Extended Mathematical Programming: Competition and Stochasticity

Michael C. Ferris

Joint work with: Michael Bussieck, Steven Dirkse, Jan Jagla and Alexander Meeraus

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University of Wisconsin, Madison

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Optimization in 2011

- Optimization models improve understanding of underlying systems and facilitate operational/strategic improvements under resource constraints
- Optimization is a mature field - many algorithms, accompanying theory, broad application.
- Who is driving the bus? data, application or optimization

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- Optimization models improve understanding of underlying systems and facilitate operational/strategic improvements under resource constraints
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- Who is driving the bus? data, application or optimization
- In practice: need (large scale) data, problem/model transformations, access to solution features
- Modeling systems (AIMMS, AMPL, ... , GAMS, ...) provide some of these needs from an optimization perspective
- Often need specialized model formats or data handling:
  - GDX container, Matlab, R, Excel,...
  - Ré and Recht (2011): specialized algorithmic primitives
  - Application conduits: e.g. PDE constrained optimization (inverse models, adjoint calculations)

What can optimization do well right now?

- Discrete and continuous models, exact and heuristic solutions
- Classical format models used extensively in applications

$$\min_{x \in X} f(x) \text{ s.t. } g(x) \leq 0, h(x) = 0$$

- Newer modeling formats provide richer capabilities:
  - Mixed integer (linear) programming
  - Semidefinite programming
  - Risk measures, robust and stochastic optimization, predictive modeling
  - Sparse optimization /compressed sensing
  - Distributed (multi-agent) models and equilibria
- Algorithms: papers, (commercial) libraries, source code
- See NEOS wiki (www.neos-guide.org) or try out NEOS solvers (www.neos-solvers.org) for extensive examples

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- Why aren't you using my *********** algorithm?
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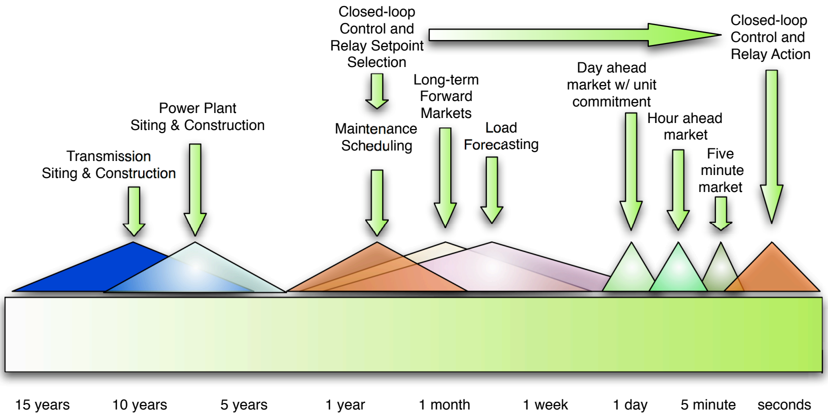
- Experts will do anything, users much less accomodating
- Show me on a problem like mine
- Must run on defaults
- Must deal graciously with poorly specified cases
- Must be usable from my environment (Matlab, R, GAMS, ...)
- Must be able to model my problem easily

Extended Mathematical Programs (EMP) provide annotations to an existing optimization model that convey new model structures to a solver

Example: The smart grid

- The next generation electric grid will be more dynamic, flexible, constrained, and more complicated.
- Decision processes (in this environment) are predominantly hierarchical.
- Models to support such decision processes must also be layered or hierachical.
- Optimization and computation facilitate adaptivity, control, treatment of uncertainties and understanding of interaction effects.
- Developing interfaces and exploiting hierarchical structure using computationally tractable algorithms will provide overall solution speed, understanding of localized effects, and value for the coupling of the system.

Representative decision-making timescales in electric power systems

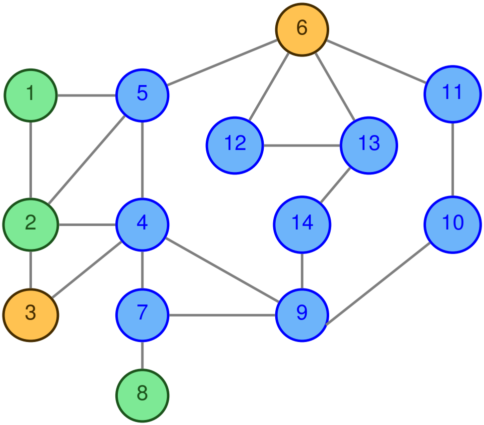


A monster model is difficult to validate, inflexible, prone to errors.

