Extended Mathematical Programming: Competition and Stochasticity

Michael C. Ferris

Joint work with: Michael Bussieck, Steven Dirkse, and Alexander Meeraus

University of Wisconsin, Madison

Initiative for Computational Economics (ICE) Conference, University of Chicago, Chicago
July 27, 2012
The PIES Model (Hogan)

\[ \begin{align*}
\min_x & \quad c^T x \\
\text{s.t.} & \quad Ax = d(p) \\
& \quad Bx = b \\
& \quad x \geq 0
\end{align*} \]

- Issue is that \( p \) is the multiplier on the “balance” constraint of LP
- Extended Mathematical Programming (EMP) facilitates annotations of models to describe additional structure
- `empinfo: dualvar p balance`
- Can solve the problem by writing down the KKT conditions of this LP, forming an LCP and exposing \( p \) to the model
- EMP does this automatically from the annotations
Model building with EMP

- Take one system of (nonlinear) equations and annotate them to:
  - form a simple nonlinear program (no annotations)
  - form a complementarity problem from an embedded optimization problem (nlp with side constraints outside of optimizers control)
  - form an equilibrium model consisting of optimality conditions of several nlp’s along with equilibrium constraints (MOPEC)
  - form a bilevel program (an optimization problem with optimization problems as constraints)
  - Can assign multipliers (prices) from one sub-model as variables in another model
  - Can reformulate nonsmooth models using duality
  - Can introduce random variables into a model

- The annotations essentially detail who controls which equations and variables
Spatial Price Equilibrium

\[ n \in \{1, 2, 3, 4, 5, 6\} \]
\[ L \in \{1, 2, 3\} \]

Supply quantity: \( S_L \)
Production cost: \( \Psi(S_L) = .. \)
Spatial Price Equilibrium

\[ n \in \{1, 2, 3, 4, 5, 6\} \]
\[ L \in \{1, 2, 3\} \]

Supply quantity: \( S_L \)
Production cost: \( \Psi(S_L) = \ldots \)
Demand: \( D_L \)
Unit demand price: \( \theta(D_L) = \ldots \)
Spatial Price Equilibrium

\[ n \in \{1, 2, 3, 4, 5, 6\} \]
\[ L \in \{1, 2, 3\} \]

Supply quantity: \( S_L \)
Production cost: \( \Psi(S_L) = .. \)
Demand: \( D_L \)
Unit demand price: \( \theta(D_L) = .. \)
Transport: \( T_{ij} \)
Unit transport cost: \( c_{ij}(T_{ij}) = .. \)

One large system of equations and inequalities to describe this (GAMS).
Nonlinear Program Model (Monopolist)

- One producer controlling all regions
- Full knowledge of demand system
- Full knowledge of transportation system

\[
\begin{align*}
\max_{(D,S,T) \in \mathcal{F}} & \quad \sum_{l \in L} \theta_l(D_l)D_l - \sum_{l \in L} \Psi_l(S_l) - \sum_{i,j} c_{ij}(T_{ij})T_{ij} \\
\text{s.t.} & \quad S_l + \sum_{i,l} T_{il} = D_l + \sum_{l,j} T_{lj}, \quad \forall l \in L
\end{align*}
\]

EMP = NLP
2 agents: NLP + VI Model (Monopolist)

- One producer controlling all regions
- Full knowledge of demand system
- Price-taker in transportation system

\[
\max_{(D,S,T) \in \mathcal{F}} \sum_{l \in L} \theta_l(D_l)D_l - \sum_{l \in L} \psi_l(S_l) - \sum_{i,j} c_{ij}(T_{ij})T_{ij} \\
\text{s.t. } S_l + \sum_{i,l} T_{il} = D_l + \sum_{l,j} T_{lj}, \quad \forall l \in L
\]

\[p_{ij} = c_{ij}(T_{ij})\]  

empinfo: vi tcDef tc
2 agents: NLP + VI Model (Monopolist)

