Stochastic Programming in GAMS

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The problem

A furniture maker can manufacture and sell four different dressers. Each dresser requires a certain number t_{c_j} of man-hours for carpentry, and a certain number t_{f_j} of man-hours for finishing, $j = 1, \ldots, 4$. In each period, there are d_c man-hours available for carpentry, and d_f available for finishing. There is a (unit) profit $\bar{c}_j$ per dresser of type j that's manufactured. The owner's goal is to maximize total profit:

$$\max_{x \geq 0} \; 12x_1 + 25x_2 + 21x_3 + 40x_4 \qquad (\textit{profit})$$

subject to

$$4x_1 + 9x_2 + 7x_3 + 10x_4 \leq 6000 \qquad (\textit{carpentry})$$
$$x_1 + x_2 + 3x_3 + 40x_4 \leq 4000 \qquad (\textit{finishing})$$

Succinctly:

$$\max_x \; c^T x \text{ s.t. } Tx \leq d, x \geq 0$$

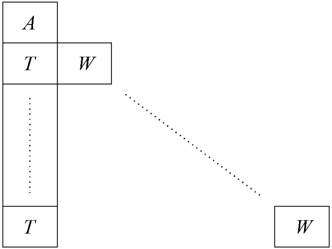
Is your time estimate that good?

- The time for carpentry and finishing for each dresser cannot be known with certainty
- Each entry in T takes on four possible values with probability $1/4$, independently
- 8 entries of T are random variables: $s = 65,536$ different T's each with same probability of occurring
- But decide "now" how many dressers x of each type to build
- Might have to pay for overtime (for carpentry and finishing)
- Can make different overtime decision y^s for each scenario s - recourse!

Stochastic recourse

- Two stage stochastic programming, x is here-and-now decision, recourse decisions y depend on realization of a random variable
- $\mathbb{R}$ is a risk measure (e.g. expectation, CVaR)

$$\begin{aligned} \text{SP:}\ \max \quad & c^\top x + \mathbb{R}[q^\top y] \\ \text{s.t.} \quad & Ax = b, \quad x \geq 0, \\ \forall \omega \in \Omega: \quad & T(\omega)x + W(\omega)y(\omega) \leq d(\omega), \\ & y(\omega) \geq 0. \end{aligned}$$



EMP/SP extensions to facilitate these models

Extended Form Problem

$$\max_{x,y} c^T x + \sum_{s=1}^{65,536} \pi_s q^T y$$

subject to

$$T^s x - y^s \leq d, \quad s = 1, \ldots, 65,536$$
$$x, y^s \geq 0$$

- Immediate profit + expected *future profit*
- Stochastic program with recourse

What do we learn?

- Deterministic solution: $x_d = (1333, 0, 0, 67)$
- Expected profit using this solution: $\$16,942$
- Expected (averaged) overtime costs: $\$1,725$
- Extensive form solution: $x_e = (257, 0, 666, 34)$ with expected profit $\$18,051$
- Deterministic solution is not optimal for stochastic program, but more significantly it isn't getting us on the right track!
- Stochastic solution suggests large number of "type 3" dressers, while deterministic solution has none!
- How to formulate model, how to solve, why it works

Models with explicit random variables

- Model transformation:
  - Write a core model as if the random variables are constants
  - Identify the random variables and decision variables and their staging
  - Specify the distributions of the random variables
- Solver configuration:
  - Specify the manner of sampling from the distributions
  - Determine which algorithm (and parameter settings) to use
- Output handling:
  - Optionally, list the variables for which we want a scenario-by-scenario report

Stochastic Programming as an EMP

- Separate solution process from model description

Three separate pieces of information (extended mathematical program) needed

1. emp.info: model transformation

```
randvar T('c','1') discrete .25  3.6 .25  3.9 ...
...
stage 2 y T profit cons obj
```

2. solver.opt: solver configuration (benders, sampling strategy, etc)

```
    4 "ISTRAT"  * solve universe problem (DECIS/Benders)
```

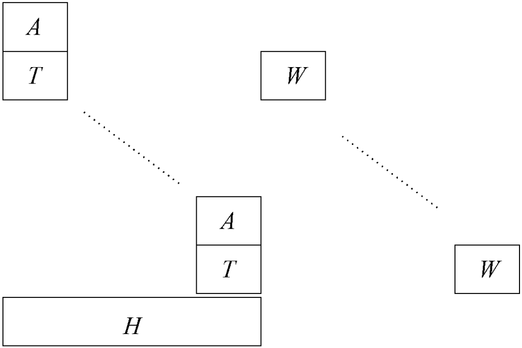
3. dictionary: output handling (where to put all the "scenario solutions")

Computation methods matter!

