Complementarity Problems and Applications

SIAM Optimization 2005

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Mixture of equalities and inequalities

$$0 \leq x \quad \perp \quad F(x) \geq 0$$

$$x' F(x) = 0$$

$$x_i F_i(x) = 0, \;\; i = 1, 2, \ldots, n$$

either $x_i = 0$ or $F_i(x) = 0$

Historical events (omitted)

- KKT conditions / Complementary slackness
- Lemke's method (bimatrix / Nash games, selfish routing, electricity pricing)
- Triangulation/homotopy
- Complexity: Lemke exponential, NP-Hard, specializations polynomial - interior points
  - Murty; Cottle, Pang and Stone;
  - Facchinei and Pang

Aims of talk

- Interplay between nonsmoothness and complementarity
- Modeling perspective: what do complementarity problems add?
- Explicit examples from economics and engineering

Thanks to...

- Todd Munson, Steve Dirkse
- Jong-Shi Pang
- Danny Ralph, Christian Kanzow

- Joe Burke, Jeff Renfro, Vincent Acary
- Andy Philpott, Jarrad Wallace, Qian Li

- Alex Meeraus,Tom Rutherford
- David Gay, Bob Fourer

Equivalent Nonsmooth Map

$$0 = F(x_+) + x - x_+$$

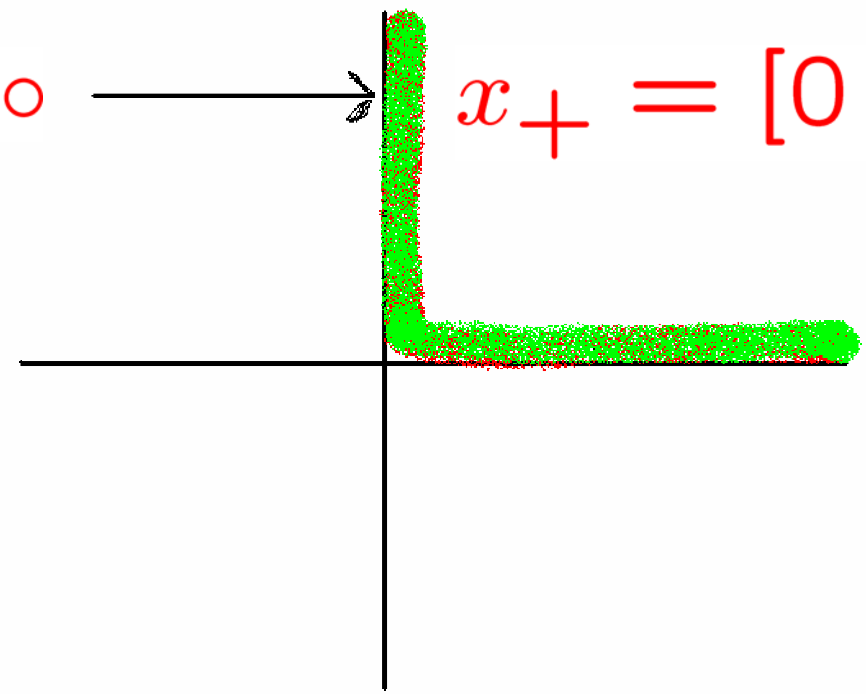
$$0 \leq x_+, \; F(x_+) = x_+ - x \geq 0$$

$$x'_+ F(x_+) = x'_+(x_+ - x) = 0$$

$$0 = F(\pi_C(x)) + x - \pi_C(x)$$

Normal Manifold (I)

