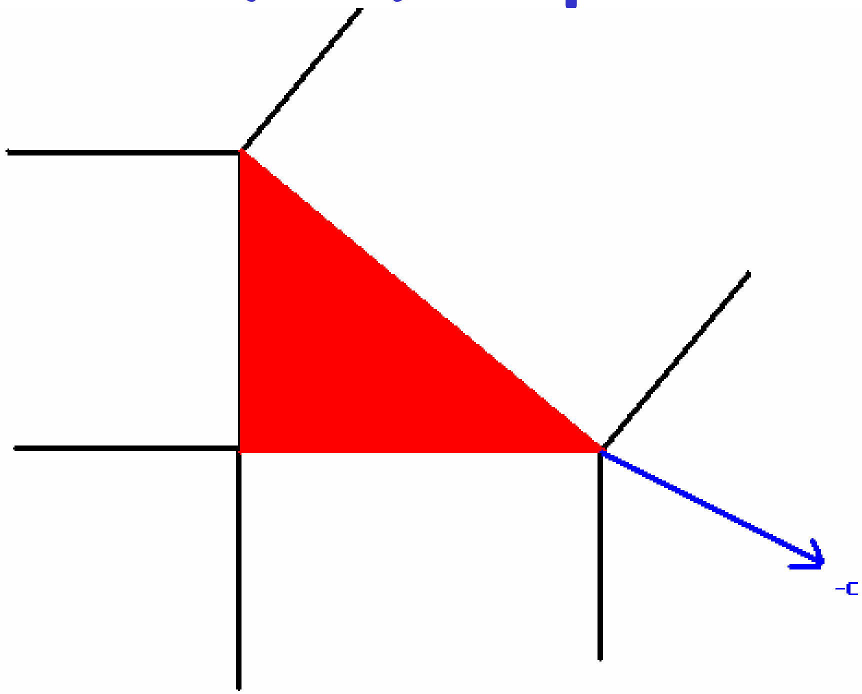
Player II loss matrix B_{ij}

$$p^* \in \arg\min_{p \in \Delta} \langle Aq^*, p \rangle$$

$$q^* \in \arg\min_{q \in \Delta} \left\langle B' p^*, q \right\rangle$$

Δ is unit simplex

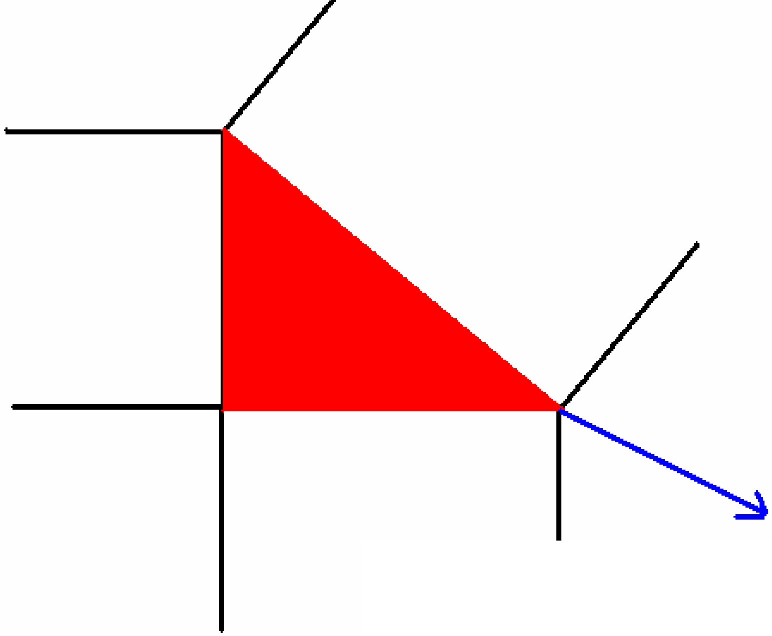
(LP) Optimality Conditions



$$\begin{aligned} -Aq^* &\in N_\Delta(p^*) \\ -B'p^* &\in N_\Delta(q^*) \end{aligned}$$

$$0 \in \begin{bmatrix} 0 & A \\ B' & 0 \end{bmatrix} \begin{bmatrix} p^* \\ q^* \end{bmatrix} + N_{\Delta \times \Delta}\left(\begin{bmatrix} p^* \\ q^* \end{bmatrix} \right)$$