Example: Transmission Line Expansion Model (1)

$$\min_{x \in X} \quad \sum_{\omega} \pi_\omega \sum_{i \in N} d_i^\omega p_i^\omega(x)$$

$$s.t. \quad Ax \leq b$$



N:	The set of nodes
X:	Line expansion set
x:	Amount of investment in given line
ω:	Demand scenarios
π_ω:	Scenario prob
d_i^ω:	Demand (load at i in scenario ω)
$p_i^\omega(x)$:	Price (LMP) at i in scenario ω as a function of x

Generator Expansion (2): $\forall f \in F$:

$$\min_{y_f} \sum_{\omega} \pi_\omega \sum_{j \in G_f} C_j(y_j, q_j^\omega) - r(h_f - \sum_{j \in G_f} y_j)$$

$$\text{s.t.} \sum_{j \in G_f} y_j \le h_f, y_f \ge 0$$

G_f: Generators of firm $f \in F$
y_j: Investment in generator j
q_j^ω: Power generated at bus j in scenario ω
C_j: Cost function for generator j
r: Interest rate

Market Clearing Model (3): $\forall \omega$:

$$\min_{z,\theta,q^\omega} \sum_f \sum_{j \in G_f} C_j(y_j, q_j^\omega) \quad \text{s.t.}$$

$$q_j^\omega - d_j^\omega = \sum_{i \in I(j)} z_{ij} \qquad \forall j \in N(\perp p_j^\omega)$$

$$z_{ij} = \Omega_{ij}(\theta_i - \theta_j) \qquad \forall (i,j) \in A$$

$$-b_{ij}(x) \le z_{ij} \le b_{ij}(x) \qquad \forall (i,j) \in A$$

$$\underline{u}_j(y_j) \le q_j^\omega \le \overline{u}_j(y_j)$$

z_{ij}: Real power flowing along line ij
q_j^ω: Real power generated at bus j in scenario ω
θ_i: Voltage phase angle at bus i
Ω_{ij}: Susceptance of line ij
$b_{ij}(x)$: Line capacity as a function of x
$\underline{u}_j(y)$, $\overline{u}_j(y)$: Generator j limits as a function of y

How to combine: Nash Equilibria

- Non-cooperative game: collection of players $a \in \mathcal{A}$ whose individual objectives depend not only on the selection of their own strategy $x_a \in C_a = \mathrm{dom} f_a(\cdot, x_{-a})$ but also on the strategies selected by the other players $x_{-a} = \{x_a : o \in \mathcal{A} \setminus \{a\}\}$.
- Nash Equilibrium Point:

$$\bar{x}_{\mathcal{A}} = (\bar{x}_a, a \in \mathcal{A}) : \forall a \in \mathcal{A} : \bar{x}_a \in \mathrm{argmin}_{x_a \in C_a} f_a(x_a, \bar{x}_{-a}).$$

1. for all $x \in \mathcal{A}$, $f_a(\cdot, x_{-a})$ is convex
2. $C = \prod_{a \in \mathcal{A}} C_a$ and for all $a \in \mathcal{A}$, C_a is closed convex.

VI reformulation

Define

$$G : \mathbb{R}^N \mapsto \mathbb{R}^N \text{ by } G_a(x_{\mathcal{A}}) = \partial_a f_a(x_a, x_{-a}), a \in \mathcal{A}$$

where ∂_a denotes the subgradient with respect to x_a. Generally, the mapping G is set-valued.

Theorem

Suppose the objectives satisfy (1) and (2), then every solution of the variational inequality

$$x_{\mathcal{A}} \in C \text{ such that } -G(x_{\mathcal{A}}) \in N_C(x_{\mathcal{A}})$$

is a Nash equilibrium point for the game.
Moreover, if C is compact and G is continuous, then the variational inequality has at least one solution that is then also a Nash equilibrium point.