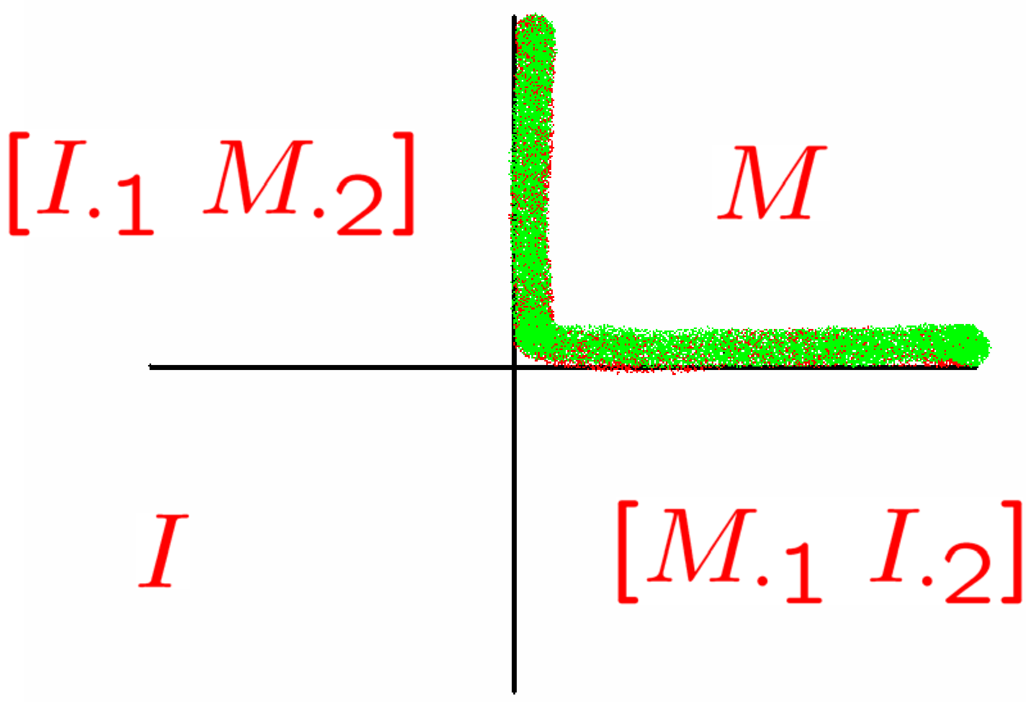
$[-1\ 1] = x \circ$



$x_+ = [0\ 1]$

$$Mx_+ + q + x - x_+$$

Normal Manifold (II)