$$0 \in Mz + N_C(z)$$



Normal Map

projection: $\pi_C(x)$

$$x - \pi_C(x) \in N_C(\pi_C(x))$$

Suppose $-M\pi_C(x) = x - \pi_C(x)$ then

$$-M\pi_C(x) \in N_C(\pi_C(x))$$

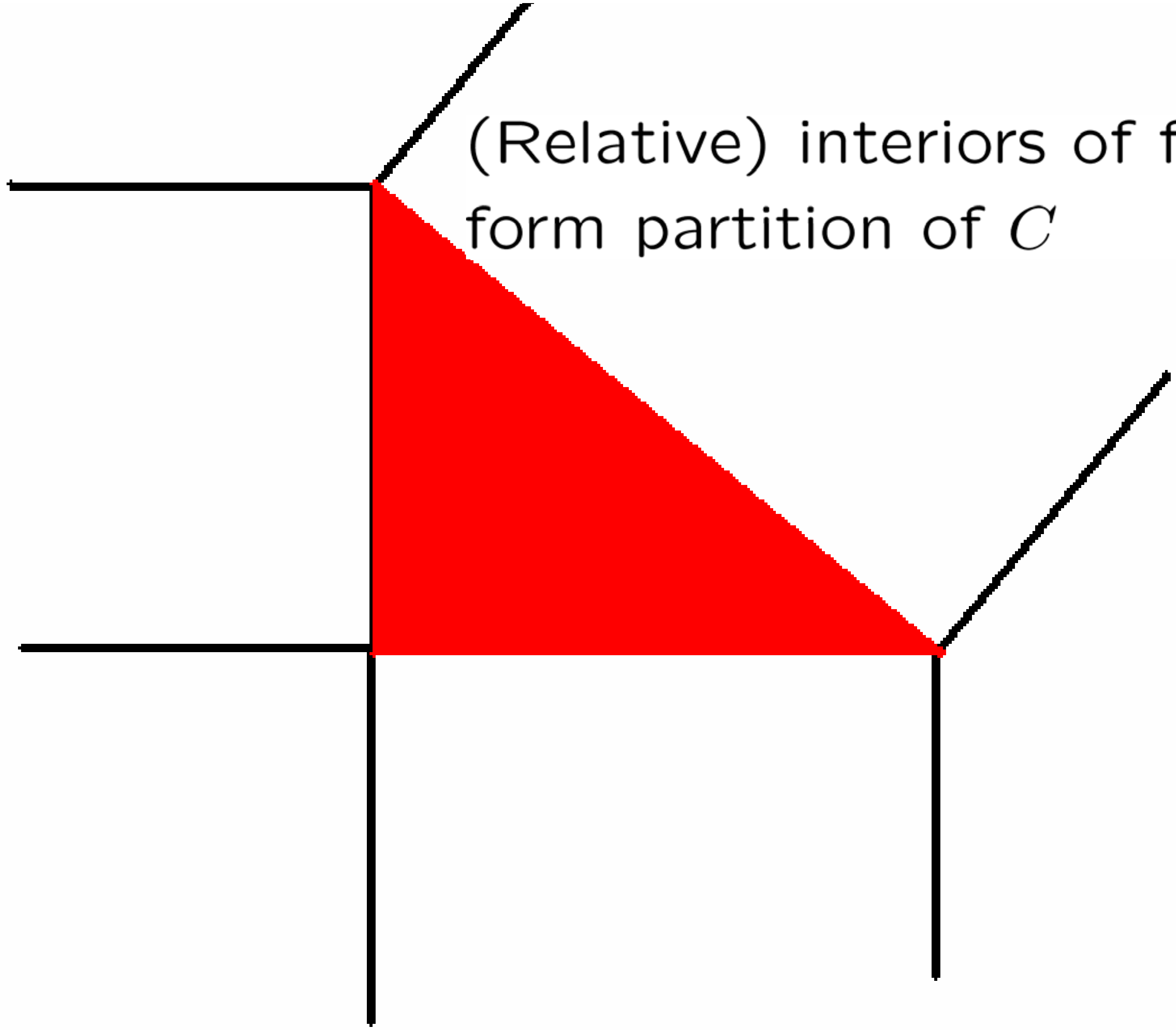
$z = \pi_C(x)$ solves our problem

Find x, a zero of the normal map:

