Optimality conditions and complementarity, Nash equilibria and games, engineering and economic application

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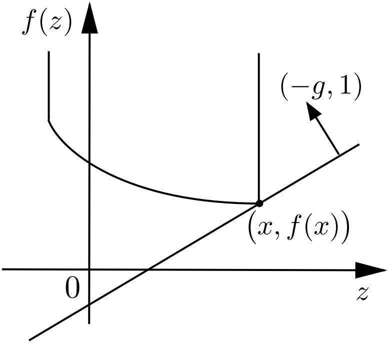
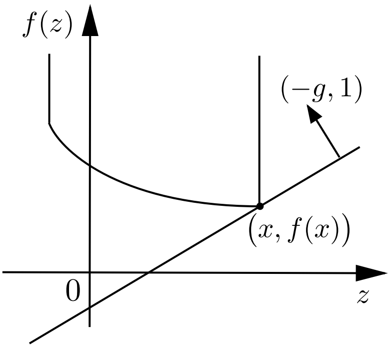
Convex subdifferentials



- Assume f is convex, then $f(z) \geq f(x) + \nabla f(x)^T (z - x)$ (linearization is below the function)
- Incorporate constraints by allowing f to take on $+\infty$ if constraint is violated $f : \mathbb{R}^n \mapsto (-\infty, +\infty]$
- $\partial f(x) = \{g : f(z) \geq f(x) + g^T (z - x), \forall z\}$, the subdifferential of f at x

Convex subdifferentials



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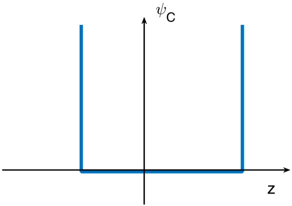
- If f is differentiable and convex, then $\partial f(x) = \{\nabla f(x)\}$
- e.g. $f(z) = \frac{1}{2}z^T Qz + p^T z$, then $\partial f(x) = \{Qx + p\}$
- x^* solves $\min f(x)$ if and only if $0 \in \partial f(x^*)$

Indicator functions and normal cones

$$\psi_{\mathcal{C}}(z) = \begin{cases} 0 & \text{if } z \in \mathcal{C} \\ \infty & \text{else} \end{cases}$$



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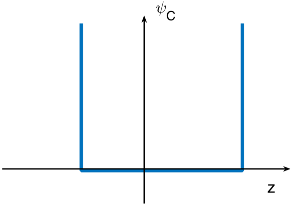

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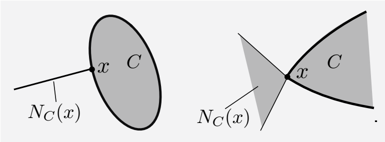

If $x \in \mathcal{C}$, then

$$g \in \partial\psi_{\mathcal{C}}(x)$$

$$\iff \psi_{\mathcal{C}}(z) \geq \psi_{\mathcal{C}}(x) + g^T(z - x), \ \forall z$$

$$\iff 0 \geq g^T(z - x), \ \forall z \in \mathcal{C}$$

Normal cone to $\mathcal{C}$ at x,

$$N_{\mathcal{C}}(x) := \partial\psi_{\mathcal{C}}(x) = \begin{cases} \{g : g^T(z - x) \leq 0, \forall z \in \mathcal{C}\} & \text{if } x \in \mathcal{C} \\ \emptyset & \text{if } x \notin \mathcal{C} \end{cases}$$

Some calculus

- $f_i : \mathbb{R}^n \mapsto (-\infty, \infty]$, $i = 1, \ldots, m$, proper, convex functions

$$F = f_1 + \cdots + f_m$$

assume $\bigcap_{i=1}^{m} \mathrm{rint}(\mathrm{dom}(f_i)) \neq \emptyset$ then (as sets)

$$\partial F(x) = \partial f_1(x) + \cdots + \partial f_m(x), \ \forall x$$

- $\mathcal{C} = \bigcap_{i=1}^{m} \mathcal{C}_i$, then $\psi_{\mathcal{C}} = \psi_{\mathcal{C}_i} + \cdots + \psi_{\mathcal{C}_m}$, so $N_{\mathcal{C}} = N_{\mathcal{C}_i} + \cdots + N_{\mathcal{C}_m}$

$$x^* \text{ solves } \min_{x \in \mathcal{C}} f(x) \iff x^* \text{ solves } \min_{x} (f + \psi_{\mathcal{C}})(x)$$
$$\iff 0 \in \partial(f + \psi_{\mathcal{C}})(x^*) \iff 0 \in \nabla f(x^*) + N_{\mathcal{C}}(x^*)$$