$$Mx_+ + q + x - x_+$$

The PATH Solver

- PATH: Newton method based on nonsmooth Normal map

$$F(x_+) + x - x_+$$

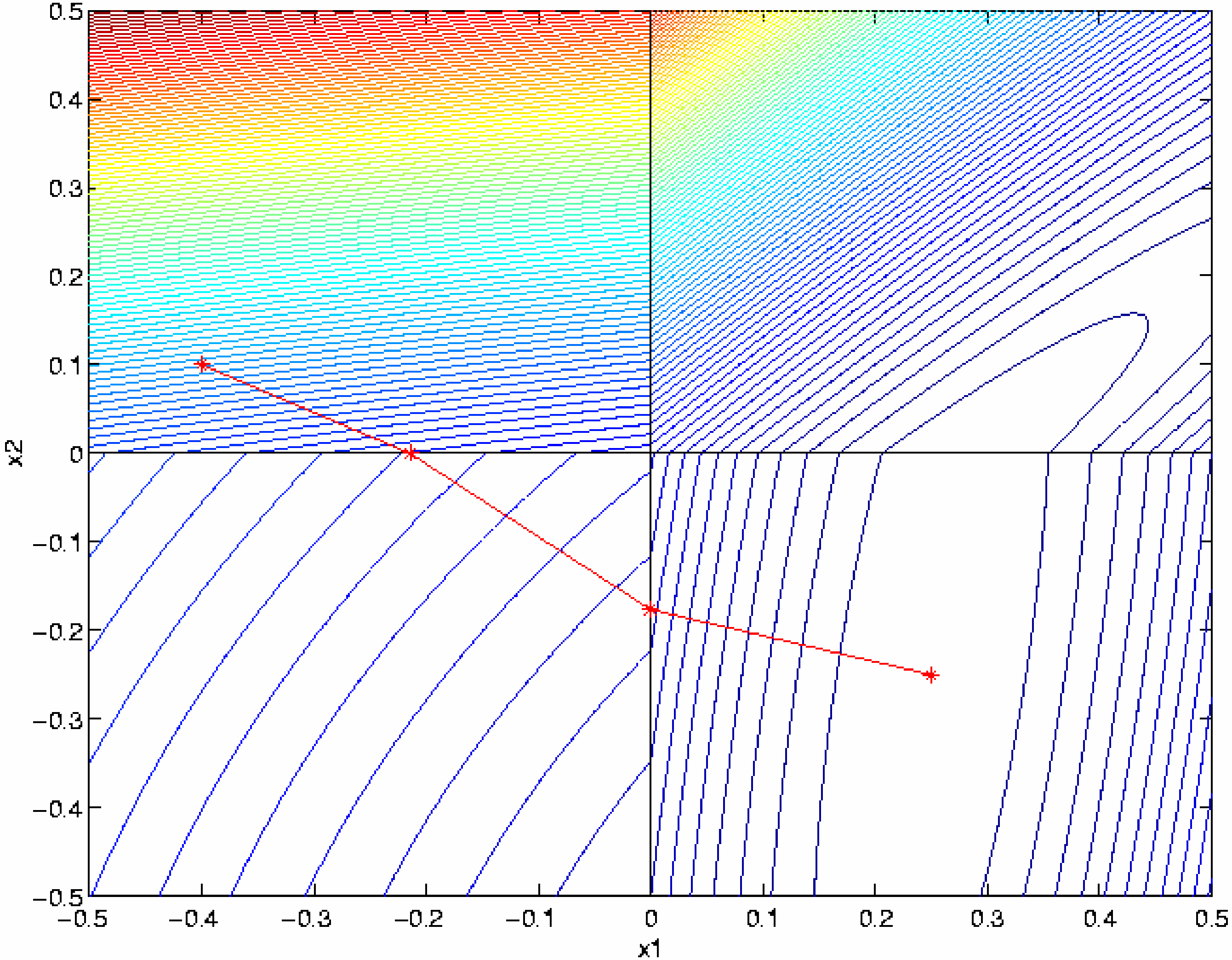
- Newton point is solution of piecewise linearization

$$F(x_+^k) + \nabla F(x_+^k)(x_+ - x_+^k) + x - x_+ = 0$$

- Uses more general projection:

$$x_+ \Leftarrow \pi_{\mathcal{B}}(x)$$

The "Newton" Step



Key solver features

- Underlying robust theory
- Large scale linear algebra
- Ease of model generation/checking
- Globalization and merit functions
- Treat singularities/ill conditioning
- Crash methods and preprocessing

- Alternative: Semismooth based Newton approaches

Market equilibrium (I)

$$Supply \quad / Production:$$

min $\quad \frac{1}{2}x'Qx + c'x$

s.t. $\quad Ax \geq b$

$$Bx \geq r^*$$

$$x \geq 0$$

$$\begin{bmatrix} Q & -A' & -B' \\ A & & \\ B & & \end{bmatrix} \begin{bmatrix} x \\ v \\ w \end{bmatrix} + \begin{bmatrix} c \\ -b \\ -r^* \end{bmatrix}$$