$$0 = M\pi_C(x) + x - \pi_C(x)$$

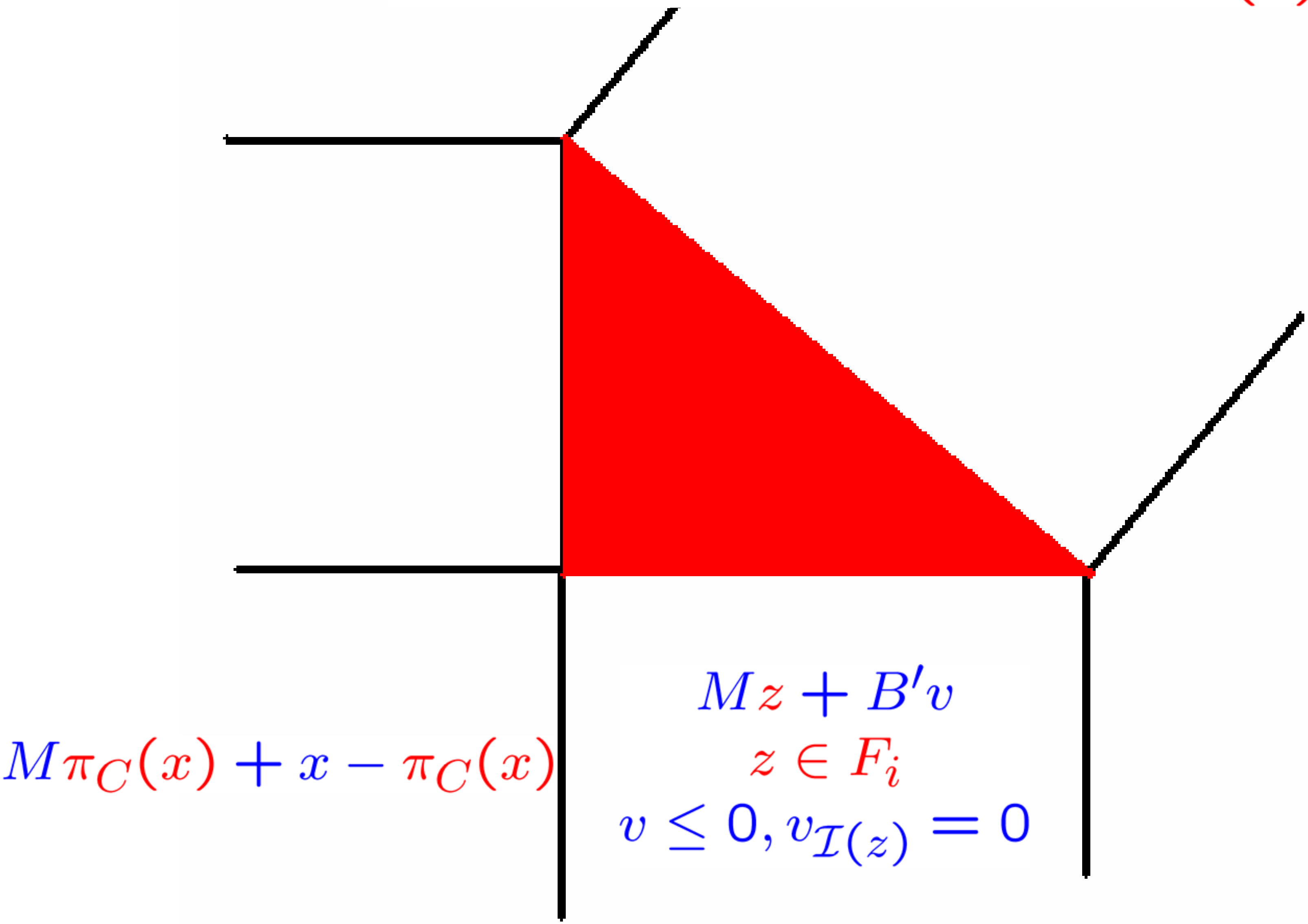
Normal manifold $= \{F_i + N_{F_i}\}$

(Relative) interiors of faces F_i form partition of C



$$C = \{z \mid Bz \geq b\}$$

$$N_C(z) = \{B'v \mid v \leq 0, v_{\mathcal{I}(z)} = 0\}$$



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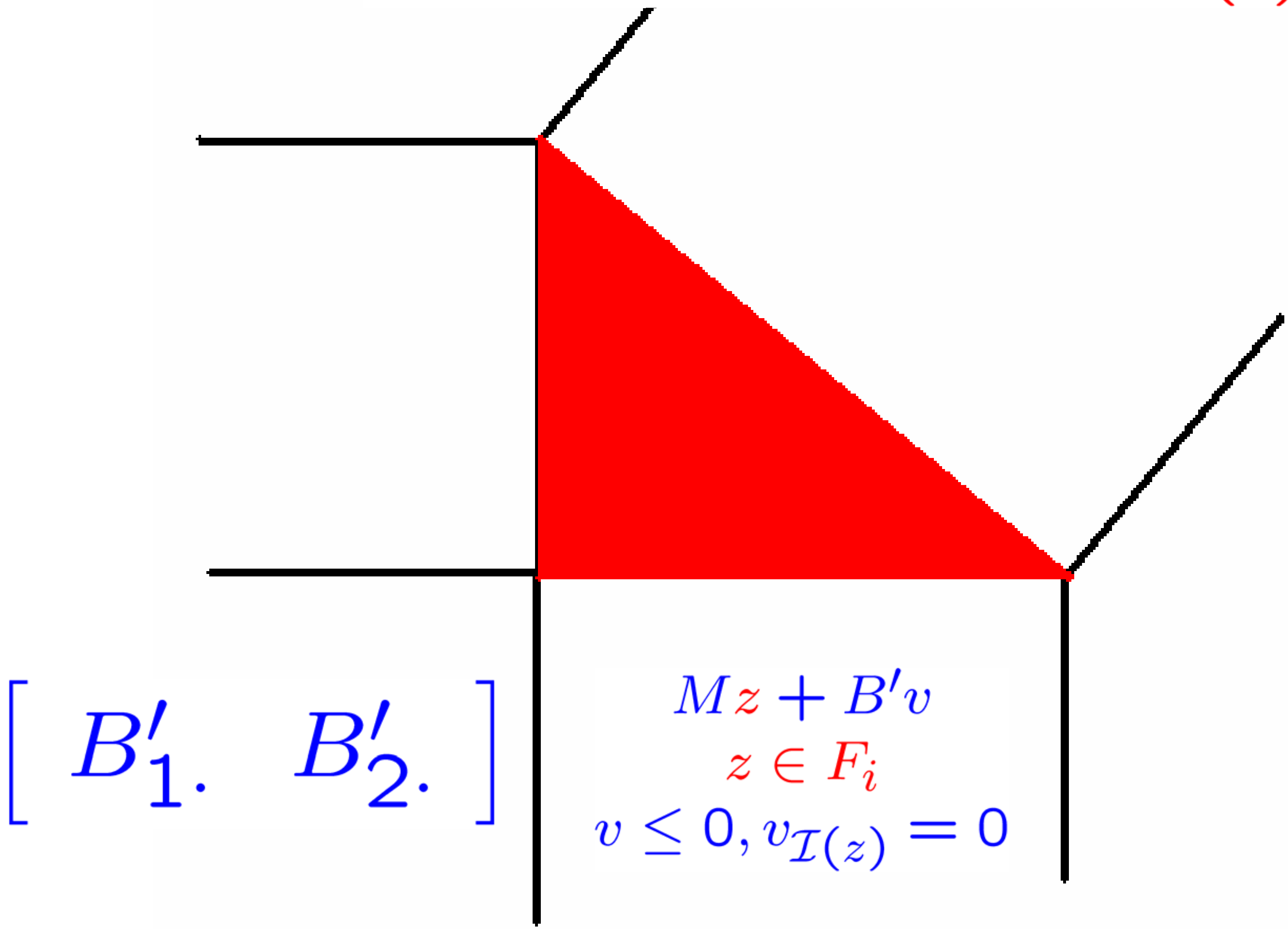