- Problem becomes large very quickly!
- Lindo solver defaults: 825 seconds
- Lindo solver barrier method: 382 seconds
- CPLEX solver barrier method: 4 seconds (8 threads)
- Models can be solved by the extensive form equivalent, existing codes such as LINDO and DECIS, or decomposition approaches such as Benders, ATR, etc - Just change the solver

Key-idea: Non-anticipativity constraints

- Replace x with $x_1, x_2, \ldots, x_K$
- Non-anticipativity: $(x_1, x_2, \ldots, x_K) \in L$ (a subspace) - the H constraints



Computational methods exploit the separability of these constraints, essentially by dualization of the non-anticipativity constraints.

- Primal and dual decompositions (Lagrangian relaxation, progressive hedging , etc)
- L shaped method (Benders decomposition applied to det. equiv.)
- Trust region methods and/or regularized decomposition

How to generate the model

1. May have multiple sources of uncertainty: e.g. man-hours d also can take on 4 values in each setting independently: $s = 1,048,576$
2. EMP/SP allows description of compositional (nonlinear) random effects in generating ω

$$\text{i.e. } \omega = \omega_1 \times \omega_2, \; T(\omega) = f(X(\omega_1), Y(\omega_2))$$

3. emp.info: model transformation

```
randvar T('c','1') discrete .25  3.60 .25  3.90 .25  4.10 .25  4.40
randvar T('c','2') discrete .25  8.25 .25  8.75 .25  9.25 .25  9.75
randvar T('c','3') discrete .25  6.85 .25  6.95 .25  7.05 .25  7.15
randvar T('c','4') discrete .25  9.25 .25  9.75 .25 10.25 .25 10.75
randvar T('f','1') discrete .25  0.85 .25  0.95 .25  1.05 .25  1.15
randvar T('f','2') discrete .25  0.85 .25  0.95 .25  1.05 .25  1.15
randvar T('f','3') discrete .25  2.60 .25  2.90 .25  3.10 .25  3.40
randvar T('f','4') discrete .25 37.00 .25 39.00 .25 41.00 .25 43.00
randvar d('c')     discrete .25 5873. .25 5967. .25 6033. .25 6127.
randvar d('f')     discrete .25 3936. .25 3984. .25 4016. .25 4064.

stage 2 y T d cost cons obj
```

4. Generates extensive form problem with over 3 million rows and columns and 29 million nonzeros
5. Solves on 24 threaded cluster machine in 262 secs

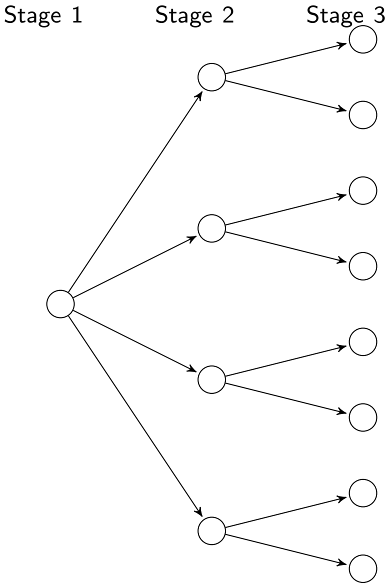
Multi-stage model: Clear Lake

- Easy to write down multi-stage problems
- Water levels $l(t)$ in dam for each month t
- Determine what to release normally $r(t)$, what then floods $f(t)$ and what to import $z(t)$
- minimize cost of flooding and import
- Change in reservoir level in period t is $\delta(t)$