Market equilibrium (II)

$$\begin{bmatrix} Q & -A' & -B' \\ A & & \\ B & & \end{bmatrix} \begin{bmatrix} x \\ v \\ w \end{bmatrix} + \begin{bmatrix} c \\ -b \\ -r^* \end{bmatrix}$$

$$Demand : r^* = q(p) = Dp + d$$

$$Equilibrium : w = p$$

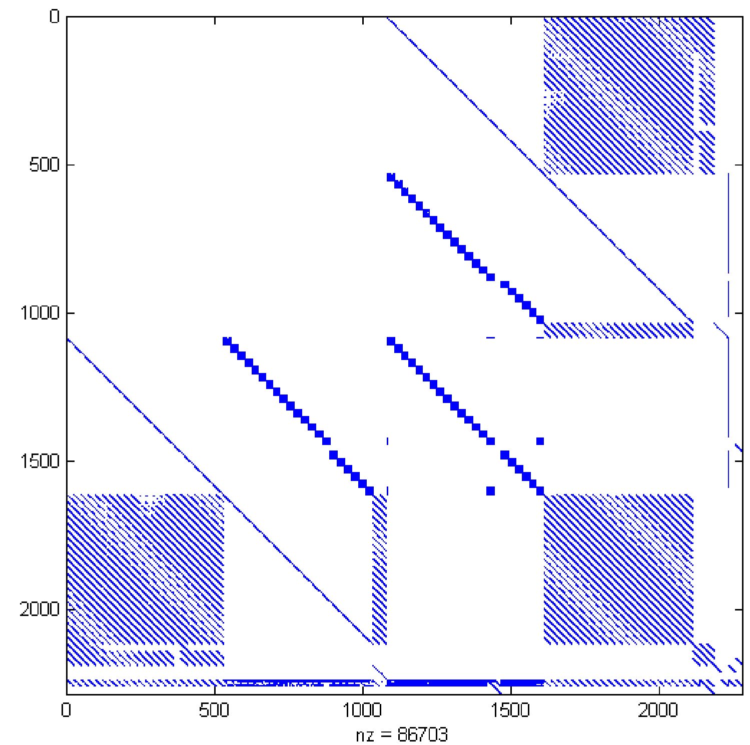
Market equilibrium (III)

$$\begin{bmatrix} Q & -A' & -B' \\ A & & \\ B & & -D \end{bmatrix} \begin{bmatrix} x \\ v \\ p \end{bmatrix} + \begin{bmatrix} c \\ -b \\ -d \end{bmatrix}$$

$$0 \leq \begin{bmatrix} \nabla \mathcal{L}(x, u, p) \\ \tilde{g}_t(x) \\ \tilde{h}_t(x) \quad -D(p) \end{bmatrix} \perp \begin{bmatrix} x \\ u \\ p \end{bmatrix} \geq 0$$

Application: Uruguay Round

- World Bank Project with Harrison and Rutherford
- 24 regions, 22 commodities
  - 2200 x 2200 (nonlinear)
- Short term gains $53 billion p.a.
  - Much smaller than previous literature
- Long term gains $188 billion p.a.
  - Number of less developed countries loose in short term
- Unpopular conclusions – forced concessions by World Bank



Pizza Cheese

- MPC (milk protein concentrate) outside of quota restrictions
- Not allowed by law for use in cheese
- "Innovate" new MPC for use in new product: Pizza Cheese
- Model determines relative prices, and explains huge increase in MPC imports

Definition of MPEC (MPCC)

$$\begin{array}{ll} \text{min} & f(x,y) \\ \text{s.t.} & g(x,y) \le 0 \end{array}$$

Add parameterization to definition of F; parameter y

$$0 \le x \perp F(x,y) \ge 0$$

Theory hard; no constraint qualification, specify in AMPL/GAMS

NCP functions

Definition: $\phi(a,b) = 0 \Leftrightarrow 0 \le a \perp b \ge 0$

Example: $\phi_{\text{min}}(a,b) := \min(a,b)$

Example: $\phi_{FB}(a,b) := \sqrt{a^2 + b^2} - a - b$

Componentwise: $\Phi_i(x) := \phi(x_i, F_i(x))$

$$\Phi(x) = 0 \iff 0 \le x \perp F(x) \ge 0$$

Aside: (semismooth) fact

$\Psi(x) := \Phi(x)'\Phi(x)$ cont. diffble.