- One producer controlling all regions
- Full knowledge of demand system
- Price-taker in transportation system

\[
\begin{align*}
\max_{(D,S,T) \in \mathcal{F}} & \quad \sum_{l \in L} \theta_l(D_l)D_l - \sum_{l \in L} \Psi_l(S_l) - \sum_{i,j} c_{ij}(T_{ij})T_{ij} \\
\text{s.t.} & \quad S_l + \sum_{i,l} T_{il} = D_l + \sum_{l,j} T_{lj}, \quad \forall l \in L \\
& \quad p_{ij} = c_{ij}(T_{ij})
\end{align*}
\]

\[\text{EMP} = \text{MOPEC} \implies \text{MCP}\]
Classic SPE Model (NLP + VI agents)

- One producer controlling all regions
- Price-taker in demand system
- Price-taker in transportation system

\[
\begin{align*}
\max_{(D,S,T) \in \mathcal{F}} & \quad \sum_{l \in L} \theta_l(D_l) D_l - \sum_{l \in L} \Psi_l(S_l) - \sum_{i,j} c_{ij}(T_{ij}) T_{ij} \\
\text{s.t.} & \quad S_l + \sum_{i,l} T_{il} = D_l + \sum_{l,j} T_{lj}, \quad \forall l \in L \\
pi_l & = \theta_l(D_l) \\
p_{ij} & = c_{ij}(T_{ij})
\end{align*}
\]

empinfo: vi tcDef tc
vi pricedef price
Classic SPE Model (NLP + VI agents)

- One producer controlling all regions
- Price-taker in demand system
- Price-taker in transportation system

\[
\begin{align*}
\max_{(D,S,T) \in F} & \quad \sum_{l \in L} \theta_l(D_l)D_l - \sum_{l \in L} \Psi_l(S_l) - \sum_{i,j} c_{ij}(T_{ij})T_{ij} \\
\text{s.t.} & \quad S_l + \sum_{i,l} T_{il} = D_l + \sum_{l,j} T_{lj}, \quad \forall l \in L \\
\end{align*}
\]

\[p_{ij} = c_{ij}(T_{ij}) \quad (2)\]
\[\pi_l = \theta_l(D_l) \quad (3)\]

\text{EMP} = \text{MOPEC} \implies \text{MCP}
Cournot-Nash equilibrium (multiple agents)

Assumes that each agent (producer):

- Treats other agent decisions as fixed
- Is a price-taker in transport and demand

**EMP info file**

```
equilibrium
max obj('one') vars('one') eqns('one')
max obj('two') vars('two') eqns('two')
max obj('three') vars('three') eqns('three')
vi tcDef tc
vi pricedef price
```

\[ \text{EMP} = \text{MOPEC} \implies \text{MCP} \]
Bilevel Program (Stackelberg)

- Assumes one leader firm, the rest follow
- Leader firm optimizes subject to expected follower behavior
- Follower firms act in a Nash manner
- All firms are price-takers in transport and demand

EMP info file

bilevel obj('one') vars('one') eqns('one')
max obj('two') vars('two') eqns('two')
max obj('three') vars('three') eqns('three')
vi tcDef tc
vi pricedef price

EMP = bilevel \implies \text{MPEC} \implies (\text{via NLPEC}) \text{NLP}(\mu)
What is EMP?

Annotates existing equations/variables/models for modeler to provide/define additional structure

- equilibrium
- vi (agents can solve min/max/vi)
- bilevel (reformulate as MPEC, or as SOCP)
- disjunction (or other constraint logic primitives)
- randvar
- dualvar (use multipliers from one agent as variables for another)
- extended nonlinear programs (library of plq functions)

Currently available within GAMS
Power Systems: Economic Dispatch

\[
\min_{(q,z,\theta) \in \mathcal{F}} \sum_k C(q_k) \quad \text{s.t.} \quad q_k - \sum_{(l,c)} z(k,l,c) = d_k
\]

- Independent System Operator (ISO) determines who generates what
- \( p_k \): Locational marginal price (LMP) at \( k \)
- Volatile in “stressed” system
- Can we shed load from consumers to smooth prices?
- FERC (regulator) writes the rules - how to implement?
Understand: demand response and FERC Order No. 745

\[ \min_{q, z, \theta, R, p} \sum_k p_k R_k \]

s.t. \( C_1 \geq \sum_k p_k d_k / \sum_k d_k \)

\[ C_2 \geq \sum_k p_k (q_k + R_k) / \sum_k (d_k - R_k) \]

\[ 0 \leq R_k \leq u_k, \]

and \((q, z, \theta)\) solves \( \min_{(q, z, \theta) \in \mathcal{F}} \sum_k C(q_k) \)

s.t. \( q_k - \sum_{(l, c)} z_{(k, l, c)} = d_k - R_k \) \hspace{1cm} (1)

where \( p_k \) is the multiplier on constraint (1)
Solution Process (F./Liu)