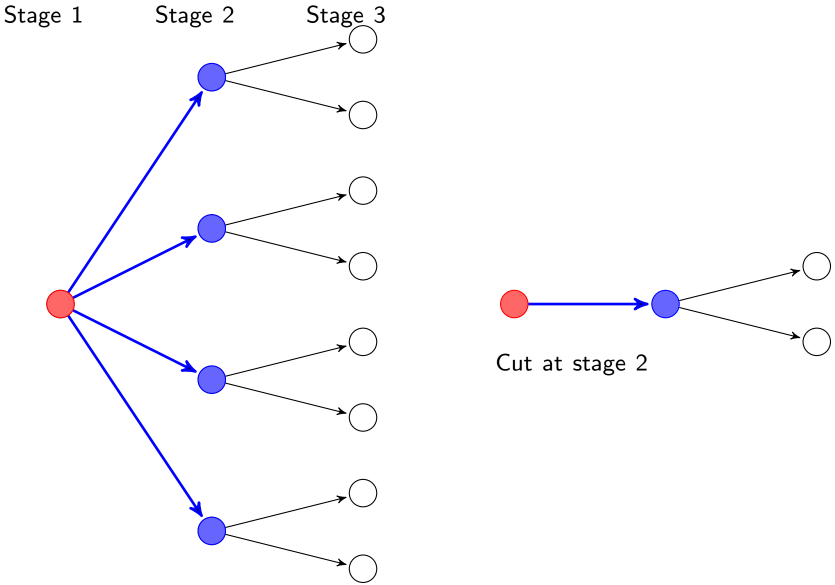
$$\begin{aligned} &\max cost = c(f, z) \\ &\text{s.t. } \; l(t) = l(t-1) + \delta(t) + z(t) - r(t) - f(t) \end{aligned}$$

- Random variables are δ, realized at stage t, $t \geq 2$.
- Variables l, r, f, z in stage t, $t \geq 2$.
- balance constraint at t in stage t.

Multi to 2 stage reformulation



Multi to 2 stage reformulation



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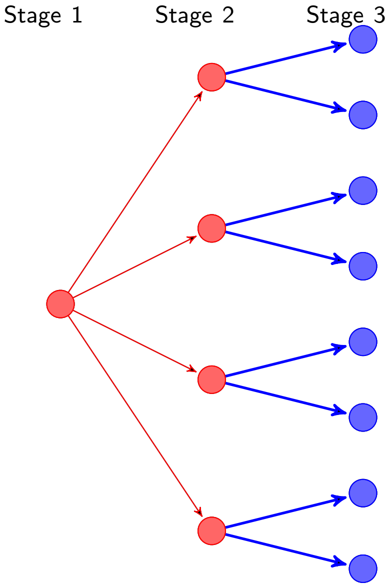
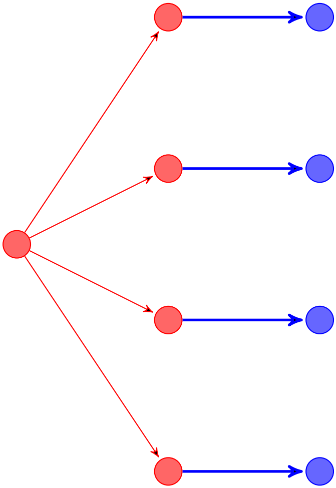




Cut at stage 3

Solution options

- Form the extensive form equivalent
- Solve using LINDO api (stochastic solver)
- Convert to two stage problem and solve using DECIS or any number of competing methods
- Problem with $3^{40} \approx 1.2 * 10^{19}$ realizations in stage 2
  - DECIS using Benders and Importance Sampling: < 1 second (and provides confidence bounds)
  - CPLEX on a presampled approximation:

sample	samp. time(s)	CPLEX time(s) for solution			cols (mil)
500	0.0	5	(4.5 barrier,	0.5 xover)	0.25
1000	0.2	18	(16 barrier,	2 xover)	0.5
10000	28	195	(44 barrier,	151 xover)	5
20000	110	1063	(98 barrier,	965 xover)	10

Chance constraints

- Use of discrete variables (in submodels) to capture logical or discrete choices (logmip - Grossmann et al)
- Chance constraints: $Prob(G(x,\xi) \leq \gamma) \geq 1 - \alpha$
- emp.info: chance E1 E2 0.95
- Use binary variable to model indicator function

$$\mathbb{E}(\mathcal{I}_{\{G(x,\xi)\leq\gamma\}}) = P(G(x,\xi) \leq \gamma) \geq 1 - \alpha$$

