Extended Mathematical Programming: Competition and Stochasticity

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The PIES Model (Hogan)

$$\begin{aligned} \min_x \quad & c^T x && \text{cost} \\ \text{s.t.} \quad & Ax = d(p) && \text{balance} \\ & Bx = b && \text{technical constr} \\ & x \geq 0 \end{aligned}$$

- Issue is that p is the multiplier on the "balance" constraint of LP
- Extended Mathematical Programming (EMP) facilitates annotations of models to describe additional structure
- Can solve the problem by writing down the KKT conditions of this LP, forming an LCP and exposing p to the model
- EMP does this automatically from the annotations

Reformulation details

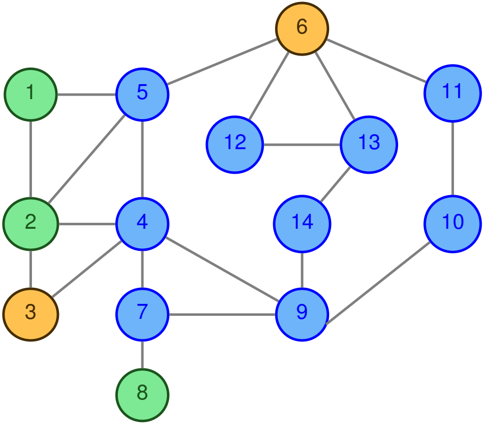
$$\begin{aligned} 0 &= Ax - d(p) & \perp \quad & \mu \\ 0 &= Bx - b & \perp \quad & \lambda \\ 0 &\leq -A^T\mu - B^T\lambda + c & \perp \quad & x \geq 0 \end{aligned}$$

- empinfo: dualvar p balance
- replaces $\mu \equiv p$
- LCP/MCP is then solvable using PATH

$$z = \begin{bmatrix} p \\ \lambda \\ x \end{bmatrix}, \quad F(z) = \begin{bmatrix} & & A \\ & & B \\ -A^T & -B^T & \end{bmatrix} z + \begin{bmatrix} -d(p) \\ -b \\ c \end{bmatrix}$$

Power Systems: Economic Dispatch

$$\min_{(q,z,\theta)\in\mathcal{F}} \sum_k C(q_k) \text{ s.t. } q_k - \sum_{(l,c)} z_{(k,l,c)} = d_k$$



- Independent System Operator (ISO) determines who generates what
- p_k: Locational marginal price (LMP) at k
- Volatile in "stressed" system
- Can we shed load from consumers to smooth prices?
- FERC (regulator) writes the rules - how to implement?

Understand: demand response and FERC Order No. 745

$$\min_{q,z,\theta,R,p} \sum_k p_k R_k$$

$$\text{s.t.}\; C_1 \geq \sum_k p_k d_k / \sum_k d_k$$

$$C_2 \geq \sum_k p_k (q_k + R_k) / \sum_k (d_k - R_k)$$

$$0 \leq R_k \leq u_k,$$

$$\text{and } (q, z, \theta) \text{ solves } \min_{(q,z,\theta) \in \mathcal{F}} \sum_k C(q_k)$$

