Extended Mathematical Programming: Structure and Solution

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Mathematical programming: modeling

- Optimization models improve understanding of underlying systems and facilitate operational/strategic improvements
- Modeling systems enable application interfacing, prototyping of optimization capability
- Data (collection) remains bottleneck in many applications
  - Tools interface to databases, spreadsheets, Matlab
- Problem format is old/traditional

$$\min_{x} f(x) \text{ s.t. } g(x) \leq 0, h(x) = 0$$

  - Support for integer, sos, semicontinuous variables
  - Limited support for logical constructs
  - Support for complementarity constraints

Complementarity Problems (MCP)

- p represents prices, x represents activity levels
- System model: given prices, (agent) i determines activities x_i

$$G_i(x_i, x_{-i}, p) = 0$$

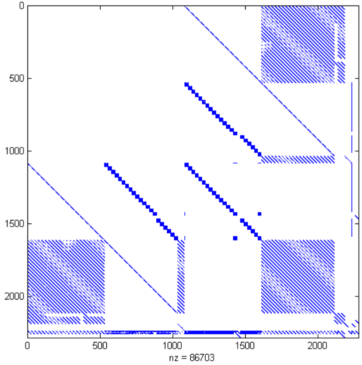
  x_{-i} are the decisions of other agents.
- Walras Law: market clearing

$$0 \leq S(x,p) - D(x,p) \perp p \geq 0$$

- Key difference: optimization assumes you control the complete system
- Complementarity determines what activities run, and who produces what

World Bank Project (Uruguay Round)

- 24 regions, 22 commodities
  - Nonlinear complementarity problem
  - Size: 2200 x 2200
- Short term gains $53 billion p.a.
  - Much smaller than previous literature
- Long term gains $188 billion p.a.
  - Number of less developed countries loose in short term
- Unpopular conclusions - forced concessions by World Bank
- Region/commodity structure not apparent to solver



MPEC: complementarity constraints

$$\begin{array}{ll} \min\limits_{x,s} & f(x,s) \\ \text{s.t.} & g(x,s) \leq 0, \\ & 0 \geq s \perp h(x,s) \leq 0 \end{array}$$

- g, h model "engineering" expertise: finite elements, etc
- $\perp$ models complementarity, disjunctions
- Complementarity "$\perp$" constraints available in AMPL and GAMS
- NLPEC: use the convert tool to automatically reformulate as a parameteric sequence of NLP's
- Solution by repeated use of standard NLP software
  - Problems solvable, local solutions, hard

Use of complementarity

- Pricing electricity markets and options
- Video games: model contact problems
  - Friction only occurs if bodies are in contact
- Structure design
  - how springy is concrete
  - optimal sailboat rig design
- Computer/traffic networks
  - The price of anarchy measures difference between "system optimal" (MPEC) and "individual optimization" (MCP)

What else can we model via CP?

$$\min\left(G(x), H(x)\right) \leq y$$

$$\min\left(F^1(x), F^2(x), \ldots, F^m(x)\right) = 0$$

$$\text{kth-largest}\left(F^1(x), F^2(x), \ldots, F^m(x)\right) = 0$$

$$\text{Switch on/off: } g(x)h(x) \leq 0, h(x) \geq 0$$

So what's my point?

- Can solve practical models due to availability of good software (e.g. PATH)
- Design (optimization) under our control
- Uncertainties treated via "scenarios"
- Competition/market effects beyond control of designer treated by complementarity

- Complementarity facilitates modeling of competition, nonsmoothness and "switching"
- Large scale models involving complementarity now solvable
- Do you (or should you) care?