- Bilevel program (hierarchical model)
- Upper level objective involves multipliers on lower level constraints
- Extended Mathematical Programming (EMP) annotates model to facilitate communicating structure to solver
  - dualvar p balance
  - bilevel R min cost q z θ balance . . .
- Automatic reformulation as an MPEC (single optimization problem with equilibrium constraints)
- Model solved using NLPEC and Conopt
- bilevel $\rightarrow$ MPEC $\rightarrow$ NLP
- Potential for solution of “consumer level” demand response
- Challenge: devise robust algorithms to exploit this structure for fast solution
Stability and feasibility

case 14, $D = 650, L = 150$

$\$/MW

Total DR (MW)

$\$/MW

Total DR (MW)

case 30, $D = 320, L = 42$

$\$/MW

Total DR (MW)

$\$/MW

Total DR (MW)

case 57, $D = 1600, L = 220$

$\$/MW

Total DR (MW)

$\$/MW

Total DR (MW)

case 118, $D = 9500, L = 540$

$\$/MW

Total DR (MW)

$\$/MW

Total DR (MW)
Operational view: LMP, Demand, Response
Alternative models: ED, avg, max, weighted avg

Fig. 7. Simulation on the 300-bus case
Fig. 8. Simulation on the 2383-bus case

• DR3: Replace the objective (12) by

$$\sum_{k \in B} \omega_k \lambda_k R_k$$

where $\omega_k$ is a weight parameter on node $k$, to incorporate a relative "reluctancy" factor regarding the dispatch of demand response at different nodes.

An illustrative experiment is performed on the 14-bus case with the results presented in Figure 9. The total demand level is set to 650 MW and the line limit is 150 MW on every line. While ED1 gives an AvgLMP of $73.14/MW, we set $C_1$ to $60/MW, as depicted by the horizontal dotted lines in the subplots. As before, we enforce no artificial bounds on $R_k$ by setting $u_R = d_k$ for each $k \in B$. For DR3, we set $\omega_2 = 2$ and $\omega_k = 1$, $\forall k \in B / \{2\}$ to express that we are relatively reluctant to dispatch demand response at node 2 compared to other nodes. For each node indicated on the horizonal axis, the light bar represents the LMP level and the dark bar represents the dispatched DR level at this node. Note that the LMP and DR levels share the same scale along the vertical axis but have different units, i.e. LMP is measured in $$/MW and DR is measured in MW.

Fig. 9. Comparison of DR model variants

As seen in the figure, DR1 was able to reduce the AvgLMP by dispatching about 31 MW of demand response at node 2.
Extension: The smart grid

- The next generation electric grid will be more dynamic, flexible, constrained, and more complicated.
- Decision processes (in this environment) are predominantly hierarchical.
- Models to support such decision processes must also be layered or hierarchical.
- Optimization and computation facilitate adaptivity, control, treatment of uncertainties and understanding of interaction effects.
- Developing interfaces and exploiting hierarchical structure using computationally tractable algorithms will provide FLEXIBILITY, overall solution speed, understanding of localized effects, and value for the coupling of the system.
Representative decision-making timescales in electric power systems

Figure 1: Representative decision-making timescales in electric power systems

A monster model is difficult to validate, inflexible, prone to errors.
Combine: Transmission Line Expansion Model (F./Tang)

\[
\min_{x \in X} \sum_{\omega} \pi_\omega \sum_{i \in N} d_i^\omega p_i^\omega(x)
\]

- Nonlinear system to describe power flows over (large) network
- Multiple time scales
- Dynamics (bidding, failures, ramping, etc)
- Uncertainty (demand, weather, expansion, etc)

\( p_i^\omega(x) \): Price (LMP) at \( i \) in scenario \( \omega \) as a function of \( x \)