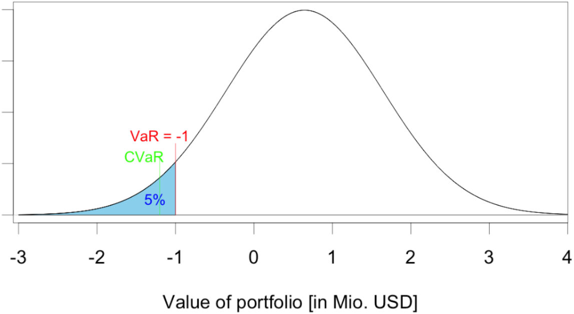
- Single or joint probabilistic constraints
- Reformulate as MIP (bigM) and adapt cuts

$$\begin{array}{ll} \max_x & c^T x \\ \text{s.t.} & A^\omega x \leq b^\omega + M^\omega(1 - y_\omega), \forall \omega \\ & \sum_{\omega\in\Omega} p_\omega y_\omega \geq 1 - \alpha, \quad x \geq 0, \quad y_\omega \in (0,1)^{|\Omega|} \end{array}$$

- Optional reformulations: convex hull or indicator constraints
- chance E1 E2 0.6 viol assigns the violation probability to the variable `viol`

Optimization of risk measures

- Determine portfolio weights w_j for each of a collection of assets
- Asset returns v are random, but jointly distributed
- Portfolio return $r(w, v)$



- Value at Risk (VaR) can be viewed as a chance constraint (hard):
- CVaR gives rise to a convex optimization problem (easy)

- mean-risk, semi-deviations, mean deviations from quantiles, VaR, CVaR

Example: Portfolio Model

- Coherent risk measures $\mathbb{E}$ and $\underline{CVaR}$ (or convex combination)
- Maximize combination of mean and mean of the lower tail (mean tail loss):

$$\begin{array}{ll} \max & 0.2 * \mathbb{E}(r) + 0.8 * \underline{CVaR}_{\alpha}(r) \\ \text{s.t.} & r = \sum_j v_j * w_j \\ & \sum_j w_j = 1, \; w \geq 0 \end{array}$$

- Jointly distributed random variables v, realized at stage 2
- Variables: portfolio weights w in stage 1, returns r in stage 2

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- Jointly distributed random variables v, realized at stage 2
- Variables: portfolio weights w in stage 1, returns r in stage 2
- Easy to add integer constraints to model (e.g. cardinality constraints)
- Alternative: mean-variance model (Markowitz)

$$\begin{array}{ll} \min & w^T \Sigma w - q \sum_j v_j * w_j \\ & \sum_j w_j = 1, \ w \geq 0 \end{array}$$

Other EMP information

- emp.info: model transformation

```
expected_value EV_r r
cvarlo         CVaR_r r alpha
stage          2 v r defr
jrandvar       v("att") v("gmc") v("usx") discrete
   table of probabilities and outcomes
```

- Variables are assigned to $\mathbb{E}(r)$ and $\underline{CVaR}_\alpha(r)$; can be used in model (appropriately) for objective, constraints, or be bounded

Reformulation details

- $\mathbb{E}(r)$ is simply a sum (probability and values generated by EMP)
- VaR is a chance constraint:

$$\mathbb{E}(\mathcal{I}_{G(x,\xi) leq \gamma}) = P(G(x,\xi) \leq \gamma) \geq 1 - \alpha \iff VaR_{1-\alpha}(G(x,\xi)) \leq \gamma$$

- CVaR transformation: can be written as convex optimization using:

$$\underline{CVaR}_{\alpha}(r) = \max_{a \in \mathbf{R}} \left\{ a - \frac{1}{\alpha} \mathbb{E}(a - r)_{+} \right\}$$

Sampling methods

But what if the number of scenarios is too big (or the probability distribution is not discrete)? use sample average approximation (SAA)

