MOPEC: Multiple Optimization Problems with Equilibrium Constraints

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The PIES Model (Hogan)

$$\begin{aligned} \min_x \quad & c^T x & \text{cost} \\ \text{s.t.} \quad & Ax = d(p) & \text{balance} \\ & Bx = b & \text{technical constr} \\ & x \geq 0 \end{aligned}$$

- Issue is that p is the multiplier on the "balance" constraint of LP
- Extended Mathematical Programming (EMP) facilitates annotations of models to describe additional structure
- Can solve the problem by writing down the KKT conditions of this LP, forming an LCP and exposing p to the model
- EMP does this automatically from the annotations

Reformulation details

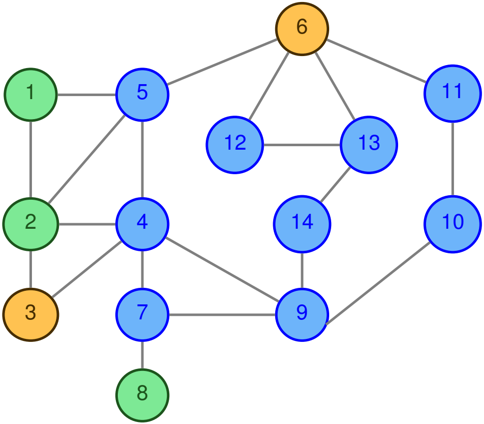
$$\begin{array}{lcl} 0 = Ax - d(p) & \perp & \mu \\ 0 = Bx - b & \perp & \lambda \\ 0 \le -A^T\mu - B^T\lambda + c & \perp & x \ge 0 \end{array}$$

- empinfo: dualvar p balance
- replaces $\mu \equiv p$
- LCP/MCP is then solvable using PATH

$$z = \begin{bmatrix} p \\ \lambda \\ x \end{bmatrix}, \quad F(z) = \begin{bmatrix} & & A \\ & & B \\ -A^T & -B^T & \end{bmatrix} \begin{bmatrix} p \\ \lambda \\ x \end{bmatrix} + \begin{bmatrix} -d(p) \\ -b \\ c \end{bmatrix}$$

Transmission Line Expansion Model (F./Tang)

$$\min_{x \in X} \sum_{\omega} \pi_{\omega} \sum_{i \in N} d_i^{\omega} p_i^{\omega}(x)$$



- Nonlinear system to describe power flows over (large) network
- Multiple time scales
- Dynamics (bidding, failures, ramping, etc)
- Uncertainty (demand, weather, expansion, etc)
- $p_i^{\omega}(x)$: Price (LMP) at i in scenario ω as a function of x
- Use other models to construct approximation of $p_i^{\omega}(x)$

Solution approach

- Use derivative free method for the upper level problem (1)
- Requires $p_i^\omega(x)$
- Construct these as multipliers on demand equation (per scenario) in an Economic Dispatch (market clearing) model
- But transmission line capacity expansion typically leads to generator expansion, which interacts directly with market clearing
- Interface blue and black models using Nash Equilibria (as EMP):

empinfo: equilibrium
forall f: min expcost(f) y(f) budget(f)
forall ω: min scencost(ω) q(ω) ...

Generator Expansion (2): $\forall f \in F$:

$$\min_{y_f} \sum_{\omega} \pi_\omega \sum_{j \in G_f} C_j(y_j, q_j^\omega) - r(h_f - \sum_{j \in G_f} y_j)$$

$$\text{s.t.} \sum_{j \in G_f} y_j \leq h_f, y_f \geq 0$$

G_f: Generators of firm $f \in F$
y_j: Investment in generator j
q_j^ω: Power generated at bus j in scenario ω
C_j: Cost function for generator j
r: Interest rate

Market Clearing Model (3): $\forall \omega$:

$$\min_{z,\theta,q^\omega} \sum_f \sum_{j \in G_f} C_j(y_j, q_j^\omega) \quad \text{s.t.}$$

$$q_j^\omega - \sum_{i \in I(j)} z_{ij} = d_j^\omega \quad \forall j \in N (\perp p_j^\omega)$$

$$z_{ij} = \Omega_{ij}(\theta_i - \theta_j) \quad \forall (i,j) \in A$$

$$-b_{ij}(x) \leq z_{ij} \leq b_{ij}(x) \quad \forall (i,j) \in A$$

$$\underline{u}_j(y_j) \leq q_j^\omega \leq \overline{u}_j(y_j)$$

z_{ij}: Real power flowing along line ij
q_j^ω: Real power generated at bus j in scenario ω
θ_i: Voltage phase angle at bus i
Ω_{ij}: Susceptance of line ij
$b_{ij}(x)$: Line capacity as a function of x
$\underline{u}_j(y)$, $\overline{u}_j(y)$: Generator j limits as a function of y