Use other models to construct approximation of \( p_i^\omega(x) \)
Generator Expansion (2): $\forall f \in F$:
\[
\min_{y_f} \sum_{\omega} \pi_\omega \sum_{j \in G_f} C_j(y_j, q_j^\omega) - r(h_f - \sum_{j \in G_f} y_j)
\]
subject to $\sum_{j \in G_f} y_j \leq h_f, y_f \geq 0$

Market Clearing Model (3): $\forall \omega$:
\[
\min_{z,\theta,q^\omega} \sum_f \sum_{j \in G_f} C_j(y_j, q_j^\omega) \quad \text{s.t.}
\]
\[
q_j^\omega - \sum_{i \in I(j)} z_{ij} = d_j^\omega \quad \forall j \in N(\perp p_j^\omega)
\]
\[
z_{ij} = \Omega_{ij}(\theta_i - \theta_j) \quad \forall (i,j) \in A
\]
\[
- b_{ij}(x) \leq z_{ij} \leq b_{ij}(x) \quad \forall (i,j) \in A
\]
\[
u_j(y_j) \leq q_j^\omega \leq \overline{u}_j(y_j)
\]
$G_f$: Generators of firm $f \in F$
$y_f$: Investment in generator $j$
$q_j^\omega$: Power generated at bus $j$ in scenario $\omega$
$C_j$: Cost function for generator $j$
$r$: Interest rate
$z_{ij}$: Real power flowing along line $ij$
$q_j^\omega$: Real power generated at bus $j$ in scenario $\omega$
$\theta_i$: Voltage phase angle at bus $i$
$\Omega_{ij}$: Susceptance of line $ij$
$b_{ij}(x)$: Line capacity as a function of $x$
$u_j(y)$, $\overline{u}_j(y)$: Generator $j$ limits as a function of $y$
Solution approach

- Use derivative free method for the upper level problem (1)
- Requires $p_i^\omega(x)$
- Construct these as multipliers on demand equation (per scenario) in an Economic Dispatch (market clearing) model
- But transmission line capacity expansion typically leads to generator expansion, which interacts directly with market clearing
- Interface blue and black models using Nash Equilibria (as EMP):

empinfo: equilibrium
forall $f$: min expcost($f$) y($f$) budget($f$)
forall $\omega$: min scencost($\omega$) q($\omega$) . . .
Feasibility

KKT of \( \min_{y_f \in Y} \sum_{\omega} \sum_{j \in G_f} C_j(y_j, q_j^\omega) - r(h_f - \sum_{j \in G_f} y_j) \) \( \forall f \in F \) \hspace{1cm} (2)

KKT of \( \min_{(z, \theta, q^\omega) \in Z(x, y)} \sum_f \sum_{j \in G_f} C_j(y_j, q_j^\omega) \) \( \forall \omega \) \hspace{1cm} (3)

- Models (2) and (3) form a complementarity problem (CP via EMP)
- Solve (3) as NLP using global solver (actual \( C_j(y_j, q_j^\omega) \) are not convex), per scenario (SNLP) this provides starting point for CP
- Solve (KKT(2) + KKT(3)) using EMP and PATH, then repeat
- Identifies CP solution whose components solve the scenario NLP’s (3) to global optimality
<table>
<thead>
<tr>
<th>Scenario</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>Demand Multiplier</td>
<td>8</td>
<td>5.5</td>
</tr>
</tbody>
</table>

**SNLP (1):**

<table>
<thead>
<tr>
<th>Scenario</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$q_3$</th>
<th>$q_6$</th>
<th>$q_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>3.05</td>
<td>4.25</td>
<td>3.93</td>
<td>4.34</td>
<td>3.39</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>4.41</td>
<td>4.07</td>
<td>4.55</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Scenario</td>
<td>$\omega_1$</td>
<td>$\omega_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>----------</td>
<td>-----------</td>
<td>-----------</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Probability</td>
<td>0.5</td>
<td>0.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Demand Multiplier</td>
<td>8</td>
<td>5.5</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**SNLP (1):**

<table>
<thead>
<tr>
<th>Scenario</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$q_3$</th>
<th>$q_6$</th>
<th>$q_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>3.05</td>
<td>4.25</td>
<td>3.93</td>
<td>4.34</td>
<td>3.39</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>4.41</td>
<td>4.07</td>
<td>4.55</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**EMP (1):**