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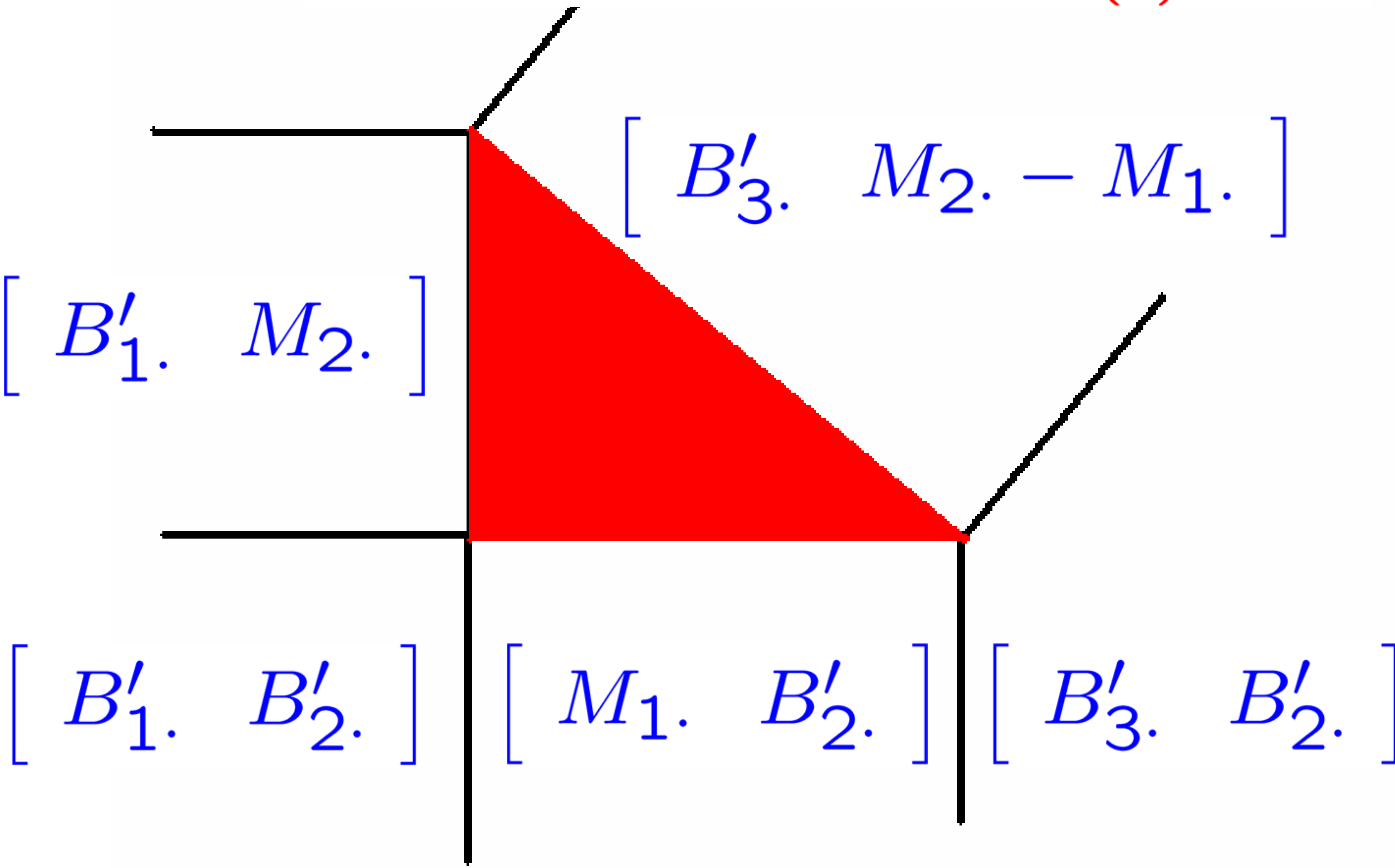
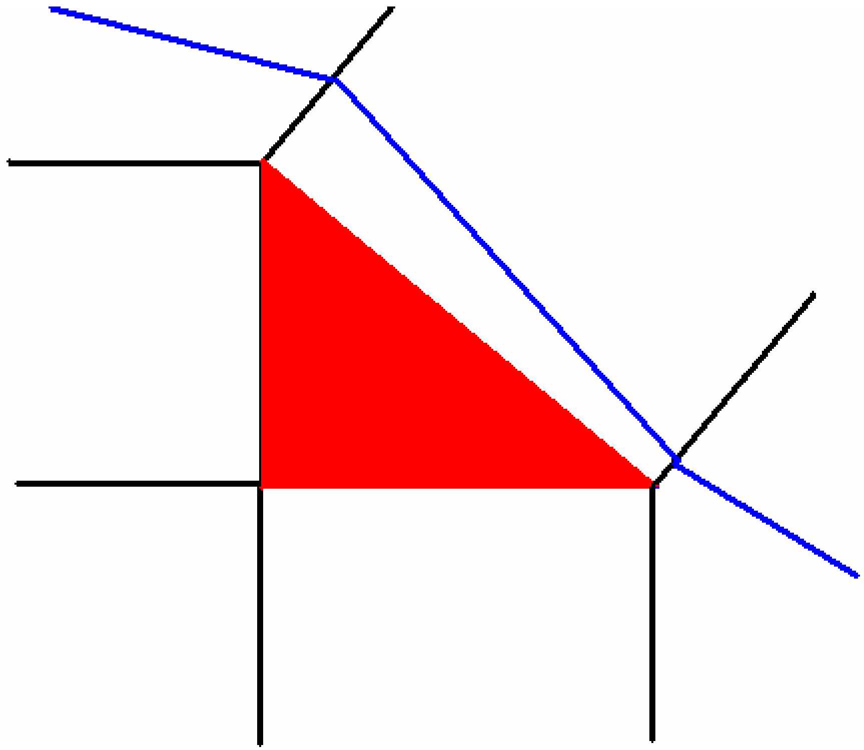