Special cases and examples

- Normal cone is a cone
- $x \in \text{int}(\mathcal{C})$, then $N_{\mathcal{C}}(x) = \{0\}$
- $\mathcal{C} = \mathbb{R}^n$, then $N_{\mathcal{C}}(x) = \{0\}, \forall x \in \mathcal{C}$

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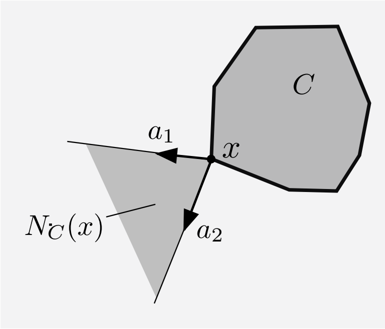


- $\mathcal{C} = \left\{z : a_i^T z \leq b_i, i = 1, \ldots, m\right\}$ polyhedral
- $N_{\mathcal{C}}(x) = \left\{ \sum_{i=1}^{m} \lambda_i a_i : 0 \leq b_i - a_i^T x \perp \lambda_i \geq 0 \right\}$
- $\perp$ makes product of items around it 0, i.e.

$$(b_i - a_i^T x)\lambda_i = 0, \; i = 1, \ldots, m$$

Combining: KKT conditions

- Example: convex optimization first-order optimality condition:

$$
\begin{aligned}
x^* \text{ solves } \min_{x \in \mathcal{C}} f(x) &\iff 0 \in \nabla f(x^*) + N_{\mathcal{C}}(x^*) \\
&\iff 0 = \nabla f(x^*) + y,\ y \in N_{\mathcal{C}}(x^*) \\
&\iff 0 = \nabla f(x^*) + y,\ y = A^T \lambda, \\
&\qquad 0 \le b - Ax^* \perp \lambda \ge 0 \\
&\iff 0 = \nabla f(x^*) + A^T \lambda, \\
&\qquad 0 \le b - Ax^* \perp \lambda \ge 0
\end{aligned}
$$

- More generally, if $\mathcal{C} = \{z : g(z) \le 0\}$, g convex, (with CQ)

$$
\begin{aligned}
x^* \text{ solves } \min_{x \in \mathcal{C}} f(x) &\iff 0 \in \nabla f(x^*) + N_{\mathcal{C}}(x^*) \\
&\iff 0 = \nabla f(x^*) + \nabla g(x^*) \lambda, \\
&\qquad 0 \le -g(x^*) \perp \lambda \ge 0
\end{aligned}
$$

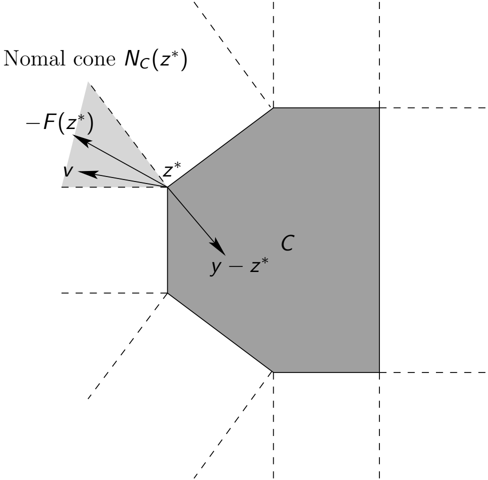
Variational Inequality (replace $\nabla f(z)$ with $F(z)$)