MPEC approaches

- Implicit: $\min f(x(y),y)$
- Auxiliary variables: $s = F(x,y)$
- NCP functions: $\Phi(s,x) = 0$
- Smoothing: $\Phi_\mu(s,x) = 0$
- Penalization: $\min f(x,y) + \mu \{s'x\}$
- Relaxation: $s'x <= \mu$
- Different problem classes require different solution techniques

Parametric algorithm NLPEC

- Reftype = FB
- Initmu = 0.01
- Numsolves = 5
- Updatefac = 0.1
- Finalmu = 0
- Slack = positive

$$NLP(\mu): \min f(x,y)$$
$$g(x,y) \leq 0$$
$$s = F(x,y)$$
$$x, s \geq 0$$
$$\phi_\mu(s_i, x_i) = 0$$

Reformulate problem and set up sequence of solves

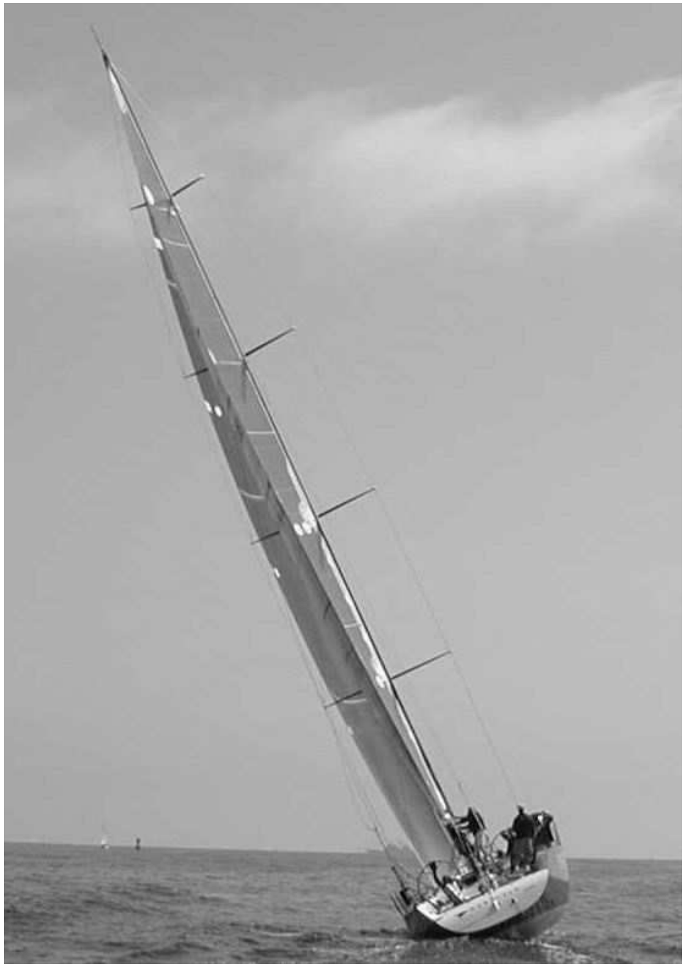
Running NLPEC

- Create the GAMS model as an MPEC
- Setup nlpec.opt
- Gams modelfile mpec=nlpec optfile=1

- Reformulated automagically
- Results returned directly to GAMS
- Modeler tip: use "convert" to get AMPL
- Modeler tip: use "kestrel" to solve GAMS models using AMPL linked solvers