MOPEC

$$\min_{x_i} \theta_i(x_i, x_{-i}, y) \text{ s.t. } g_i(x_i, x_{-i}, y) \leq 0, \forall i$$

```
equilibrium
min theta(1) x(1) g(1)
...
min theta(m) x(m) g(m)
```

is solved in a Nash manner

- Allows multipliers from one problem to be used in another problems

  ```
  dualvar p g(1)
  ```

Feasibility

$$\text{KKT of } \min_{y_f \in Y} \sum_{\omega} \pi_\omega \sum_{j \in G_f} C_j(y_j, q_j^\omega) - r(h_f - \sum_{j \in G_f} y_j) \quad \forall f \in F \quad (2)$$

$$\text{KKT of } \min_{(z,\theta,q^\omega) \in Z(x,y)} \sum_{f} \sum_{j \in G_f} C_j(y_j, q_j^\omega) \quad \forall \omega \quad (3)$$

- Models (2) and (3) form a complementarity problem (CP via EMP)
- Solve (3) as NLP using global solver (actual $C_j(y_j, q_j^\omega)$ are not convex), per scenario (SNLP) this provides starting point for CP
- Solve (KKT(2) + KKT(3)) using EMP and PATH, then repeat
- Identifies CP solution whose components solve the scenario NLP's (3) to global optimality

Scenario	ω_1	ω_2
Probability	0.5	0.5
Demand Multiplier	8	5.5

SNLP (1):

Scenario	q_1	q_2	q_3	q_6	q_8
ω_1	3.05	4.25	3.93	4.34	3.39
ω_2		4.41	4.07	4.55	

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SNLP (1):

Scenario	q_1	q_2	q_3	q_6	q_8
ω_1	3.05	4.25	3.93	4.34	3.39
ω_2		4.41	4.07	4.55	

EMP (1):

Scenario	q_1	q_2	q_3	q_6	q_8
ω_1	2.86	4.60	4.00	4.12	3.38
ω_2		4.70	4.09	4.24	

Firm	y_1	y_2	y_3	y_6	y_8
f_1	167.83	565.31			266.86
f_2			292.11	207.89	

Scenario	ω_1	ω_2
Probability	0.5	0.5
Demand Multiplier	8	5.5

SNLP (2):

Scenario	q_1	q_2	q_3	q_6	q_8
ω_1	0.00	5.35	4.66	5.04	3.91
ω_2		4.70	4.09	4.24	

Scenario	ω_1	ω_2
Probability	0.5	0.5
Demand Multiplier	8	5.5

SNLP (2):

Scenario	q_1	q_2	q_3	q_6	q_8
ω_1	0.00	5.35	4.66	5.04	3.91
ω_2		4.70	4.09	4.24	

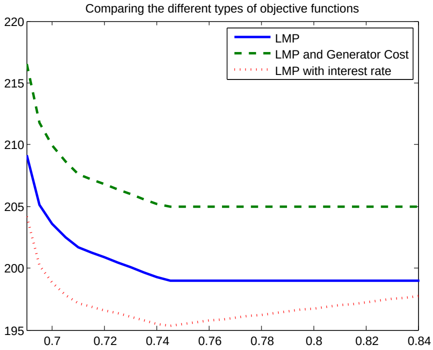
EMP (2):

Scenario	q_1	q_2	q_3	q_6	q_8
ω_1	0.00	5.34	4.62	5.01	3.99
ω_2		4.71	4.07	4.25	

Firm	y_1	y_2	y_3	y_6	y_8
f_1	0.00	622.02			377.98
f_2			283.22	216.79	

Observations

- But this is simply one function evaluation for the outer "transmission capacity expansion" problem
- Number of critical arcs typically very small
- But in this case, p_j^ω are very volatile
- Outer problem is small scale, objectives are open to debate, possibly ill conditioned



- Economic dispatch should use AC power flow model
- Structure of market open to debate
- Types of "generator expansion" also subject to debate
- Suite of tools is very effective in such situations

EMP: variational inequalities

Allows (GAMS) models to be manipulated to form other problems of interest via a simple EMP info file:

- VI(f, C):

$$0 \in f(x) + N_C(x)$$

  ```
  vi f x cons
  ```

  generates a variational inequality where C defined by 'cons'
- Either generates the equivalent complementarity (KKT) problem, or provides problem structure for algorithmic exploitation
- Extension of (square) nonlinear systems and mixed complementarity problems
- QVI can be specified in the same manner

MOPEC

$$\min_{x_i} \theta_i(x_i, x_{-i}, p) \text{ s.t. } g_i(x_i, x_{-i}, p) \leq 0, \forall i$$

and

$$p \text{ solves VI}(h(x, \cdot), C)$$

```
equilibrium
min theta(1) x(1) g(1)
...
min theta(m) x(m) g(m)
vi h p cons
```

is solved in a Nash manner

MOPEC

$$\min_{x_i} \theta_i(x_i, x_{-i}, p) \text{ s.t. } g_i(x_i, x_{-i}, p) \leq 0, \forall i$$

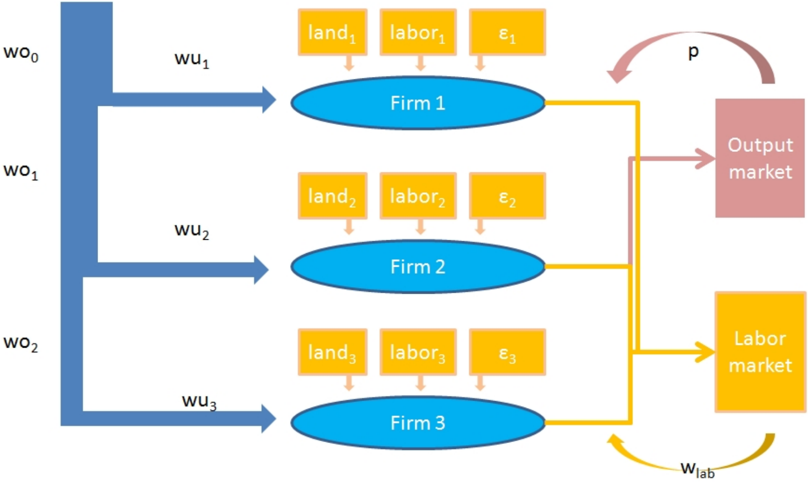
and

$$h(x, p) = 0$$

```
equilibrium
min theta(1) x(1) g(1)
...
min theta(m) x(m) g(m)
vi h p cons
```

is solved in a Nash manner

Water rights pricing (Britz/F./Kuhn)



The model AO

$$\max_{q_i, x_i, wo_i \geq 0} \sum_i \left(q_i \cdot p - \sum_{f \in \{int, lab\}} x_{i,f} \cdot w_f \right)$$

$$\text{s.t.} \quad q_i \leq \prod_f (x_{i,f} + e_{i,f})^{\epsilon_{i,f}}$$

$$x_{i,land} \leq e_{i,land}$$

$$wo_{i-1} = x_{i,wat} + wo_i$$

$$0 \leq \sum_i q_i - d(p) \quad \perp \quad p \geq 0$$

$$0 \leq \sum_i e_{i,lab} - \sum_i x_{i,lab} \quad \perp \quad w_{lab} \geq 0$$

The model IO

$$\max_{q_i, x_i, wo_i \geq 0} \quad \left(q_i \cdot p - \sum_f x_{i,f} \cdot w_f \right)$$

$$\text{s.t.} \quad q_i \leq \prod_f \left(x_{i,f} + e_{i,f} \right)^{\epsilon_{i,f}}$$

$$x_{i,land} \leq e_{i,land}$$

$$wo_{i-1} = x_{i,wat} + wo_i$$

$$0 \leq \sum_i q_i - d(p) \quad \perp \quad p \geq 0$$

$$0 \leq \sum_i e_{i,lab} - \sum_i x_{i,lab} \quad \perp \quad w_{lab} \geq 0$$

The model IO

$$\max_{q_i, x_i, wo_i, wr_i^b, wr_i^s \geq 0} \left(q_i \cdot p - \sum_f x_{i,f} \cdot w_f - wr_i^b \cdot (w_{wr} + \tau) + wr_i^s \cdot w_{wr} \right)$$