- $F : \mathbb{R}^n \to \mathbb{R}^n$
- Ideally: $\mathcal{C} \subseteq \mathbb{R}^n$ – constraint set; Often: $\mathcal{C} \subseteq \mathbb{R}^n$ – simple bounds

$$VI(F, \mathcal{C}) : \quad 0 \in F(z) + N_{\mathcal{C}}(z)$$

- VI generalizes many problem classes
- Nonlinear Equations: $F(z) = 0$ set $\mathcal{C} \equiv \mathbb{R}^n$
- Convex optimization: $F(z) = \nabla f(z)$
- For NCP: $0 \leq F(z) \perp z \geq 0$, set $\mathcal{C} \equiv \mathbb{R}^n_+$
- For MCP (rectangular VI), set $\mathcal{C} \equiv [l, u]^n$.
- For LP, set $F(z) \equiv \nabla f(z) = p$ and $\mathcal{C} = \{z : Az = a, Hz \leq h\}$.

VI: $0 \in F(z) + \mathcal{N}_{\mathcal{C}}(z)$



Many applications where F is not the derivative of some f

Other applications of complementarity

- Economics: Walrasian equilibrium (supply equals demand), taxes and tariffs, computable general equilibria, option pricing (electricity market), airline overbooking
- Transportation: Wardropian equilibrium (shortest paths), selfish routing, dynamic traffic assignment
- Applied mathematics: Free boundary problems
- Engineering: Optimal control (ELQP)
- Mechanics: Structure design, contact problems (with friction)
- Geology: Earthquake propogation

Nash problems

$$\min_{x,t} \theta(x,t) \text{ s.t. } F(x,t) = 0,\ t \in T$$

- x represents “state”, prices, production levels, etc
- t represents “taxes/tariffs”, strategy or design variables
- Often: $F(x,t) = 0 \iff x = x(t)$ (implicit function)

Nash problems

$$\min_{x_1, t_1} \theta_1(x_1, t) \text{ s.t. } F(x_1, t) = 0,\ t = (t_1, t_2) \in T_1$$

- x represents "state", prices, production levels, etc
- t represents "taxes/tariffs", strategy or design variables
- Agent 1 problem is parameterized by agent 2's variable

Nash problems

$$\min_{x_1,t_1} \theta_1(x_1,t) \text{ s.t. } F(x_1,t)=0,\ t=(t_1,t_2)\in T_1$$

$$\min_{x_2,t_2} \theta_2(x_2,t) \text{ s.t. } F(x_2,t)=0,\ t=(t_1,t_2)\in T_2$$

- x represents "state", prices, production levels, etc
- t represents "taxes/tariffs", strategy or design variables
- Solution is pair (x_1,t_1), (x_2,t_2) so that each agent cannot improve when other agents strategy remains fixed

Bimatrix Games: Golden Balls

- VI can be used to formulate many standard problem instances corresponding to special choices of M and $\mathcal{C}$.
- Nash game: two players have I and J pure strategies.
- p and q (strategy probabilities) belong to unit simplex $\triangle_I$ and $\triangle_J$ respectively.
- Payoff matrices $A \in R^{J \times I}$ and $B \in R^{I \times J}$, where $A_{j,i}$ is the profit received by the first player if strategy i is selected by the first player and j by the second, etc.
- The expected profit for the first and the second players are $q^T Ap$ and $p^T Bq$ respectively.
- A Nash equilibrium is reached by the pair of strategies (p^*, q^*) if and only if

$$p^* \in \arg \min_{p \in \triangle_I} \langle Aq^*, p \rangle \quad \text{and} \quad q^* \in \arg \min_{q \in \triangle_J} \langle B^T p^*, q \rangle$$

Optimal Sanctions (Boehringer/F./Rutherford)