Cao/Ferris Path (Eaves)



- Start in cell that has interior (face is an extreme point)
- Move towards a zero of affine map in cell
- Update direction when hit boundary
- Solves or determines infeasible if M is copositive-plus on rec(C)
- Nails 2-person game

3-person game

Probabilities p, q, r

Loss matrices $A_{ijk}, B_{ijk}, C_{ijk}$

$$p^* \in \arg\min_{p \in \Delta} \sum A_{ijk} p_i q_j^* r_k^*$$
$$q^* \in \arg\min_{q \in \Delta} \sum B_{ijk} p_i^* q_j r_k^*$$
$$r^* \in \arg\min_{r \in \Delta} \sum C_{ijk} p_i^* q_j^* r_k$$

$$0 \in F(p^*, q^*, r^*) + N_{\Delta \times \Delta \times \Delta}(p^*, q^*, r^*)$$

Newton Method

- Linearize F and solve the AVI
  - Start path at previous Newton point
  - Globalize using generalized linesearch
- C = a box: essentially PATH solver - MCP
  - Underlying robust theory
  - Large scale linear algebra
  - Treat singularities/ill conditioning
  - Crash methods and preprocessing
- Special case: C = positive orthant – NCP
- Special case: C = whole space - $F(x) = 0$

Simple Dynamic Games

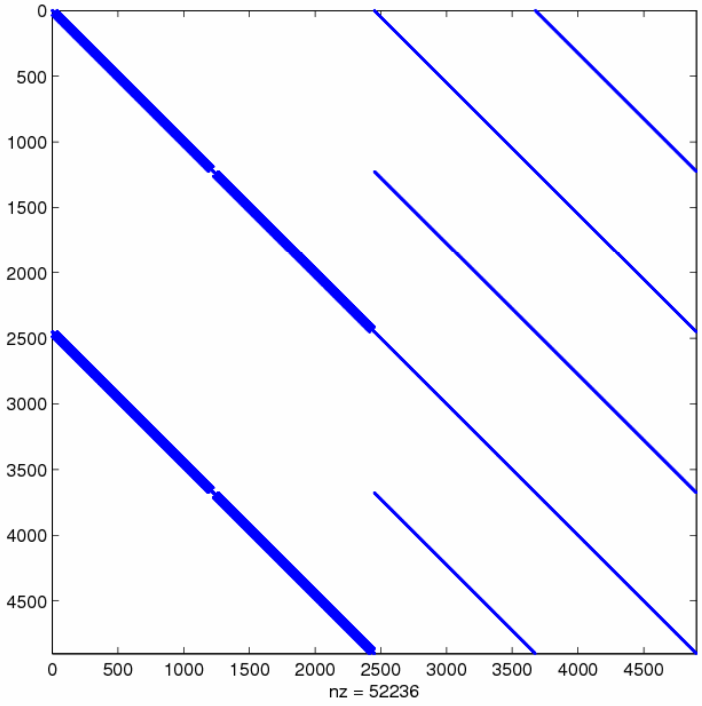
- Two players, infinite (discrete) time
- Prototype (Cournot) example on grid
  - (i,j) = players machines
- Investment and depreciation affects probabilities of changing state
- Variables are investment levels, quantities, prices; nonlinear F