EMP(i): Embedded models

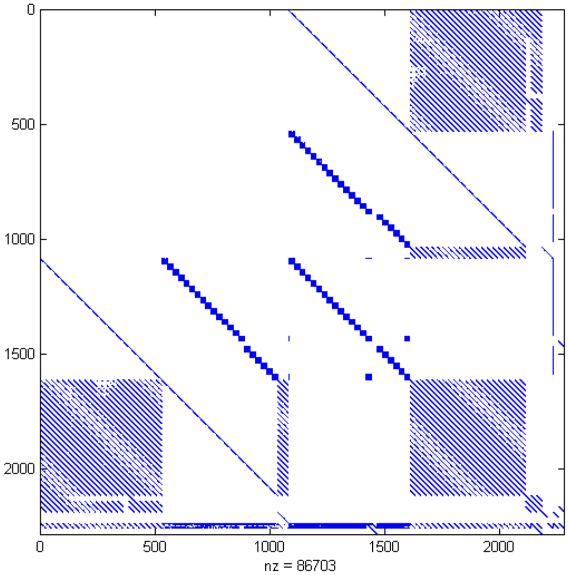
- Model has the format:

$$\begin{array}{lll} \min\limits_{x} & f(x,y) & \\ \text{s.t.} & g(x,y) \leq 0 & (\perp \lambda \geq 0) \\ & & \\ H(x,y,\lambda) = 0 & & (\perp y \text{ free}) \end{array}$$

- Difficult to implement correctly, particularly when multiple optimization models present
- Can do automatically - simply annotate equations
- EMP tool automatically creates an MCP

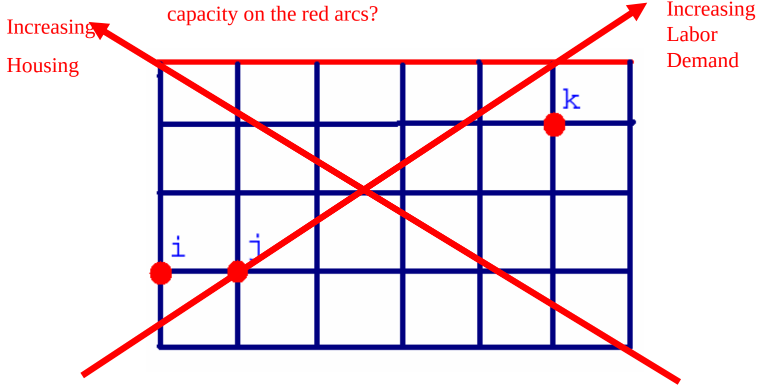
$$\nabla_x f(x,y) + \lambda^T \nabla g(x,y) = 0$$
$$0 \leq -g(x,y) \perp \lambda \geq 0$$
$$H(x,y,\lambda) = 0$$

Recall: World Bank Example



Walras meets Wardrop

What is the effect on housing prices of increasing capacity on the red arcs?



Features

- We buy a house to “optimize” some measure
  - Price driven by market
  - We compete against each other
- Driver’s choose routes to “optimize” travel time
  - Choices affect congestion
  - Your choice affects me!
- Production processes are “optimized”
- But the road designer does not control any of these!

Simplified AGE model

$$(P): \min_{y \geq 0} \quad c^T y$$
$$\text{s.t. } Ay \geq d \quad (\perp p \geq 0)$$

$$(C): \max_{d \geq 0} \quad u(d)$$
$$\text{s.t. } p^T d \leq I$$

- In equilibrium, the optimal demand d from (C) will be the demand in (P), and the sales price p in (C) will be the marginal price on production from (P)
- Complementarity conditions of (P) and (C) have both primal and dual variables
- Optimization models linked by variables and multipliers
- Equilibrium problem solvable as a complementarity problem
- Can add "other features" such as taxation, transportation, tolls.

The Hollywood perspective: game theory

- Nash Games: x^* is a Nash Equilibrium if

$$x_i^* \in \arg\min_{x_i \in X_i} \ell_i(x_i, x_{-i}^*, q), \forall i \in \mathcal{I}$$

x_{-i} are the decisions of other players.

- Quantities q given exogenously, or via complementarity:

$$0 \leq H(x, q) \perp q \geq 0$$

- EMP reformulates automatically for appropriate solvers
- Applications: Discrete-Time Finite-State Stochastic Games. Specifically, the Ericson & Pakes (1995) model of dynamic competition in an oligopolistic industry.