Solution approach (Tang)

- Use derivative free method for the upper level problem (1)
- Requires $p_i^{\omega}(x)$
- Construct these as multipliers on demand equation (per scenario) in an Economic Dispatch (market clearing) model
- But transmission line capacity expansion typically leads to generator expansion, which interacts directly with market clearing
- Interface blue and black models using Nash Equilibria (as EMP):

empinfo: equilibrium
forall f: min expcost(f) y(f) budget(f)
forall ω: min scencost(ω) q(ω) ...

Flow of information

$$\min_{x \in X} \quad \sum_{\omega} \pi_\omega \sum_{i \in N} d_i^\omega p_i^\omega(x)$$

$$\text{s.t.} \quad \min_{y_f \in Y} \sum_{\omega} \pi_\omega \sum_{j \in G_f} C_j(y_j, q_j^\omega) - r(h_f - \sum_{j \in G_f} y_j) \qquad \forall f \in F$$

$$\min_{z,\theta,q^\omega} \sum_{f} \sum_{j \in G_f} C_j(y_j, q_j^\omega) \qquad \forall \omega$$

$$s.t. \quad q_j^\omega - d_j^\omega = \sum_{i \in I(j)} z_{ij} \qquad \forall j \in N (\perp p_j^\omega(x))$$

$$z_{ij} = \Omega_{ij}(\theta_i - \theta_j) \qquad \forall (i,j) \in A$$

$$-b_{ij}(x) \leq z_{ij} \leq b_{ij}(x) \qquad \forall (i,j) \in A$$

$$\underline{u}_j(y_j) \leq q_j^\omega \leq \overline{u}_j(y_j) \qquad \forall j \in N$$

Feasibility

$$\text{KKT of } \min_{y_f \in Y} \sum_{\omega} \pi_\omega \sum_{j \in G_f} C_j(y_j, q_j^\omega) - r(h_f - \sum_{j \in G_f} y_j) \quad \forall f \in F \quad (2)$$

$$\text{KKT of } \min_{(z,\theta,q^\omega) \in Z(x,y)} \sum_{f} \sum_{j \in G_f} C_j(y_j, q_j^\omega) \quad \forall \omega \quad (3)$$

- Models (2) and (3) form an MCP/VI (via EMP)
- Solve (3) as NLP using global solver (actual $C_j(y_j, q_j^\omega)$ are not convex), per scenario (SNLP) this provides starting point for MCP
- Solve (KKT(2) + KKT(3)) using EMP and PATH, then repeat
- Identifies MCP solution whose components solve the scenario NLP's (3) to global optimality

Scenario	ω_1	ω_2
Probability	0.5	0.5
Demand Multiplier	8	5.5

SNLP (1):

Scenario	q_1	q_2	q_3	q_6	q_8
ω_1	3.05	4.25	3.93	4.34	3.39
ω_2		4.41	4.07	4.55	

EMP (1):

Scenario	q_1	q_2	q_3	q_6	q_8
ω_1	2.86	4.60	4.00	4.12	3.38
ω_2		4.70	4.09	4.24	

Firm	y_1	y_2	y_3	y_6	y_8
f_1	167.83	565.31			266.86
f_2			292.11	207.89	

Scenario	ω_1	ω_2
Probability	0.5	0.5
Demand Multiplier	8	5.5

SNLP (2):

Scenario	q_1	q_2	q_3	q_6	q_8
ω_1	0.00	5.35	4.66	5.04	3.91
ω_2		4.70	4.09	4.24	

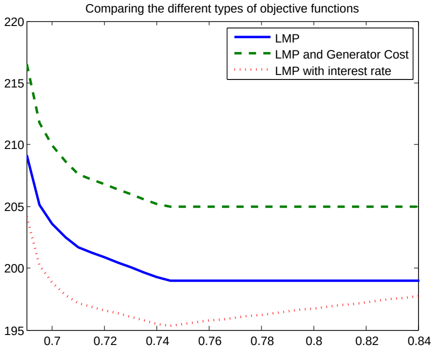
EMP (2):

Scenario	q_1	q_2	q_3	q_6	q_8
ω_1	0.00	5.34	4.62	5.01	3.99
ω_2		4.71	4.07	4.25	

Firm	y_1	y_2	y_3	y_6	y_8
f_1	0.00	622.02			377.98
f_2			283.22	216.79	

Observations

- But this is simply one function evaluation for the outer "transmission capacity expansion" problem
- Number of critical arcs typically very small
- But in this case, p_j^ω are very volatile
- Outer problem is small scale, objectives are open to debate, possibly ill conditioned