Large scale issues

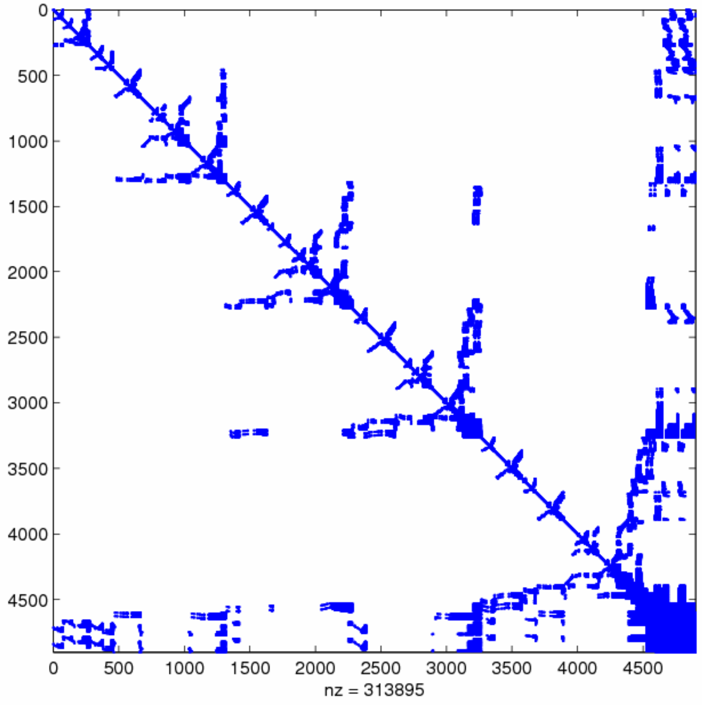
Problem			LUSOL	
n	dim	nnz	time	pct LU
20	1600	68171	0.418	77.0%
50	10000	587112	9.166	91.6%
100	40000	2773928	49.308	93.2%

Problem			UMFPACK	
n	dim	nnz	time	pct LU
20	1600	66684	0.218	56.4%
50	10000	658755	2.268	66.3%
100	40000	2778235	11.520	73.2%

Sparsity pattern



Jacobian



LU factors of Jacobian

Extended NLP in GAMS

$$\min_{x \in X} f_0(x) + \theta(f_1(x), \ldots, f_m(x))$$

$$\theta(u) = \sup_{y \in Y} \{y'u - k(y)\}$$

$$\mathcal{L}(x, y) = f_0(x) + \sum_{i=1}^{m} y_i f_i(x) - k(y)$$

$$x \in X, y \in Y$$

MCP formulation (FOC)

$$\begin{aligned} 0 &\in \nabla_x \mathcal{L}(x,y) &+N_X(x) \\ 0 &\in -\nabla_y \mathcal{L}(x,y) &+N_Y(y) \end{aligned}$$

```
model enlp /  gradLx.x,
              -gradLy.y /;
solve enlp using mcp;
```

Extensions:

```
model enlp /  PARTIAL(L,x).x,
              PARTIAL(-L,y).y /;
```

Extend X and Y beyond simple bounds

Other applications

- Option pricing (electricity market)
- Contact problems (with friction)
- Free boundary problems
- Optimal control (ELQP)
- Electronics, internet design
- Structure design
- Dynamic traffic assignment

Chemical Phase Equilibrium

$$f(\alpha) = \sum_i y_i - x_i$$
$$y_i = K_i x_i,\ x_i = \frac{z_i}{K_i\alpha+1-\alpha}$$

$$Vapor : f(\alpha) \geq 0,\ \alpha = 1$$
$$TwoPhase : f(\alpha) = 0,\ 0 \leq \alpha \leq 1$$
$$Liquid : f(\alpha) \leq 0,\ \alpha = 0$$

$$f(\alpha) \in N_{[0,1]}(\alpha)$$

Definition of MPEC (MPCC)

$$\begin{array}{ll} \min & f(x,y) \\ \text{s.t.} & g(x,y) \leq 0 \end{array}$$

Add parameterization to definition of F; parameter y

$$0 \in F(x,y) + N_{\mathbf{R}^n_+}(x)$$

Theory hard; no constraint qualification, specify in AMPL/GAMS

MPEC approaches

- Implicit: $\min f(x(y), y)$
- Auxiliary variables: $s = F(x,y)$
- NCP functions: $\Phi(s,x) = 0$
- Smoothing: $\Phi_\mu(s,x) = 0$
- Penalization: $\min f(x,y) + \mu \{s'x\}$
- Relaxation: $s'x <= \mu$
- Different problem classes require different solution techniques

Parametric algorithm NLPEC

- Reftype mult
- Aggregate none
- Constraint inequ
- Initmu = 0.01
- Numsolves = 5
- Updatefac = 0.1
- Finalmu = 0

$$NLP(\mu) : \min f(x,y)$$
$$g(x,y) \leq 0$$
$$s = F(x,y)$$
$$x, s \geq 0$$
$$s_i x_i \leq \mu,\ \forall i$$