<table>
<thead>
<tr>
<th>Scenario</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$q_3$</th>
<th>$q_6$</th>
<th>$q_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>2.86</td>
<td>4.60</td>
<td>4.00</td>
<td>4.12</td>
<td>3.38</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>4.70</td>
<td>4.09</td>
<td>4.24</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Firm</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
<th>$y_6$</th>
<th>$y_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1$</td>
<td>167.83</td>
<td>565.31</td>
<td></td>
<td></td>
<td>266.86</td>
</tr>
<tr>
<td>$f_2$</td>
<td></td>
<td></td>
<td>292.11</td>
<td>207.89</td>
<td></td>
</tr>
<tr>
<td>Scenario</td>
<td>$\omega_1$</td>
<td>$\omega_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>------------</td>
<td>------------</td>
<td>------------</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Probability</td>
<td>0.5</td>
<td>0.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Demand Multiplier</td>
<td>8</td>
<td>5.5</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**SNLP (2):**

<table>
<thead>
<tr>
<th>Scenario</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$q_3$</th>
<th>$q_6$</th>
<th>$q_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>0.00</td>
<td>5.35</td>
<td>4.66</td>
<td>5.04</td>
<td>3.91</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>4.70</td>
<td>4.09</td>
<td>4.24</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Scenario</td>
<td>$\omega_1$</td>
<td>$\omega_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>----------</td>
<td>------------</td>
<td>------------</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Probability</td>
<td>0.5</td>
<td>0.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Demand Multiplier</td>
<td>8</td>
<td>5.5</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**SNLP (2):**

<table>
<thead>
<tr>
<th>Scenario</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$q_3$</th>
<th>$q_6$</th>
<th>$q_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>0.00</td>
<td>5.35</td>
<td>4.66</td>
<td>5.04</td>
<td>3.91</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>4.70</td>
<td>4.09</td>
<td>4.24</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**EMP (2):**

<table>
<thead>
<tr>
<th>Scenario</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$q_3$</th>
<th>$q_6$</th>
<th>$q_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>0.00</td>
<td>5.34</td>
<td>4.62</td>
<td>5.01</td>
<td>3.99</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>4.71</td>
<td>4.07</td>
<td>4.25</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Firm</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
<th>$y_6$</th>
<th>$y_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1$</td>
<td>0.00</td>
<td>622.02</td>
<td>283.22</td>
<td>216.79</td>
<td>377.98</td>
</tr>
</tbody>
</table>
Observations

- But this is simply one function evaluation for the outer “transmission capacity expansion” problem
- Number of critical arcs typically very small
- But in this case, $p_{j}^{\omega}$ are very volatile
- Outer problem is small scale, objectives are open to debate, possibly ill-conditioned
- Economic dispatch should use AC power flow model
- Structure of market open to debate
- Types of “generator expansion” also subject to debate
- Suite of tools is very effective in such situations
Water rights pricing (Britz/F./Kuhn)

Models firms behavior with market to determine water rights
Agents have stochastic recourse?

- Two stage stochastic programming, $x$ is here-and-now decision, recourse decisions $y$ depend on realization of a random variable $R$
- $R$ is a risk measure (e.g. expectation, CVaR)

SP: $\min c^T x + R[q^T y]$

s.t. $Ax = b, \quad x \geq 0,$

$\forall \omega \in \Omega : \quad T(\omega)x + W(\omega)y(\omega) \leq d(\omega),$

$y(\omega) \geq 0.$
Design: Stochastic competing agent models (F./Wets)

- Competing agents (consumers, or generators in energy market)
- Each agent minimizes objective independently (cost)
- Market prices are function of all agents activities
- Additional twist: model must “hedge” against uncertainty
- Facilitated by allowing contracts bought now, for goods delivered later
- Conceptually allows to transfer goods from one period to another (provides wealth retention or pricing of ancilliary services in energy market)
- Can investigate new instruments to mitigate risk, or move to system optimal solutions from equilibrium (or market) solutions
Example as MOPEC: agents solve a Stochastic Program

Each agent minimizes:

\[ u_a = (\kappa - f(q_{a,0,*}))^2 + \sum_s \pi_s (\kappa - f(q_{a,s,*}))^2 \]