- Economic dispatch should use AC power flow model
- Structure of market open to debate
- Types of "generator expansion" also subject to debate
- Suite of tools is very effective in such situations

The benefits of sub-model building

- Coupling collections of (sub)-models with well defined (information sharing) interfaces facilitates:
  - appropriate detail and consistency of sub-model formulation (each of which may be very large scale, of different types (mixed integer, semidefinite, nonlinear, variational, etc) with different properties (linear, convex, discrete, smooth, etc))
  - ability for individual subproblem solution verification and engagement of decision makers
  - ability to treat uncertainty by stochastic and robust optimization at submodel level and with evolving resolution
  - ability to solve submodels to global optimality (by exploiting size, structure and model format specificity)

(A monster model that mixes several modeling formats loses its ability to exploit the underlying structure and provide guarantees on solution quality)

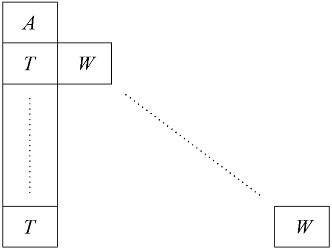
II: Stochastic programming and risk measures

$$\text{SP:} \quad \min \quad c^\top x + \mathbb{R}[d^\top y]$$

$$\text{s.t.} \quad Ax = b, \quad x \geq 0,$$

$$\forall \omega \in \Omega : \quad T(\omega)x + W(\omega)y(\omega) \geq h(\omega),$$

$$y(\omega) \geq 0.$$



- Two stage stochastic programming, x is here-and-now decision, recourse decisions y depend on realization of a random variable
- $\mathbb{R}$ is a risk measure (e.g. expectation, CVaR)
- Deterministic equivalent is a large scale optimization problem
- Multi-stage problems are natural generalization

Models with explicit random variables

- Model transformation:
  - Write a core model as if the random variables are constants
  - Identify the random variables and decision variables and their staging
  - Specify the distributions of the random variables
- Solver configuration:
  - Specify the manner of sampling from the distributions
  - Determine which algorithm (and parameter settings) to use
- Output handling:
  - Optionally, list the variables for which we want a scenario-by-scenario report

Example: Farm Model (core model)

- Allocate land (L) for planting crops x to max (p/wise lin) profit
- Yield rate per crop c is $F{*}Y(c)$
- Can purchase extra crops b and sell s, but must have enough crops d to feed cattle

$$\begin{array}{ll} \max_{x,b,s\geq 0} & profit = p(x,b,s) \\ \text{s.t.} & \sum_c x(c) \leq L, \\ & F{*}Y(c) * x(c) + b(c) - s(c) \geq d(c) \end{array}$$

- Random variables are F, realized at stage 2: structured $T(\omega)$
- Variables x stage 1, b and s stage 2.
- landuse constraints in stage 1, requirements in stage 2.

Can now generate the *deterministic equivalent* problem or pass on directly to specialized solver

Stochastic Programming in GAMS

Three separate pieces of information needed

1. emp.info: model transformation

```
randvar F 2 discrete 0.25 0.8 // below
                     0.50 1.0 // avg
                     0.25 1.2 // above

stage 2 b s req
```

2. solver.opt: solver configuration (benders, sampling strategy, etc)

```
4 "ISTRAT"  * solve universe problem (DECIS/Benders)
```

3. dictionary: output handling (where to put all the "scenario solutions")

How does this help?

- Clarity/simplicity of model
- Separates solution process from model description
- Models can be solved by deterministic equivalent, exisiting codes such as LINDO and DECIS, or decomposition approaches such as Benders, ATR, etc
- Allows description of compositional (nonlinear) random effects in generating ω

$$\text{i.e. } \omega = \omega_1 \times \omega_2, \; T(\omega) = f(X(\omega_1), Y(\omega_2))$$

- Easy to write down multi-stage problems
- Automatically generates "COR", "TIM" and "STO" files for Stochastic MPS (SMPS) input

Example: Portfolio Model (core model)

- Determine portfolio weights w_j for each of a collection of assets
- Asset returns v are random, but jointly distributed
- Portfolio return $r(w, v)$
- Minimize a "risk" measure

$$\begin{array}{ll} \max & 0.2 * \mathbb{E}(r) + 0.8 * CVaR(r) \\ \text{s.t.} & r = \sum_j v_j * w_j \\ & \sum_j w_j = 1, \; w \geq 0 \end{array}$$