$$\text{s.t.} \quad q_i \leq \prod_f (x_{i,f} + e_{i,f})^{\epsilon_{i,f}}$$

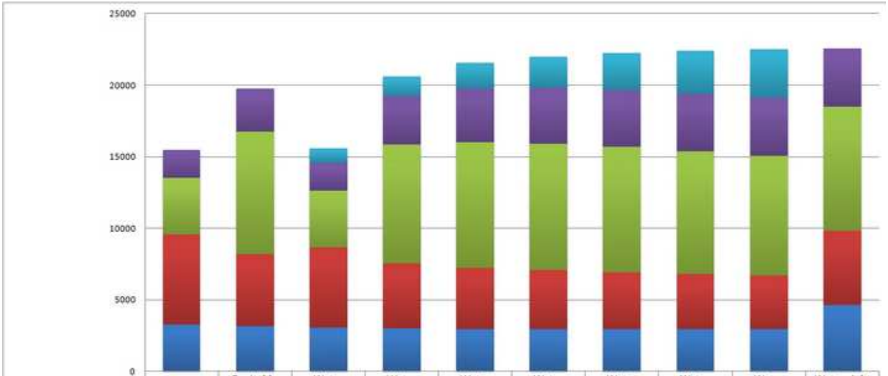
$$x_{i,land} \leq e_{i,land}$$

$$wo_{i-1} = x_{i,wat} + wo_i$$

$$wr_i + wr_i^b \geq x_{i,wat} + wr_i^s$$

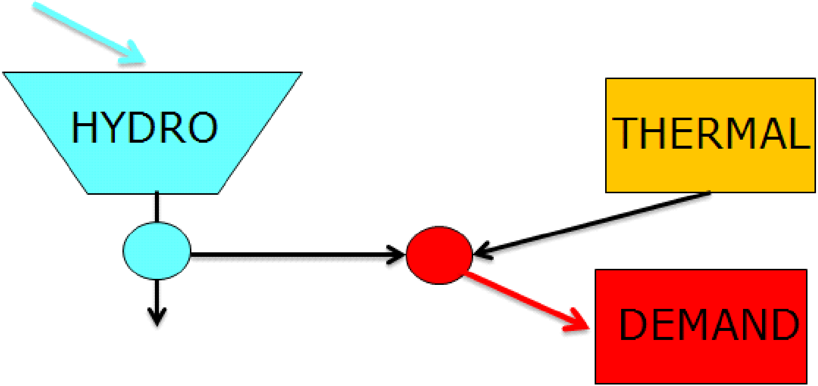
$$\begin{array}{lcll} 0 \leq & \sum_i q_i - d(p) & \perp & p \geq 0 \\ 0 \leq & \sum_i e_{i,lab} - \sum_i x_{i,lab} & \perp & w_{lab} \geq 0 \\ 0 \leq & \sum_i wr_i^s - \sum_i wr_i^b & \perp & w_{wr} \geq 0 \end{array}$$

Different Management Strategies



	No institution	Equitable rights	Water charges 10	Water charges 14	Water charges 18	Water charges 22	Water charges 26	Water charges 30	Water charges 34	Water right trade
Water charge revenue	0	0	1000	1400	1800	2200	2600	3000	3400	0
Consumer surplus	1943	3040	1969	3398	3705	3865	3959	4018	4054	4099
Profit Firm 3	3983	8513	3979	8279	8776	8869	8784	8601	8356	8642
Profit Firm 2	6306	5033	5580	4557	4281	4097	3960	3853	3766	5217
Profit Firm 1	3255	3176	3068	2995	2972	2957	2947	2940	2936	4617

Hydro-Thermal System (Philpott/F./Wets)



Simple electricity system optimization problem

$$\text{SSP:} \quad \min \quad \sum_{j\in\mathcal{T}} C_j(v(j)) - \sum_{i\in\mathcal{H}} V_i(x(i))$$

$$\begin{aligned}\text{s.t.} \quad & \sum_{i\in\mathcal{H}} U_i\left(u(i)\right) + \sum_{j\in\mathcal{T}} v(j) \geq d, \\ & x(i) = x_0(i) - u(i), \quad i \in \mathcal{H} \\ & u(i), v(j), x(i) \geq 0.\end{aligned}$$

- $u(i)$ water release of hydro reservoir $i \in \mathcal{H}$
- $v(j)$ thermal generation of plant $j \in \mathcal{T}$
- production function U_i (strictly concave) converts water release to energy
- water level reservoir $i \in \mathcal{H}$ is denoted $x(i)$
- $C_j(v(j))$ denote the cost of generation by thermal plant
- $V_i(x(i))$ to be the future value of terminating the period with storage x (assumed separable)

