An Extended Mathematical Programming Framework

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Extended Mathematical Programs

- Optimization models improve understanding of underlying systems and facilitate operational/strategic improvements under resource constraints
- Problem format is old/traditional

\[
\min_{x} f(x) \text{ s.t. } g(x) \leq 0, \ h(x) = 0
\]

- Extended Mathematical Programs allow annotations of constraint functions to augment this format.
- This talk will give several examples of how to use this modeling framework
But who cares?

- Why aren’t you using my ************ algorithm? (Michael Ferris, Boulder, CO, 1994)
But who cares?

- Why aren’t you using my ************ algorithm? (Michael Ferris, Boulder, CO, 1994)
- Show me on a problem like mine
- Must run on defaults
- Must deal graciously with poorly specified cases
- Must be usable from my environment (Matlab, R, GAMS, …)
- Must be able to model my problem easily

EMP provides annotations to an existing optimization model that convey new model structures to a solver
NEOS is soliciting case studies that show how to do the above, and will provide some tools to help
The PIES Model (Hogan)

\[
\begin{align*}
\min_x & \quad c^T x \\
\text{s.t.} & \quad Ax = q(p) \\
& \quad Bx = b \\
& \quad x \geq 0
\end{align*}
\]

- Issue is that \( p \) is the multiplier on the dembal constraint of LP
- Can solve the problem by writing down the KKT conditions of this LP, forming an LCP and exposing \( p \) to the model
- \textbf{EMP}: dualvar \( p \) dembal
Example: Bimatrix Games

- Nash game: two players have $I$ and $J$ pure strategies.
- $p$ and $q$ (strategy probabilities) belong to unit simplex $\triangle_I$ and $\triangle_J$ respectively.
- Payoff matrices $A \in \mathbb{R}^{J \times I}$ and $B \in \mathbb{R}^{I \times J}$, where $A_{j,i}$ is the profit received by the first player if strategy $i$ is selected by the first player and $j$ by the second, etc.
- The expected profit for the first and the second players are $q^T A p$ and $p^T B q$ respectively.
- A Nash equilibrium is reached by the pair of strategies $(p^*, q^*)$ if and only if

$$p^* \in \arg \min_{p \in \triangle_I} \langle A q^*, p \rangle \quad \text{and} \quad q^* \in \arg \min_{q \in \triangle_J} \langle B^T p^*, q \rangle$$

- EMP: facilitates modeling of Nash Equilibria
Complementarity Problems in Economics (MCP)

- $p$ represents prices, $x$ represents activity levels
- System model: given prices, (agent) $i$ determines activities $x_i$

\[ G_i(x_i, x_{-i}, p) = 0 \]

$x_{-i}$ are the decisions of other agents.
- Walras Law: market clearing

\[ 0 \leq S(x, p) - D(x, p) \perp p \geq 0 \]

- Key difference: optimization assumes you control the complete system
- Complementarity determines what activities run, and who produces what
Nash Equilibria

- Nash Games: $x^*$ is a Nash Equilibrium if

$$x_i^* \in \arg \min_{x_i \in X_i} \ell_i(x_i, x^*_i, q), \forall i \in \mathcal{I}$$

$x_{-i}$ are the decisions of other players.

- Quantities $q$ given exogenously, or via complementarity:

$$0 \leq H(x, q) \perp q \geq 0$$

- **empinfo**: equilibrium
  min loss(i) x(i) cons(i)
  vifunc H q

General Equilibrium models

\((C) : \max_{x_k \in X_k} U_k(x_k) \text{ s.t. } p^T x_k \leq i_k(y, p)\)

\((I) : i_k(y, p) = p^T \omega_k + \sum_j \alpha_{kj} p^T g_j(y_j)\)

\((P) : \max_{y_j \in Y_j} p^T g_j(y_j)\)

\((M) : \max_{p \geq 0} p^T \left( \sum_k x_k - \sum_k \omega_k - \sum_j g_j(y_j) \right) \text{ s.t. } \sum_l p_l = 1\)
General Equilibrium models

\[(C) : \max_{x_k \in X_k} U_k(x_k) \text{ s.t. } p^T x_k \leq i_k(y, p)\]

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\[(P) : \max_{y_j \in Y_j} p^T g_j(y_j)\]

\[(M) : \max_{p \geq 0} \begin{pmatrix} \sum_k x_k - \sum_k \omega_k - \sum_j g_j(y_j) \end{pmatrix} \text{ s.t. } \sum_l p_l = 1\]