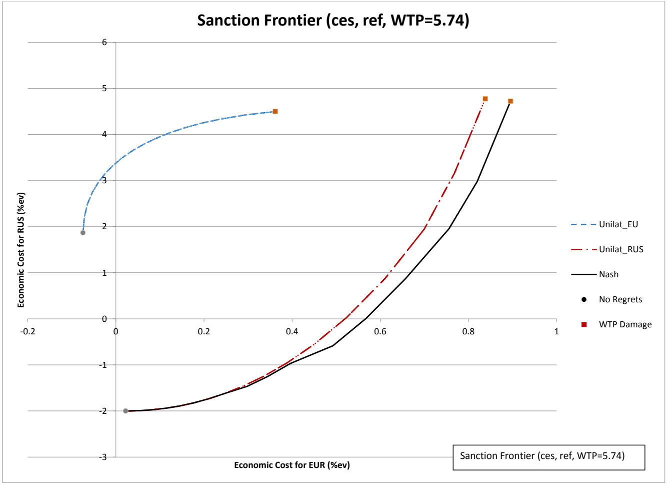
- Sanctions can be modeled using similar formulations used for tariff calculations
- Model as a Nash equilibrium with players being countries (or a coalition of countries)
- Demonstrate the actual effects of different policy changes and the power of different economic instruments
- GTAP global production/trade database: 113 countries, 57 goods, 5 factors
- Coalition members strategically choose trade taxes to
  1. maximize their welfare (no regrets) or
  2. *minimize* Russian welfare (with willingness to pay weights - WTP)
- Russia chooses trade taxes to *maximize* Russian welfare in response
- Explore unilateral vs Nash setting, and limits on Russian response

Nash problems

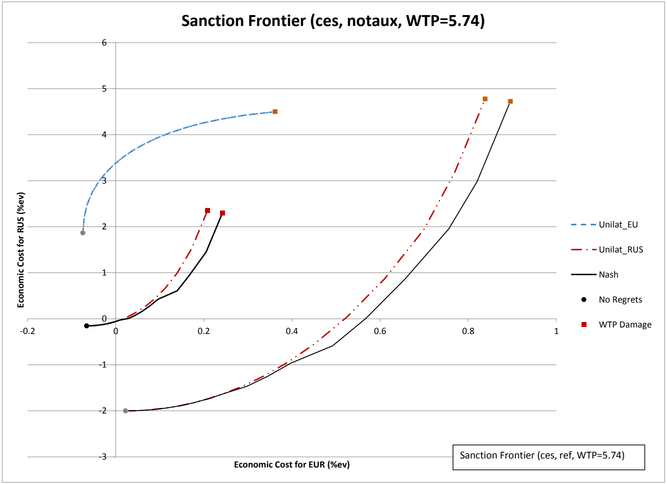
$$0 \in \nabla_{(x_1,t_1)}\theta_1(x_1,t) + N_{\mathcal{C}_1(t_2)}(x_1,t_1)$$

$$0 \in \nabla_{(x_2,t_2)}\theta_1(x_2,t) + N_{\mathcal{C}_2(t_1)}(x_2,t_2)$$

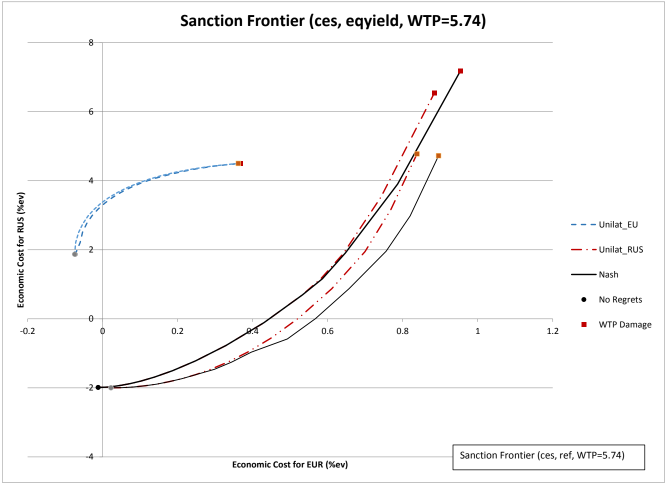
- x represents "state", prices, production levels, etc
- t represents "taxes/tariffs", strategy or design variables
- Solution is pair (x_1,t_1), (x_2,t_2) so that each agent cannot improve when other agents strategy remains fixed
- Note: implicit function $x = x(t)$ results in $x_1 = x_2$ at solution
- In convex case, can replace optimization problem by its KKT conditions, leading to MCP that is not KKT of a single optimization