SSP equivalent to CE

Thermal plants solve

$$\text{TP}(j): \quad \max \quad \pi v(j) - C_j(v(j))$$

$$\text{s.t.} \quad v_1(j) \geq 0.$$

The hydro plants $i \in \mathcal{H}$ solve

$$\text{HP}(i): \quad \max \quad \pi U_i\left(u(i)\right) + V_i(x(i))$$

$$\begin{aligned} \text{s.t.} \quad & x(i) = x_0(i) - u(i) \\ & u(i), x(i) \geq 0. \end{aligned}$$

Perfectly competitive (Walrasian) equilibrium is a MOPEC

$$\begin{aligned} \text{CE:} \quad & u(i), x(i) \in \arg\max \text{HP}(i), & i \in \mathcal{H}, \\ & v(j) \in \arg\max \text{TP}(j), & j \in \mathcal{T}, \\ & 0 \leq \left(\sum_{i \in \mathcal{H}} U_i\left(u(i)\right) + \sum_{j \in \mathcal{T}} v(j)\right) - d \perp \pi \geq 0, \end{aligned}$$

GAMS/EMP: Stochastic programming tools

- GAMS has extended mathematical programming tools to build "models of models"
- Given the core model, can annotate parameters as "random variables"
- Automatically solves expected value problem
- Can solve using deterministic equivalent or specialized solvers (including Bender's decomposition, importance sampling (DECIS), etc)
- Also allows for a variety of new constructs (such as risk measures and chance constraints)

$$\mathbb{R}_\omega \left[c^0(x) + \sum_{t=0}^{T} p_{\omega t}(q^+_{\omega t} - q^-_{\omega t}) + c^1(q^+_{\omega t} + q^-_{\omega t}) \right]$$

Two stage problems

$$
\begin{aligned}
\text{TP}(j): \quad & \max \quad \pi_1 v_1(j) - C_j(v_1(j)) + \\
& \qquad R_\omega[\pi_2(\omega) v_2(j,\omega) - C_j\,(v_2(j,\omega))] \\
\text{s.t.} \quad & v_1(j) \geq 0, \quad v_2(j,\omega) \geq 0, \qquad \text{for all } \omega \in \Omega.
\end{aligned}
$$

$$
\begin{aligned}
\text{HP}(i): \quad & \max \quad \pi_1 U_i(u_1(i)) + \\
& \qquad R_\omega[\pi_2(\omega) U_i(u_2(i,\omega)) + V_i(x_2(i,\omega))] \\
\text{s.t.} \quad & x_1(i) = x_0(i) - u_1(i) + h_1(i), \\
& x_2(i,\omega) = x_1(i) - u_2(i,\omega) + h_2(i,\omega), \qquad \text{for all } \omega \in \Omega, \\
& u_1(i), x_1(i) \geq 0, \quad u_2(i,\omega), x_2(i,\omega) \geq 0, \qquad \text{for all } \omega \in \Omega.
\end{aligned}
$$

Results

- Suppose every agent is risk neutral and has knowledge of all deterministic data, as well as sharing the same probability distribution for inflows. SP solution is same as CE solution
- Using coherent risk measure (weighted sum of expected value and conditional variance at risk), 10 scenarios for rain

1. High initial storage: risk-averse central plan (RSP) and the risk-averse competitive equilibrium (RCE) have same solution (but different to risk neutral case)
2. Low initial storage: RSP and RCE are very different. Since the hydro generator and the system do not agree on a worst-case outcome, the probability distributions that correspond to an equivalent risk neutral decision will not be common.
3. Extension: Construct MOPEC models for trading risk

What is EMP?

Annotates existing equations/variables/models for modeler to provide/define additional structure

- equilibrium
- vi (agents can solve min/max/vi)
- bilevel (reformulate as MPEC, or as SOCP)
- disjunction (or other constraint logic primitives)
- randvar
- dualvar (use multipliers from one agent as variables for another)
- extended nonlinear programs (library of plq functions)

Currently available within GAMS

Conclusions

- Optimization helps understand what drives a system
- Collections of models needed for specific decisions
- Modern optimization within applications requires multiple model formats, computational tools and sophisticated solvers
- EMP model type is clear and extensible, additional structure available to solver
- Uncertainty is present everywhere
- We need not only to quantify it, but we need to hedge/control/ameliorate it
- Stochastic MOPEC models capture behavioral effects (as an EMP)
- Policy implications addressable using Stochastic MOPEC
- Extended Mathematical Programming available within the GAMS modeling system