Can reformulate as embedded problem (Ermoliev et al):

\[\max_{x \in X, y \in Y} \sum_k \frac{t_k}{\beta_k} \log U_k(x_k)\]

\[\text{ s.t. } \sum_k x_k \leq \sum_k \omega_k + \sum_j g_j(y_j)\]

\[t_k = i_k(y, p) \text{ where } p \text{ is multiplier on NLP constraint}\]
Sequential Joint Maximization

\[
\max_{x \in X, y \in Y} \sum_k \frac{t_k}{\beta_k} \log U_k(x_k)
\]

s.t. \[
\sum_k x_k \leq \sum_k \omega_k + \sum_j g_j(y_j)
\]

\[t_k = i_k(y, p)\] where \(p\) is multiplier on NLP constraint

- Embedded model often solves faster as an MCP than the original MCP from Nash game
- Can exploit structure to improve computational performance further
Sequential Joint Maximization

$$\max_{x \in X, y \in Y} \sum_k \frac{t_k}{\beta_k} \log U_k(x_k)$$

s.t. $$\sum_k x_k \leq \sum_k \omega_k + \sum_j g_j(y_j)$$

$$t_k = i_k(y, p)$$ where $p$ is multiplier on NLP constraint

- Embedded model often solves faster as an MCP than the original MCP from Nash game
- Can exploit structure to improve computational performance further
- Can iterate (on $m$) $t^m_k = i_k(y^m, p^m)$, and solve sequence of NLP's

$$\max_{x \in X, y \in Y} \sum_k \frac{t^m_k}{\beta_k} \log U_k(x_k)$$

s.t. $$\sum_k x_k \leq \sum_k \omega_k + \sum_j g_j(y_j)$$
Stochastic competing agent models (with Wets)

- Competing agents (consumers, or generators in energy market)
- Each agent maximizes objective independently (utility)
- Market prices are function of all agents activities
- Additional twist: model must “hedge” against uncertainty
- Facilitated by allowing contracts bought now, for goods delivered later
- Conceptually allows to transfer goods from one period to another (provides wealth retention or pricing of ancilliary services in energy market)
The model details: c.f. Brown, Demarzo, Eaves

Each agent maximizes:

\[ u_h = - \sum_s \pi_s \left( \kappa - \prod_l c_{h,s,l}^{\alpha_{h,l}} \right) \]

Time 0:

\[ \sum_l p_{0,l} c_{h,0,l} + \sum_k q_k z_{h,k} \leq \sum_l p_{0,l} e_{h,0,l} \]

Time 1:

\[ \sum_l p_{s,l} c_{h,s,l} \leq \sum_l p_{s,l} \sum_k D_{s,l,k} z_{h,k} + \sum_l p_{s,l} e_{h,s,l} \]

Additional constraints (complementarity) outside of control of agents:

\[ 0 \leq - \sum_h z_{h,k} \perp q_k \geq 0 \]

\[ 0 \leq - \sum_h d_{h,s,l} \perp p_{s,l} \geq 0 \]
Stochastic programming and risk measures

$$\text{SP: min } \begin{array}{l} c^\top x + \mathbb{R}[d^\top y] \\ \text{s.t. } A x = b \\ T(\omega)x + W(\omega)y(\omega) \geq h(\omega), \quad \text{for all } \omega \in \Omega, \\ x \geq 0, \quad y(\omega) \geq 0, \quad \text{for all } \omega \in \Omega. \end{array}$$

Annotations are slightly more involved but straightforward:

- Need to describe probability distribution
- Define (multi-stage) structure (what variables and constraints belong to each stage)
- Define random parameters and process to generate scenarios
- Can also define risk measures on variables

Automatic reformulation (deterministic equivalent), solvers such as DECIS, etc.
An Overview of the Power Systems Network Extended Mathematical Programs: Hierarchical Models

...as well as AC power flow proxy constraints in the ISO problems.

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Transmission Line Capacity Expansion

Ferris (Univ. Wisconsin)

Implementing the EMP Framework framework, prudent when accounting for randomness in long-term planning.

between models (2) and (3) return price as a response variable to model (1). Abstracting Model (1) determines the parameters in model (3)'s constraints, and the interplay of the optimalization problem of each individual firm and model (3) represents the day-ahead model (1), which considers price as a response variable. This feedback is the outcome of the interaction across firms (green and blue), and there are three demand nodes (orange).

In this example, there are four generating nodes that belong to two generating firms, and there are three demand nodes. There are line flow limits and the demand scales.

Finally note that the interactions across firms and across scenarios are limited, yielding a problem (MCP) and thus can be solved by a number of algorithms, e.g. PATH.

The equilibrium problem can be treated by juxtaposing the first-order optimality conditions of a single convex nonlinear program (NLP): an approach to an MCP may fail to converge, but in our case the MCP corresponds to the equilibrium model of the individual optimalization problems that make up the equilibrium model. This is the outcome of the day-ahead market clearing (3) (budget constraints) (gen capacity) (line data).