- Resulting Nash equilibrium with different coalition costs - all have big impact on Russia

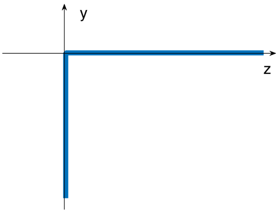


- Restricting instruments can change effects
- Russia export taxes fixed at benchmark level



- Restricting instruments can change effects
- Equal yield constraint on Russian trade taxes

Complementarity Problems via Graphs



- $\mathcal{T} = \mathcal{N}_{\mathbb{R}_+} = (\mathbb{R}_+ \times \{0\}) \bigcup (\{0\} \times \mathbb{R}_-)$
- $\mathcal{T}$ is "monotone"

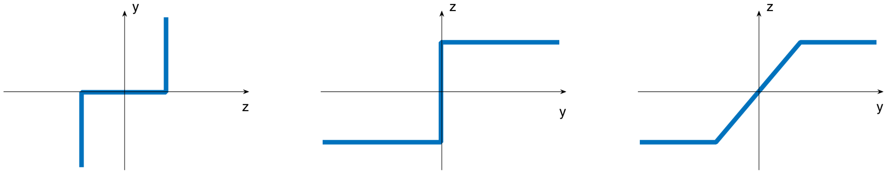
$$-y \in \mathcal{T}(z) \iff (z, -y) \in \mathcal{T} \iff 0 \leq y \perp z \geq 0$$

By approximating (smoothing) graph can generate interior point algorithms for example $yz = \epsilon, y, z > 0$

$$0 \in F(z) + \mathcal{N}_{\mathbb{R}^n_+}(z) \iff (z, -F(z)) \in \mathcal{T}^n \iff 0 \leq F(z) \perp z \geq 0$$

Operators and Graphs ($\mathcal{C} = [-1, 1], \mathcal{T} = \mathcal{N}_\mathcal{C}$)

$$z_i = -1, -F_i(z) \leq 0 \text{ or } z_i \in (-1, 1), -F_i(z) = 0 \text{ or } z_i = 1, -F_i(z) \geq 0$$



$\mathcal{T}(z)$ $\quad$ $\mathcal{T}^{-1}(y)$ $\quad$ $(\mathcal{I} + \mathcal{T})^{-1}(y) = P_\mathcal{T}(y)$

$P_\mathcal{T}(y)$ is the projection of y onto $[-1, 1]$

Normal Map

- Suppose $\mathcal{T}$ is a maximal monotone operator

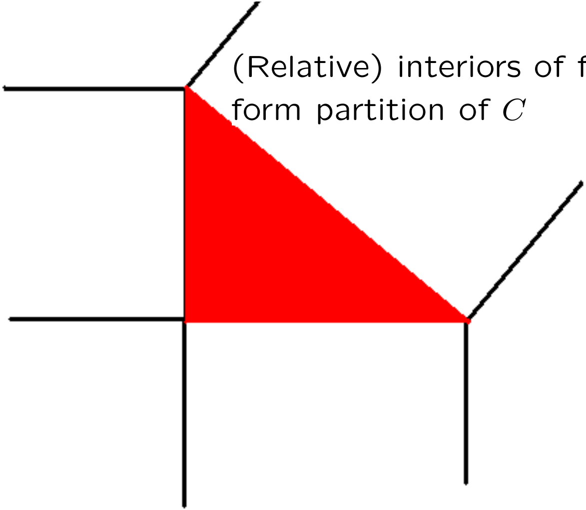
$$0 \in F(z) + \mathcal{T}(z) \qquad (GE)$$

- Define $P_{\mathcal{T}} = (I + \mathcal{T})^{-1}$ (continuous, single-valued, non-expansive)