Reformulate problem and set up sequence of solves

Grid Computing for MCP

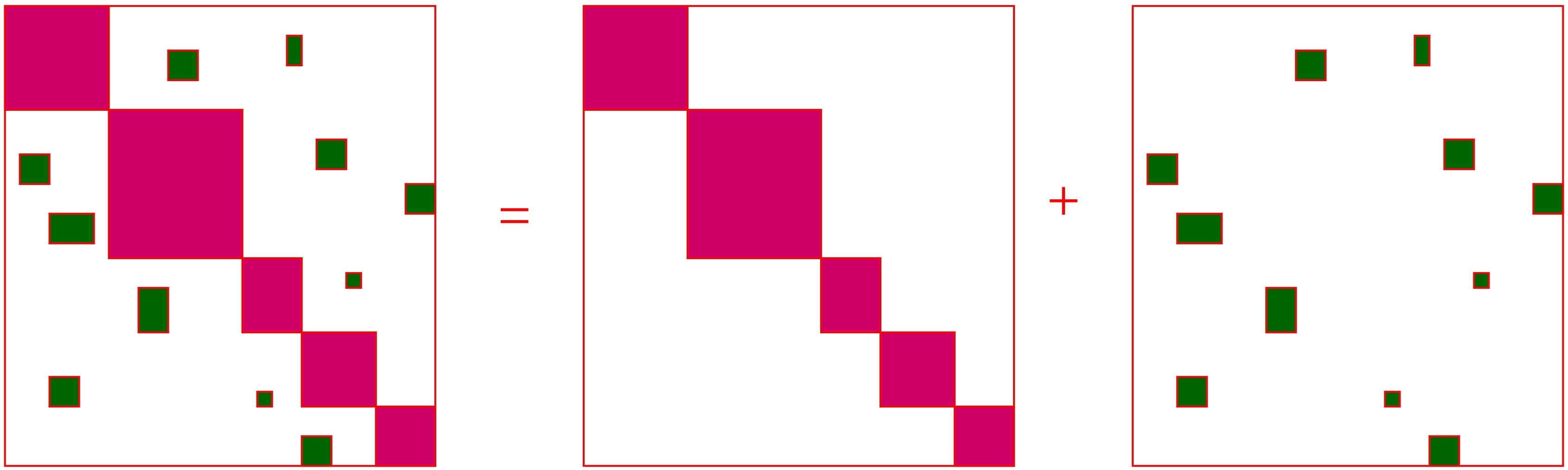
- Uses dedicated clusters and cycles from desktop workstations (> 1000 machines available for "ferris")
- Heterogeneous machines, with or without shared file system
- Machines updated regularly
- Condor is fault tolerant
- GAMS/Grid: available for download today
  - enlp.solvelink = 3

Use for grid computation

- Global Optimization
- Enhance speed (or size) of computation model
  - Linear algebra
    - May not have LU exploitable structure
  - Decomposition approaches
    - Benders, Dantzig-Wolfe, Lagrangian Duality
    - Jacobi, Gauss-Seidel

Trade/Policy Model (MCP)