Budget time 0:

\[ \sum_i p_{0,i} q_{a,0,i} + \sum_j v_j y_{a,j} \leq \sum_i p_{0,i} e_{a,0,i} \]

Budget time 1:

\[ \sum_i p_{s,i} q_{a,s,i} \leq \sum_i p_{s,i} \sum_j D_{s,i,j} y_{a,j} + \sum_i p_{s,i} e_{a,s,i} \]

Additional constraints (complementarity) outside of control of agents:

(contract) \[ 0 \leq - \sum_a y_{a,j} \perp v_j \geq 0 \]

(walras) \[ 0 \leq - \sum_a d_{a,s,i} \perp p_{s,i} \geq 0 \]
Model and solve

- Can model financial instruments such as “financial transmission rights”, “spot markets”, “reactive power markets”
- Reduce effects of uncertainty, not simply quantify
- Use structure in preconditioners
  - Use nonsmooth Newton methods to formulate complementarity problem
  - Solve each “Newton” system using GMRES
  - Precondition using “individual optimization” with fixed externalities

\[
\begin{align*}
\begin{array}{c}
\text{Region 1} \\
\text{Region 2} \\
\text{Region 3}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Region 1} \\
\text{Region 2} \\
\text{Region 3}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Region 1} \\
\text{Region 2} \\
\text{Region 3}
\end{array}
\end{align*}
\]

= +
The problem
A furniture maker can manufacture and sell four different dressers. Each dresser requires a certain number $t$ of man-hours for carpentry, and a certain number $t_{fj}$ of man-hours for finishing, $j = 1, \ldots, 4$. In each period, there are $d_c$ man-hours available for carpentry, and $d_f$ available for finishing. There is a (unit) profit $\bar{c}_j$ per dresser of type $j$ that’s manufactured. The owner’s goal is to maximize total profit:

$$\max_{x \geq 0} \quad 12x_1 + 25x_2 + 21x_3 + 40x_4 \quad \quad (profit)$$

subject to

$$4x_1 + 9x_2 + 7x_3 + 10x_4 \leq 6000 \quad \quad (carpentry)$$
$$x_1 + x_2 + 3x_3 + 40x_4 \leq 4000 \quad \quad (finishing)$$

Succinctly:

$$\max_{x} \quad c^T x \text{ s.t. } Tx \leq d, x \geq 0$$
Is your time estimate that good?

- The time for carpentry and finishing for each dresser cannot be known with certainty.
- Each entry in $T$ takes on four possible values with probability $1/4$, independently.
- 8 entries of $T$ are random variables: $s = 65,536$ different $T$’s each with same probability of occurring.
- But decide “now” how many dressers $x$ of each type to build.
- Might have to pay for overtime (for carpentry and finishing).
- Can make different overtime decision $y^s$ for each scenario $s$ - recourse!
Models with explicit random variables

- **Model transformation:**
  - Write a core model as if the random variables are constants
  - Identify the random variables and decision variables and their staging
  - Specify the distributions of the random variables
  - emp.info: model transformation

```plaintext
randvar T('c','1') discrete .25 3.60 .25 3.90 .25 4.10 .25 4.40
randvar T('c','2') discrete .25 8.25 .25 8.75 .25 9.25 .25 9.75
randvar T('c','3') discrete .25 6.85 .25 6.95 .25 7.05 .25 7.15
randvar T('c','4') discrete .25 9.25 .25 9.75 .25 10.25 .25 10.75
randvar T('f','1') discrete .25 0.85 .25 0.95 .25 1.05 .25 1.15
randvar T('f','2') discrete .25 0.85 .25 0.95 .25 1.05 .25 1.15
randvar T('f','3') discrete .25 2.60 .25 2.90 .25 3.10 .25 3.40
randvar T('f','4') discrete .25 37.00 .25 39.00 .25 41.00 .25 43.00
randvar d('c') discrete .25 5873. .25 5967. .25 6033. .25 6127.
randvar d('f') discrete .25 3936. .25 3984. .25 4016. .25 4064.
```

- **Solver configuration:**
  - Specify the manner of sampling from the distributions
  - Determine which algorithm (and parameter settings) to use

- **Output handling:**
  - Optionally, list the variables for which we want a scenario-by-scenario report
What do we learn?