Key point: models generated correctly solve quickly

Here S is mesh spacing parameter

S	Var	rows	non-zero	dense(%)	Steps	RT (m:s)
20	2400	2568	31536	0.48	5	0 : 03
50	15000	15408	195816	0.08	5	0 : 19
100	60000	60808	781616	0.02	5	1 : 16
200	240000	241608	3123216	0.01	5	5 : 12

Convergence for $S = 200$ (with new basis extensions in PATH)

Iteration	Residual
0	1.56(+4)
1	1.06(+1)
2	1.34
3	2.04(−2)
4	1.74(−5)
5	2.97(−11)

Security Constrained Optimal Power Flow

- least cost energy dispatch (uses prices and quantities)
- subject to: physical grid constraints
  - power flow equations (Ohm's law)
  - line and bus voltage limits
- and contingency constraints
- and "generators dispatch to optimize profits" at given prices (supply/price bid curves)

- Leader/follower game: Stackleberg
- Supply chains with "market leader"

EMP(ii): Heirarchical models

- Bilevel programs:

$$\begin{array}{ll} \min_{x,y} & f(x,y) \\ \text{s.t.} & g(x,y) \le 0, \\ & y \text{ solves } \min_{s} v(x,s) \text{ s.t. } h(x,s) \le 0 \end{array}$$

- Model as:
  model bilev /deff,defg,defv,defh/;
  plus empinfo: bilevel y min v defh
- EMP tool automatically creates the MPEC
- Note that heirarchical structure is available to solvers for decomposition approaches

EMP(iii): Other new types of constraints

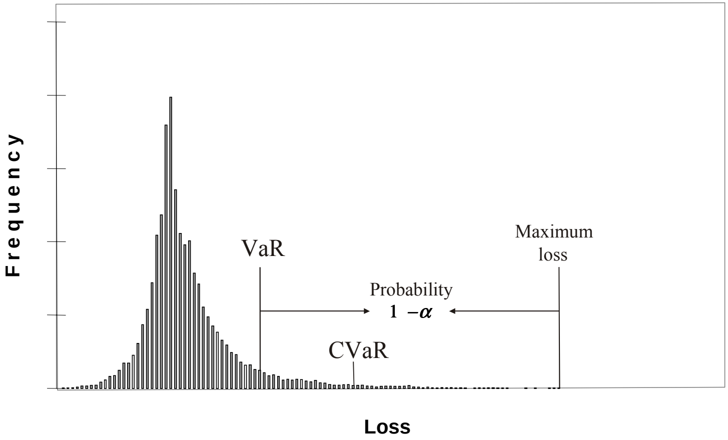
- range constraints $L \leq Ax - b \leq U$
- indicator constraints
- disjunctive programming
- soft constraints
- rewards and penalties
- robust programming (probability constraints, stochastics)

$$f(x, \xi) \leq 0, \forall \xi \in \mathcal{U}$$

- conic programming $a_i^T x - b_i \in K_i$

Some constraints can be reformulated easily, others not!

CVaR constraints: mean excess dose (radiotherapy)



Move mean of tail to the left!

Example: Robust Linear Programming

Data in LP not known with certainty:

$$\min c^T x \text{ s.t. } a_i^T x \leq b_i, i = 1, 2, \ldots, m$$

Suppose the vectors a_i are known to be lie in the ellipsoids

$$a_i \in \varepsilon_i := \{\bar{a}_i + P_i u : \|u\|_2 \leq 1\}$$

where $P_i \in \mathbf{R}^{n \times n}$ (and could be singular, or even 0).
Conservative approach: robust linear program

$$\min c^T x \text{ s.t. } a_i^T x \leq b_i, \text{ for all } a_i \in \varepsilon_i, i = 1, 2, \ldots, m$$