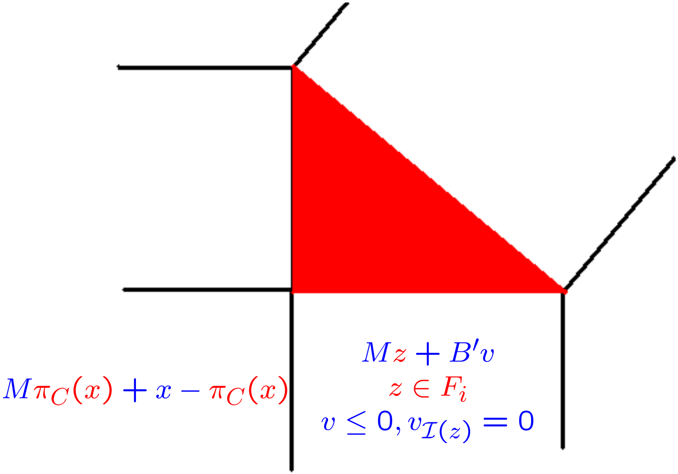
$$\begin{aligned}
0 \in F(z) + \mathcal{T}(z) \quad &\iff \quad z \in F(z) + \mathcal{I}(z) + \mathcal{T}(z) \\
&\iff \quad z - F(z) = x \text{ and } x \in (\mathcal{I} + \mathcal{T})(z) \\
&\iff \quad z - F(z) = x \text{ and } P_{\mathcal{T}}(x) = z \\
&\iff \quad P_{\mathcal{T}}(x) - F(P_{\mathcal{T}}(x)) = x \\
&\iff \quad 0 = F(P_{\mathcal{T}}(x)) + x - P_{\mathcal{T}}(x)
\end{aligned}$$

This is the so-called Normal Map Equation

Normal manifold $= \{F_i + N_{F_i}\}$



$$C = \{z | Bz \geq b\}, N_C(z) = \{B'v | v \leq 0, v_{\mathcal{I}(z)} = 0\}$$



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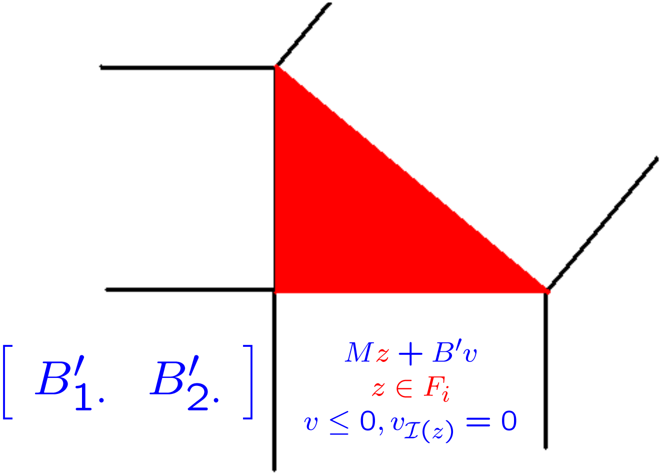
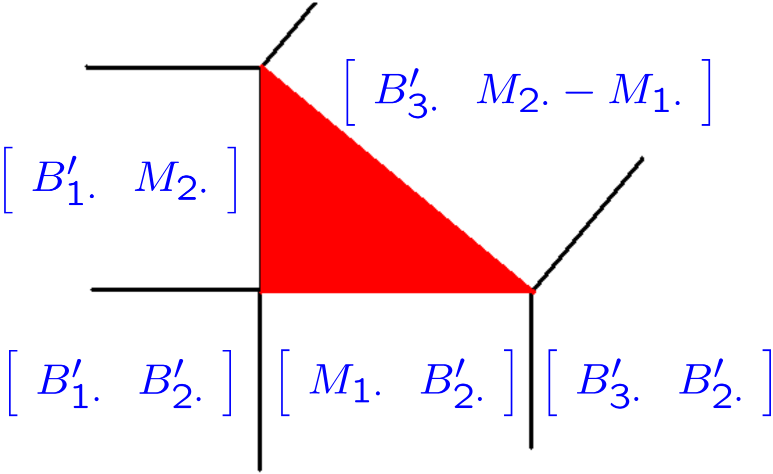
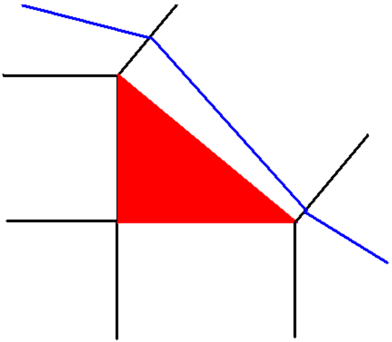