- Jointly distributed random variables v, realized at stage 2
- Variables: portfolio weights w in stage 1, returns r in stage 2
- Coherent risk measures $\mathbb{E}$ and $CVaR$

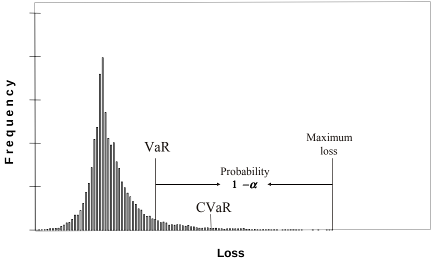
Risk Measures

$\overline{CVaR}_\alpha$: mean of upper tail beyond α-quantile (e.g. $\alpha = 0.95$)

- Classical: utility/disutility $u(\cdot)$:

$$\min_{x \in X} f(x) = \mathbb{E}[u(F(x, \xi))]$$

- Modern approach to modeling risk aversion uses concept of risk measures



- mean-risk, semi-deviations, mean deviations from quantiles, VaR, CVaR
- Römisch, Schultz, Rockafellar, Urasyev (in Math Prog literature)
- Much more in mathematical economics and finance literature
- Optimization approaches still valid, different objectives

EMP version

- emp.info: model transformation

```
expected_value EV_r r
cvarlo         CVaR_r r alpha
stage          2 r defr
jrandvar       v("att") v("gmc") v("usx") 2 discrete
   table of probabilities and outcomes
```

- Variables are assigned to $\mathbb{E}(r)$ and $\underline{CVaR}_\alpha(r)$; can be used in model (appropriately) for objective, constraints, or be bounded
- Problem transformation: Theorem states this expression can be written as convex optimization using:

$$\underline{CVaR}_\alpha(r) = \max_{a \in \mathbb{R}} \left\{ a - \frac{1}{\alpha} \sum_{j=1}^{N} Prob_j * (a - r_j)_+ \right\}$$

Example: Clear Lake Model (core model)

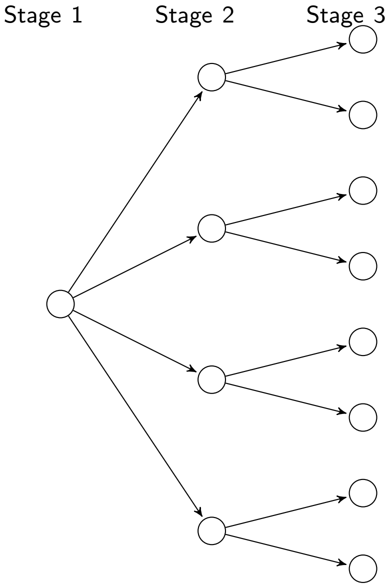
- Water levels $l(t)$ in dam for each month t
- Determine what to release normally $r(t)$, what then floods $f(t)$ and what to import $z(t)$
- minimize cost of flooding and import
- Change in reservoir level in period t is $\delta(t)$

$$\begin{aligned} &\max cost = c(f, z) \\ &\text{s.t. } l(t) = l(t-1) + \delta(t) + z(t) - r(t) - f(t) \end{aligned}$$

- Random variables are δ, realized at stage t, $t \geq 2$.
- Variables l, r, f, z in stage t, $t \geq 2$.
- balance constraint at t in stage t.

Example of a multi-stage stochastic program.