Robust Linear Programming as SOCP/ENLP

The constraints can be rewritten as:

$$\begin{aligned} b_i &\geq \sup \left\{ a_i^T x : a_i \in \varepsilon_i \right\} \\ &= \bar{a}_i^T x + \sup \left\{ u^T P_i^T x : \|u\|_2 \leq 1 \right\} = \bar{a}_i^T x + \left\| P_i^T x \right\|_2 \end{aligned}$$

Thus the robust linear program can be written as

$$\min c^T x \text{ s.t. } \bar{a}_i^T x + \left\| P_i^T x \right\|_2 \leq b_i, i = 1, 2, \ldots, m$$

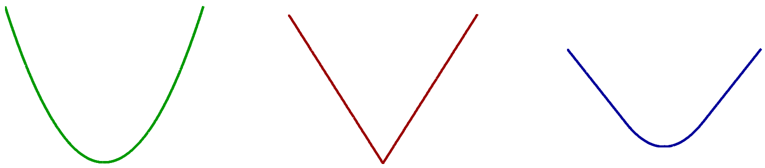
$$\min c^T x + \sum_{i=1}^{m} \psi_C(b_i - \bar{a}_i^T x, P_i^T x)$$

where C represents the second-order cone. Our extension allows automatic reformulation and solution (as SOCP) by Mosek or Conopt.

EMP(iv): Extended nonlinear programs

$$\min_{x \in X} f_0(x) + \theta(f_1(x), \ldots, f_m(x))$$

Examples of different θ



least squares, absolute value, Huber function

Solution reformulations are very different

Huber function used in robust statistics.

Choices for θ

$$\inf_{x \in X} f_0 + \theta[f(x)], \quad \theta(u) = \sup_{y \in Y}\{y^T u - k(y)\}$$

θ is convex with values in $(-\infty, +\infty]$; may be nonsmooth

- L_2: $k(u) = \frac{1}{4\lambda}u^2$, $Y = (-\infty, +\infty)$
- L_1: $k(u) = 0$, $Y = [-\rho, \rho]$
- L_∞: $k(u) = 0$, $Y = \Delta$, unit simplex in $\mathbf{R}^m_+$
- Linear-quad (Huber 1981): $k(u) = \frac{1}{4\lambda}u^2$, $Y = [-\rho, \rho]$
- Second order cone constraint: $k(y) = 0$, $Y = C^\circ$
- The new feature here is implementation and solution within the GAMS modeling language framework, which produces a tool usable without advanced knowledge in convex analysis and without cumbersome "hand tailoring" to accommodate different penalizations [Ferris, Dirkse, Jagla, and Meeraus 2008]
- This makes the theoretical benefits accessible to users from a wide variety of different fields (examples later)

Solution Procedures

- Solution uses reformulation. One way (using e.g. PATH):

$$\begin{array}{lll} 0 \in & \nabla_x \mathcal{L}(x,u) & + \quad N_X(x) \\ 0 \in & -\nabla_u \mathcal{L}(x,u) & + \quad N_Y(u) \end{array}$$

  $N_X(x)$ is the normal cone to the closed convex set X at x.

- EMP: allows "annotation" of constraints to facilitate library of different θ functions to be applied
- EMP tool automatically creates an MCP
- Available!
- To do: extend solvers to exploit X and Y beyond simple bound sets

Conclusions

- Model is clearer, structure available to solver
- Large scale complementarity problems reliably solvable
- Complementarity constraints within optimization problems
- Extended Mathematical Programming available within a modeling system
- System can easily formulate and solve second order cone programs, risk measures, robust optimization, soft constraints via piecewise linear penalization (with strong supporting theory)
- Embedded optimization models automatically reformulated for appropriate solution engine
- Enhance library of (implemented) θ functions (what do you want?)
- Exploit structure in solvers
- Extend application usage of complementarity solvers
- Extend complementarity solvers to VI solvers

The good, the bad and the ugly

- Formulate extensions, convert automatically, provide structure, solve with specific solvers, underlying theory
- Uncertainty, global properties, large-scale issues
- Reformulations may be suboptimal, grid solution strategies may not be reproduceable, requires “knowledgeable” user (for annotations), needs exercising!