$C = \{z | Bz \geq b\}, F(z) = Mz + q$



The PATH algorithm

- Start in cell that has interior (face is an extreme point)
- Move towards a zero of affine map in cell
- Update direction when hit boundary (pivot)
- Solves or determines infeasible if M is copositive-plus on rec(C)
- Solves 2-person bimatrix games, 3-person games too, but these are nonlinear



But algorithm has exponential complexity (von Stengel et al)

Theorem

Suppose $\mathcal{C}$ is a polyhedral convex set and M is an L–matrix with respect to $\mathrm{rec}\mathcal{C}$ which is invertible on the lineality space of $\mathcal{C}$. Then exactly one of the following occurs:

- *PATHAVI solves (AVI)*
- *the following system has no solution*

$$Mz + q \in (\mathrm{rec}\mathcal{C})^D, \qquad z \in \mathcal{C}. \tag{1}$$

Corollary

If M is copositive–plus with respect to $\mathrm{rec}\mathcal{C}$, then exactly one of the following occurs:

- *PATHAVI solves (AVI)*
- *(1) has no solution*

Note also that if $\mathcal{C}$ is compact, then any matrix M is an L–matrix with respect to $\mathrm{rec}\mathcal{C}$. So always solved.

Experimental results: AVI vs MCP

PATH is a solver for MCP (mixed complementarity problem).

- Run PathAVI over AVI formulation.
- Run PATH over AVI in MCP form (poorer theory as $\mathrm{rec}\mathcal{C}$ larger).
- Data generation
  - M is an $n \times n$ symmetric positive definite/indefinite matrix.
  - A has m randomly generated bounded inequality constraints.

(m, n)	PathAVI		PATH		% negative eigenvalues
	status	# iterations	status	# iterations	
(180,60)	S	55	S	72	0
(180,60)	S	45	S	306	20
(180,60)	S	2	F	9616	60
(180,60)	S	1	F	10981	80
(360,120)	S	124	S	267	0
(360,120)	S	55	S	1095	20
(360,120)	S	2	F	10020	60
(360,120)	S	1	F	7988	80

What is EMP?

Annotates existing equations/variables/models for modeler to provide/define additional structure

- equilibrium
- vi (agents can solve min/max/vi)
- bilevel (reformulate as MPEC, or as SOCP)
- dualvar (use multipliers from one agent as variables for another)
- QS functions (both in objectives and constraints)

- Currently available within GAMS (full license available to course participants until August 8, 2016 - contact me!)
- Some solution algorithms implemented in modeling system - limitations on size, decomposition and advanced algorithms
- QS extensions to Moreau-Yoshida regularization, compositions, composite optimization

Splitting Methods

- Suppose $\mathcal{T}$ is a maximal monotone operator

$$0 \in F(z) + \mathcal{T}(z) \qquad (GE)$$

- Can devise Newton methods (e.g. SQP) that treat F via calculus and $\mathcal{T}$ via convex analysis
- Alternatively, can split $F(z) = A(z) + B(z)$ (and possibly $\mathcal{T}$ also) so we solve solve (GE) by solving a sequence of problems involving just

$$\mathcal{T}_1(z) = A(z) \text{ and } \mathcal{T}_2(z) = B(z) + \mathcal{T}(z)$$

where each of these is "simpler"

- Forward-Backward splitting (or ADMM):

$$z^{k+1} = (I + c_k T_2)^{-1} (I - c_k T_1) \left(z^k \right),$$