- Take sample $\xi_1, \ldots, \xi_N$ of N realizations of random vector ξ
  - viewed as historical data of N observations of ξ, or
  - generated via Monte Carlo sampling
- for any $x \in X$ estimate $f(x)$ by averaging values $F(x, \xi_j)$

$$\text{(SAA):} \min_{x \in X} \left\{ \hat{f}_N(x) := \frac{1}{N} \sum_{j=1}^{N} F(x, \xi_j) \right\}$$

- Nice theoretical asymptotic properties
- Can use standard optimization tools to solve the SAA problem
- EMP = SLP $\Longrightarrow$ SAA $\Longrightarrow$ (large scale) LP
- Continuous distributions, sampling functions, density estimation

Convergence

N	Time(s)	Soln	Profit
1000	0.6	(265,0,662,34)	18050
2000	1.0	(254,0,668,34)	18057
3000	1.6	(254,0,668,34)	18057
4000	2.3	(255,0,662,34)	18058
5000	3.1	(257,0,666,34)	18054
6000	3.9	(262,0,663,34)	18051
7000	5.0	(257,0,666,34)	18054
8000	6.1	(262,0,663,34)	18048
9000	7.3	(257,0,666,34)	18051
1m	262.0	(257,0,666,34)	18051

SAA can work well, but this is a 4 variable problem and distributions are discrete

Continuous distributions

- Can sample from continuous distributions (configurable on options)

  ```
  randvar d normal 45 10
  sample d 9
  setSeed 101
  ```

  The second line determines the size of the sample of the distribution of the random variable D to be 9.
- Variance reduction: e.g. LINDO solver provides three methods for reducing the variance: Monte Carlo sampling, Latin Square sampling and Antithetic sampling
- Can use user supplied sampling libraries
- Sampling can be separated from solution (i.e. can generate discrete approximation and then solve using existing algorithms)

Parametric distributions supported

Distribution	Parameter 1	Parameter 2	Parameter 3
Beta	shape 1	shape 2	
Cauchy	location	scale	
Chi_Square	deg. of freedom		
Exponential	lambda		
Gamma	shape	scale	
...	...	...	...
LogNormal	mean	std dev	
Normal	mean	std dev	
StudentT	deg. of freedom		
Triangular	low	mid	high
Uniform	low	high	
Weibull	shape	scale	
Binomial	n	p	
Geometric	p		
Logarithmic	p-factor		
Poisson	lambda		

What is EMP?

Annotates existing equations/variables/models for modeler to provide/define additional structure

- equilibrium
- vi (agents can solve min/max/vi)
- bilevel (reformulate as MPEC, or as SOCP)
- disjunction (or other constraint logic primitives)
- randvar
- dualvar (use multipliers from one agent as variables for another)
- extended nonlinear programs (library of plq functions)

Currently available within GAMS

Extension to MOPEC: agents solve a Stochastic Program

Buy y_i contracts in period 1, to deliver $D(\omega)y_i$ in period 2, scenario ω

Each agent i:

$$\begin{aligned} \min \quad & C(x_i^1) + \sum_\omega \pi_\omega C(x_i^2(\omega)) \\ \text{s.t.} \quad & p^1 x_i^1 + v y_i \leq p^1 e_i^1 & \text{(budget time 1)} \\ & p^2(\omega) x_i^2(\omega) \leq p^2(\omega)(D(\omega) y_i + e_i^2(\omega)) & \text{(budget time 2)} \end{aligned}$$

$$0 \leq v \perp -\sum_i y_i \geq 0 \qquad \text{(contract)}$$

$$0 \leq p^1 \perp \sum_i (e_i^1 - x_i^1) \geq 0 \qquad \text{(walras 1)}$$

$$0 \leq p^2(\omega) \perp \sum_i \left(D(\omega) y_i + e_i^2(\omega) - x_i^2(\omega)\right) \geq 0 \qquad \text{(walras 2)}$$

Conclusions

- Uncertainty is present everywhere. We need not only to quantify it, but we need to hedge/control/ameliorate it
- EMP model type available within GAMS, is clear and extensible, additional structure available to solver (cf COR, STOC files, etc)
- Provides SP technology to application domains (cf AMPL and Pyomo extensions)
- Add links to specialized solvers beyond current group (e.g. stochastic integer programming)
- More general stochastic structure needed at modeling level
- Specialized methodology for probabilistic contraints
- Create environment where advanced SP can be used by modelers