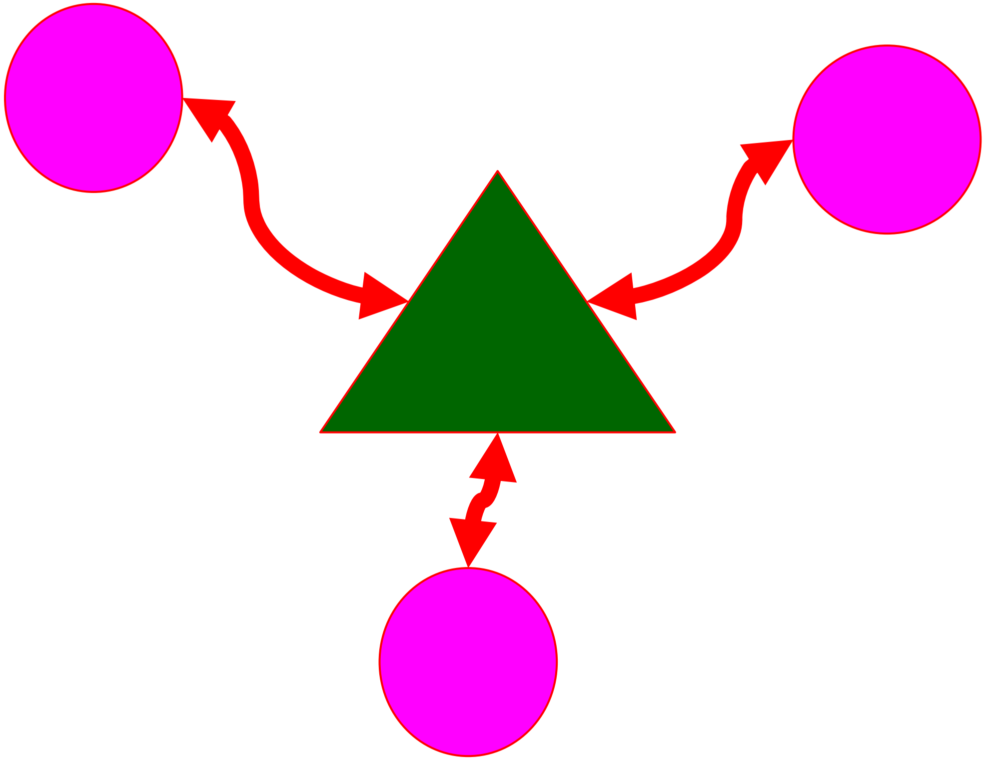
- Split model (18,000 vars) via region



- Gauss-Seidel, Jacobi, Asynchronous
- 87 regional subprobs, 592 solves

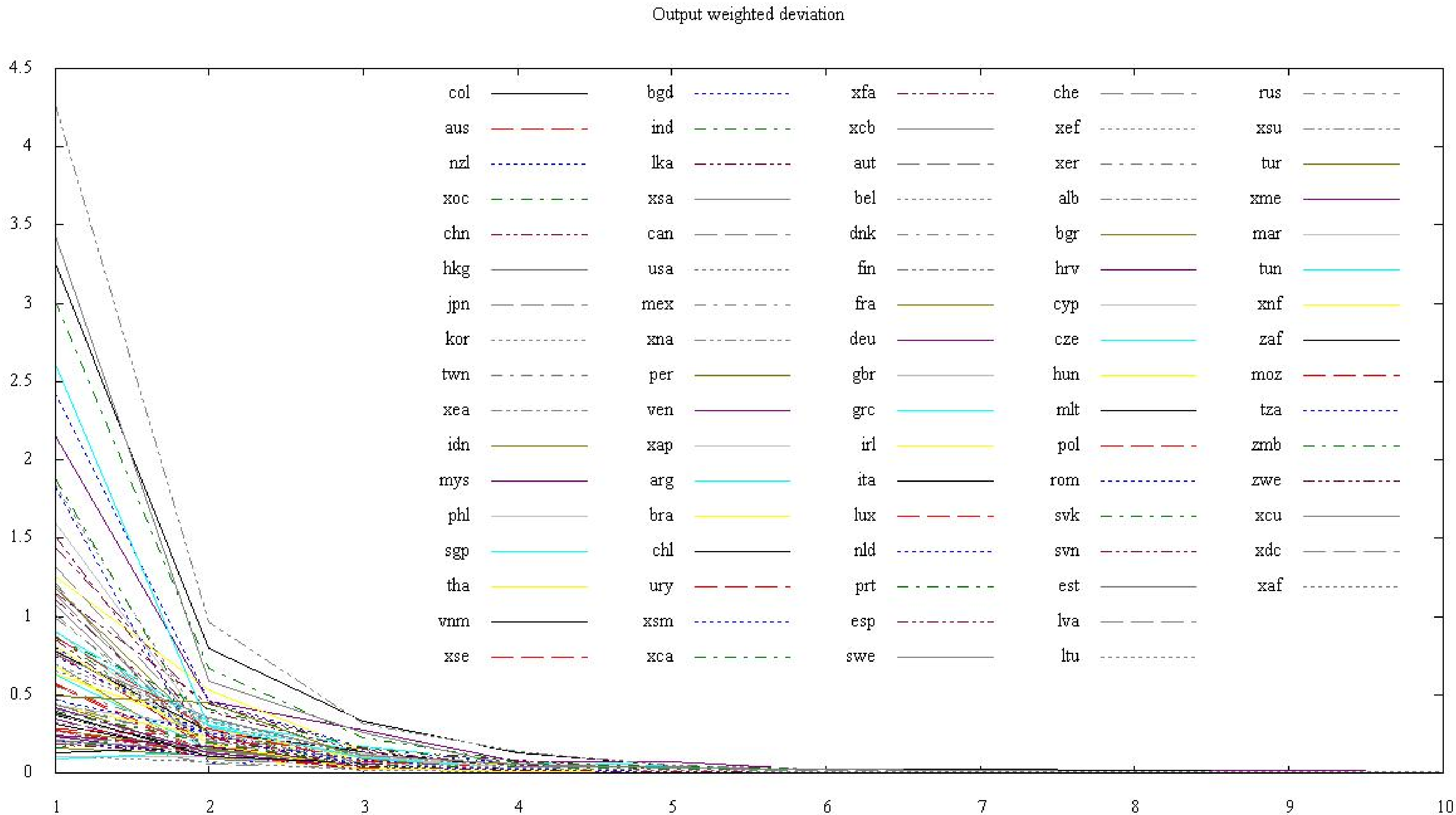
Model knowledge decomposition

- Pink model - open economy (regions)
- Green model - (partial) spatial equilibrium (commodities)
- Links are imports and exports



Calibrate supply and demand functions to points, and communicate functional forms, not points

Deviations by iteration



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Future Challenges

- MPEC/EPEC
  - theory and computation
- All solutions
  - Structure failure, Nash equilibria
- Large scale iterative solvers
  - Factors not available in RAM
- Complementarity Systems / Projected dynamical systems
- New application areas