- Deterministic solution: $x_d = (1333, 0, 0, 67)$
- Expected profit using this solution: $\$16,942$
- Expected (averaged) overtime costs: $\$1,725$
- Extensive form solution: $x_s = (257, 0, 666, 34)$ with expected profit $\$18,051$

- Deterministic solution is not optimal for stochastic program, but more significantly it isn’t getting us on the right track!

- Stochastic solution suggests large number of “type 3” dressers, while deterministic solution has none!
Computation methods matter!

- Lindo solver defaults: 825 seconds
- Lindo solver barrier method: 382 seconds
- CPLEX solver barrier method: 4 seconds (8 threads)

1. May have multiple sources of uncertainty: e.g. man-hours $d$ also can take on 4 values in each setting independently: $s = 1,048,576$

2. Generates extensive form problem with over 3 million rows and columns and 29 million nonzeros

3. Solves on 24 threaded cluster machine in 262 secs
Sampling methods

But what if the number of scenarios is too big (or the probability distribution is not discrete)? use sample average approximation (SAA)

- Take sample \( \xi_1, \ldots, \xi_N \) of \( N \) realizations of random vector \( \xi \)
  - viewed as historical data of \( N \) observations of \( \xi \), or
  - generated via Monte Carlo sampling

- for any \( x \in X \) estimate \( f(x) \) by averaging values \( F(x, \xi_j) \)

\[
(SAA): \min_{x \in X} \left\{ \hat{f}_N(x) := \frac{1}{N} \sum_{j=1}^{N} F(x, \xi_j) \right\}
\]

- Nice theoretical asymptotic properties
- Can use standard optimization tools to solve the SAA problem
- \( \text{EMP} = \text{SLP} \implies \text{SAA} \implies \text{(large scale) LP} \)
SAA can work well, but this is a 4 variable problem and distributions are discrete.
Continuous distributions: News vendor problems (F./Liu)

<table>
<thead>
<tr>
<th>$N$</th>
<th>Derand Mean</th>
<th>Derand Stddev</th>
<th>SAA Mean</th>
<th>SAA Stddev</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>16.85</td>
<td>2.185</td>
<td>16.94</td>
<td>3.615</td>
</tr>
<tr>
<td>5</td>
<td>14.84</td>
<td>1.369</td>
<td>14.92</td>
<td>2.791</td>
</tr>
<tr>
<td>10</td>
<td>14.23</td>
<td>1.127</td>
<td>14.57</td>
<td>2.248</td>
</tr>
<tr>
<td>20</td>
<td>14.03</td>
<td>0.797</td>
<td>14.18</td>
<td>1.635</td>
</tr>
<tr>
<td>100</td>
<td>14.01</td>
<td>0.100</td>
<td>14.48</td>
<td>0.745</td>
</tr>
</tbody>
</table>

1. As the sample size $N$ increases, the optimal solutions obtained by both methods converge to the true solution, i.e. 14.

2. For a given sample size $N$, new sampling method (derand) is always (slightly) closer to the true solution.

3. But standard deviation of the optimal solutions obtained by derand is significantly smaller than the SAA method.
Additional techniques requiring extensive computation

- Continuous distributions, sampling functions, density estimation
- Chance constraints: \( \text{Prob}(T_i x + W_i y_i \geq h_i) \geq 1 - \alpha \) - can reformulate as MIP and adapt cuts (Luedtke) empinfo: chance E1 E2 0.95
- Use of discrete variables (in submodels) to capture logical or discrete choices (logmip - Grossmann et al)
- Robust or stochastic programming
- Decomposition approaches to exploit underlying structure identified by EMP
- Nonsmooth penalties and reformulation approaches to recast problems for existing or new solution methods (ENLP)
- Conic or semidefinite programs - alternative reformulations that capture features in a manner amenable to global computation
Conclusions

- Optimization helps understand what drives a system
- Modern optimization within applications requires multiple model formats, computational tools and sophisticated solvers
- EMP model type is clear and extensible, additional structure available to solver
- Extended Mathematical Programming available within the GAMS modeling system
- Uncertainty is present everywhere (the world is not “normal”)
- We need not only to quantify it, but we need to hedge/control/ameliorate it
- Modeling, optimization, and computation embedded within the application domain is critical