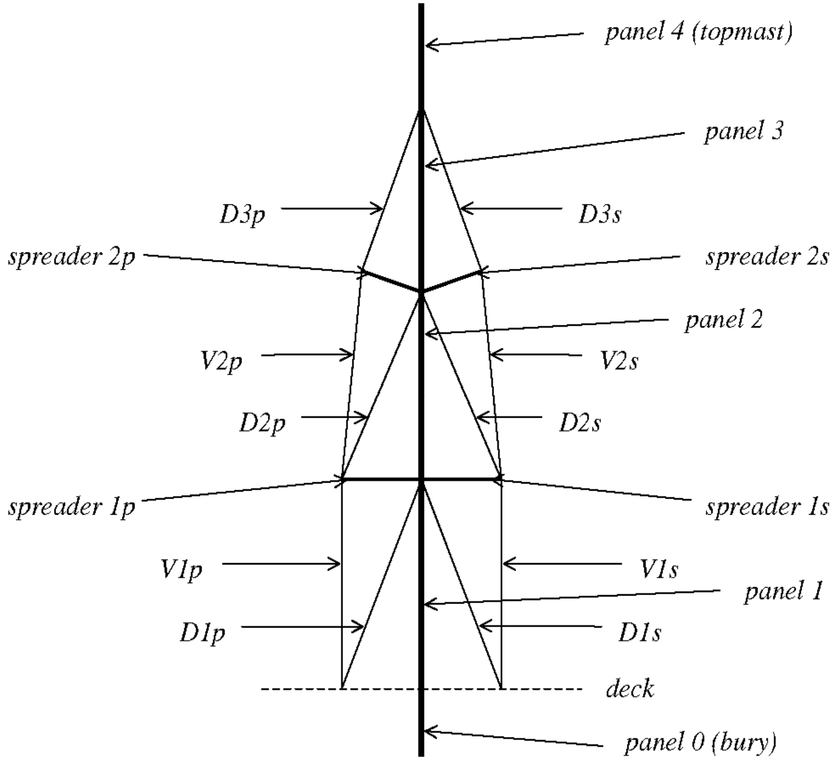
Optimal Yacht Rig Design

- Current mast design trends use a large primary spar that is supported laterally by a set of tension and compression members, generally termed the rig
- Complementarity determines member loads and deflections for given geometry and design variables
- Reduction in either the weight of the rig or the height of the VCG will improve performance



Complementarity Feature

- Stays are tension-only members (in practice) - Hookes Law
- When tensile load becomes zero, the stay goes slack (low material stiffness)



$$0 \geq s \perp s - k * dl \leq 0$$

s: axial load
k: member spring constant
dl: member length extension

MPEC extension for design

- TransPac 52' (TP52)
- Optimal rig design minimum weight problem using NLPEC
- One/two-spreader rig
- NLP starting value is a solution from CP
- Optimal val = 10.0873

Benefits/Drawbacks

- Easy to adapt existing models
- Large-scale potential
- Customizable solution to problem
- Available within GAMS right now
- Models hard to solve
- Local solutions found
- Alternatives: Filter, LOQO, KNITRO

What can we model via CP?

$$\min(G(x), H(x)) \leq y$$

$$\min(F^1(x), F^2(x), \ldots, F^m(x)) = 0$$

$$\text{kth-largest}(F^1(x), F^2(x), \ldots, F^m(x)) = 0$$

Switch off: $g(x)h(x) \leq 0, \ h(x) \geq 0$

Variational Inequality: $VI(F, C)\}$

Chemical Phase Equilibrium

$$f(\alpha) = \sum_i y_i - x_i$$
$$y_i = K_i x_i,\ x_i = \frac{z_i}{K_i\alpha + 1 - \alpha}$$

$$Vapor : f(\alpha) \geq 0,\ \alpha = 1$$
$$TwoPhase : f(\alpha) = 0,\ 0 \leq \alpha \leq 1$$
$$Liquid : f(\alpha) \leq 0,\ \alpha = 0$$

$$median\{\alpha, \alpha - 1, -f(\alpha)\} = 0$$

Other applications

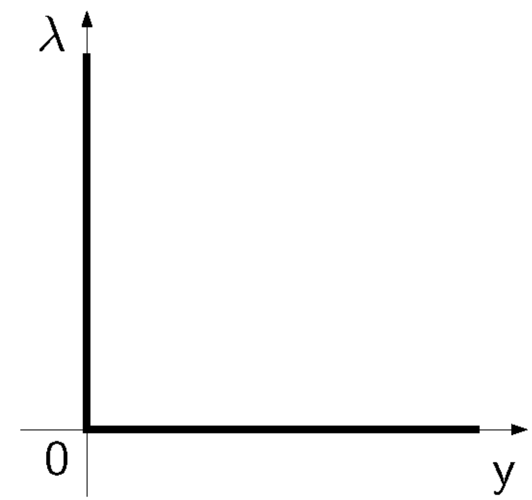
- Option pricing (electricity market)
- Contact problems (with friction)
- Free boundary problems
- Optimal control (ELQP)
- Earthquake propogation
- Structure design
- Dynamic traffic assignment