An Illustrated Example

The equilibrium MCP may be quite large, so it is natural to investigate decomposition algorithms. The most natural such algorithm for this problem is to solve successively each problem (MCP) and thus can be solved by a number of algorithms, e.g. PATH.

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First-order optimality conditions of a single convex nonlinear program (NLP): an approach to an MCP may fail to converge, but in our case the MCP corresponds to the equilibrium model of the individual optimalization problems that make up the equilibrium model. This is the outcome of the day-ahead market clearing (3) (budget constraints) (gen capacity) (line data).

Well accepted estimates cite a 35% growth in electricity demand over the next 20 years. Planning and operating the Next Generation Electricity Grid involves decisions across varying mescales, from mescales as short as a second to upward of a decade.

In order to realize the benefits of integrating a mix of energy resources (e.g. wind, solar, natural gas, etc.) and to manage variability and uncertainty in the system, power system operators need models that are both highly accurate and computationally efficient.

Parameter estimation is an integral part of model development and validation. Model parameterization must represent the underlying process accurately and be robust to uncertainty and variability in the data. The parameter estimation process is iterative and involves the use of optimization algorithms to fit the model to observed data.

Current modeling frameworks may be too rigid to sufficiently address the complexity of modern power systems. The need for flexible modeling frameworks that can capture the dynamic behavior of interconnected systems motivates the development of advanced modeling and optimization techniques.

There are three separate models. At the highest level is the Transmission Line Planning model, which considers the impact of network expansion on the price of electricity to the consumer. The next level is the Real-time Management model, which includes information about current system conditions and operational constraints. The lowest level is the Day-ahead Market Clearing model, which determines the market clearing price and dispatches power to meet load requirements.

The model hierarchy is designed to capture the interaction between planning, real-time management, and day-ahead market clearing. This allows for a more comprehensive understanding of the power system and enables more effective decision-making.

Abstracting Model (1) determines the parameters in model (3)'s constraints, and the interplay of the optimalization problem of each individual firm and model (3) represents the day-ahead model (1), which considers price as a response variable. This feedback is the outcome of the day-ahead market clearing (3) (budget constraints) (gen capacity) (line data).
### Transmission Line Planning (1)

\[
\min_{x \in \mathcal{X}} \sum_{\omega} \pi_\omega \sum_{i \in \mathcal{N}} d_{i}^{\omega} p_{i}^{\omega}(x)
\]

(budget constraints)

\[
s.t. \quad Ax \leq b
\]

\[
x \geq 0
\]

### Generator Expansion (2)

\[
\forall f \in \mathcal{F} : \min_{y_f} \sum_{\omega} \pi_\omega \sum_{j \in \mathcal{G}_f} C_f(y_j, q_j^{\omega})
\]

(budget constraints)

\[
s.t. \quad \sum_{j \in \mathcal{G}_f} y_j \leq h_f
\]

\[
y \geq 0
\]

### Day Ahead Market Clearing (3)

\[
\forall \omega : \min_{(z, \theta, q)} \sum_f \sum_{j \in \mathcal{G}} C_j(y_j, q_j^{\omega})
\]

(balance flow)

\[
s.t. \quad q_j^{\omega} - d_j^{\omega} = \sum_{i \in \mathcal{I}(j)} z_{ij} \quad \forall j \in \mathcal{N} \quad (\perp p_j^{\omega})
\]

(line data)

\[
z_{ij} = \Omega_{ij} * (\theta_i - \theta_j) \quad \forall (i, j) \in \mathcal{A}
\]

(line capacity)

\[
-b_{ij}(x) \leq z_{ij} \leq b_{ij}(x) \quad \forall (i, j) \in \mathcal{A}
\]

(gen capacity)

\[
\underline{u}_j \leq q_j \leq \bar{u}_j
\]

\[
\theta, z_{ij} \text{ free}
\]
Solution method

- Use derivative free method for the upper level problem (1)
- Constraints (2) and (3) form an MCP (via EMP)
- Can show (due to specific problem structure that there is a (convex) NLP whose KKT conditions are that MCP
- Useful for theoretical analysis
- Resulting problem is too large for NLP solvers
- Can show that “Gauss-Seidel/Jacobi” method on problems in (2) and (3) converges in this case - decoupling makes problem tractable for large scale instances
Conclusions

- Modern optimization within applications requires multiple model formats, computational tools and sophisticated solvers.
- EMP model type is clear and extensible, additional structure available to solver.
- Extended Mathematical Programming available within the GAMS modeling system.
- Able to pass additional (structure) information to solvers.
- Embedded optimization models automatically reformulated for appropriate solution engine.
- Exploit structure in solvers.
- Extend application usage further.