Multi to 2 stage reformulation



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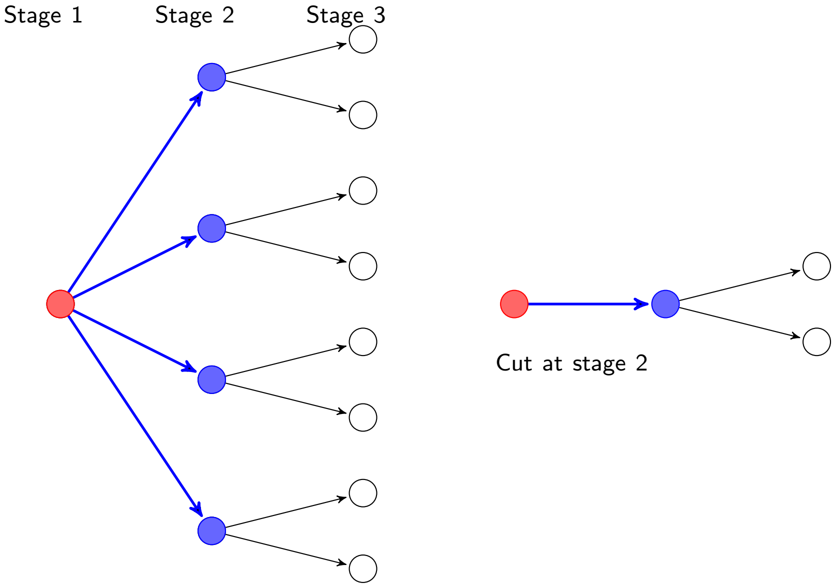

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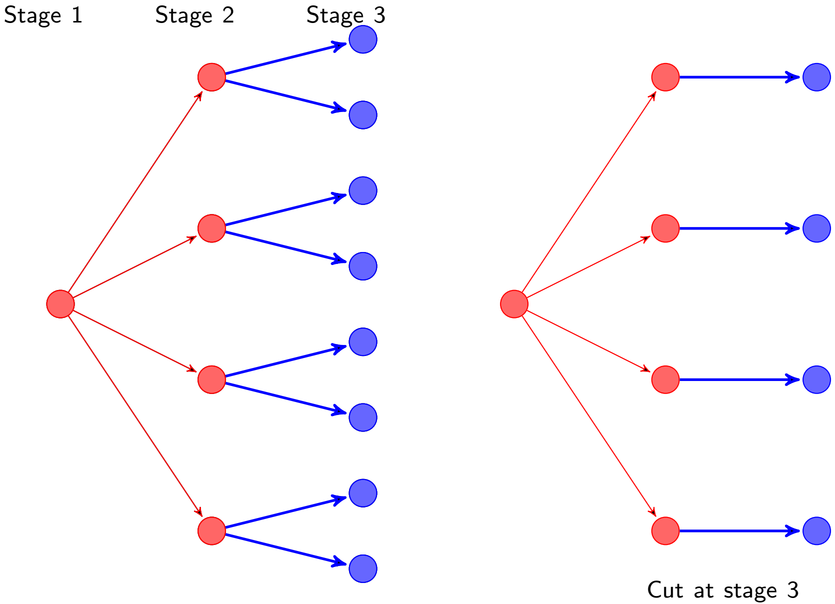

Solution options

- Form the deterministic equivalent
- Solve using LINDO api (stochastic solver)
- Convert to two stage problem and solve using DECIS or any number of competing methods
- Problem with $3^{40} \approx 1.2 * 10^{19}$ realizations in stage 2
  - DECIS using Benders and Importance Sampling: < 1 second (and provides confidence bounds)
  - CPLEX on a presampled deterministic equivalent:

sample	samp. time(s)	CPLEX time(s) for solution			cols (mil)
500	0.0	5	(4.5 barrier,	0.5 xover)	0.25
1000	0.2	18	(16 barrier,	2 xover)	0.5
10000	28	195	(44 barrier,	151 xover)	5
20000	110	1063	(98 barrier,	965 xover)	10

Additional techniques requiring extensive computation

- Continuous distributions
- Chance constraints: $Prob(T_i x + W_i y_i \geq h_i) \geq 1 - \alpha$ - can reformulate as MIP and adapt cuts (Luedtke)
- Use of discrete variables (in submodels) to capture logical or discrete choices (logmip - Grossmann et al)
- Optimization of simulation or noisy functions
- Robust or stochastic programming
- Decomposition approaches to exploit underlying structure identified by EMP
- Nonsmooth penalties and reformulation approaches to recast problems for existing or new solution methods (ENLP)
- Conic or semidefinite programs - alternative reformulations that capture features in a manner amenable to global computation

Conclusions

- Modern optimization within applications requires multiple model formats, computational tools and sophisticated solvers
- EMP model type is clear and extensible, additional structure available to solver
- Extended Mathematical Programming available within the GAMS modeling system
- Able to pass additional (structure) information to solvers
- Embedded optimization models automatically reformulated for appropriate solution engine
- Exploit structure in solvers
- Extend application usage further

slides available at http://www.cs.wisc.edu/∼ferris/talks