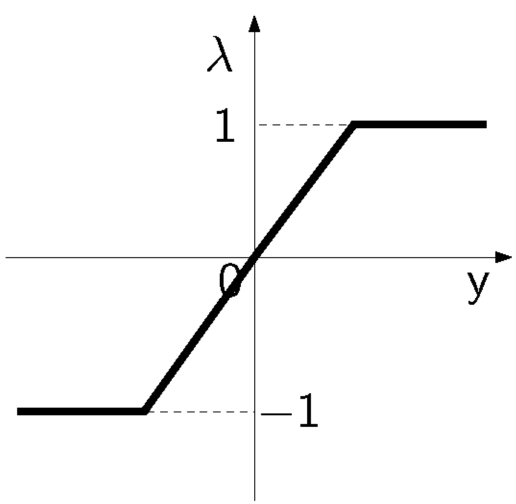
Complementarity Systems

$$\frac{dx}{dt}(t) = f(x(t), \lambda(t))$$

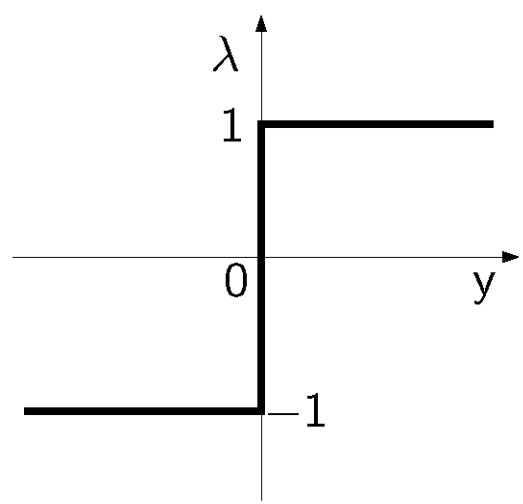
$$y(t) = h(x(t), \lambda(t))$$

$$0 \leq y(t) \perp \lambda(t) \geq 0$$

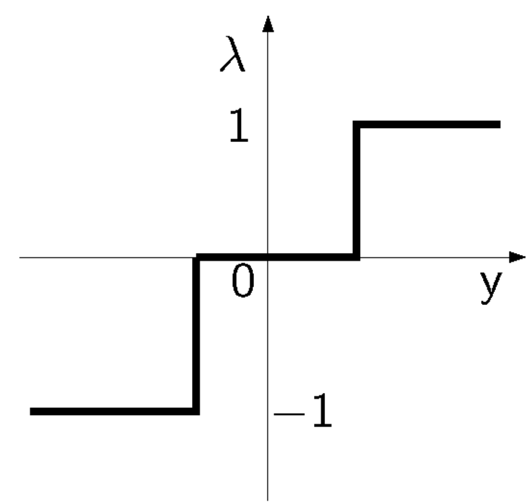




saturation



Relay



Relay with dead zone

Future Challenges

- MPEC/EPEC
  - theory and computation
- All solutions
  - Structure failure, Nash equilibria
- Large scale iterative solvers
  - Factors not available in RAM
- Complementarity Systems / Projected dynamical systems
- New application areas

Solver/Example Availability

- Student version downloadable (full license downloadable yearly)
- AMPL/GAMS (also MILES, NLPEC)
- Matlab, Callable library, NEOS

- MCPLIB
- GAMSWORLD