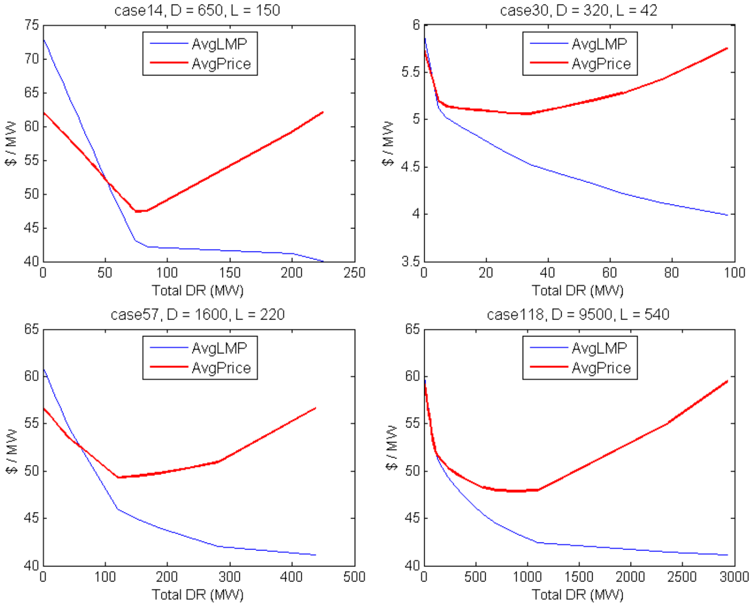
$$\text{s.t.} \quad q_k - \sum_{(l,c)} z_{(k,l,c)} = d_k - R_k \tag{1}$$

where p_k is the multiplier on constraint (1)

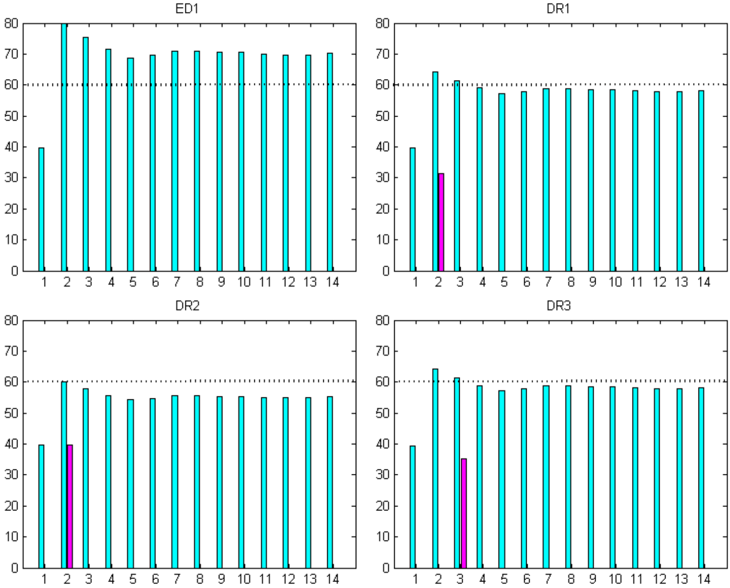
Solution Process (F./Liu)

- Bilevel program (hierarchical model)
- Upper level objective involves multipliers on lower level constraints
- Extended Mathematical Programming (EMP) annotates model to facilitate communicating structure to solver
  - dualvar p balance
  - bilevel R min cost q z θ balance ...
- Automatic reformulation as an MPEC (single optimization problem with equilibrium constraints)
- Model solved using NLPEC and Conopt
- bilevel $\implies$ MPEC $\implies$ NLP
- Potential for solution of "consumer level" demand response
- Challenge: devise robust algorithms to exploit this structure for fast solution

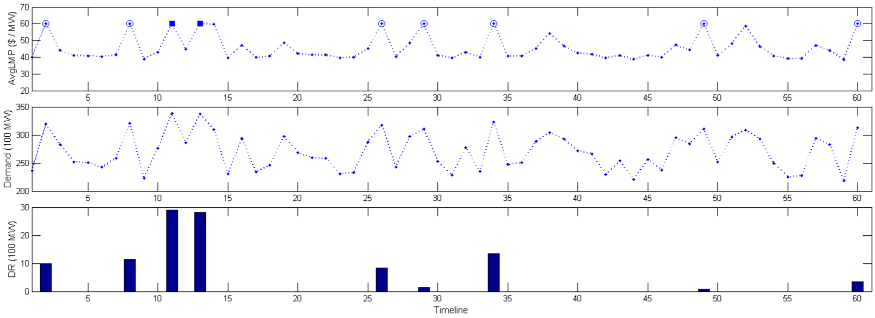
Stability and feasibility (vary C_1)



Alternative models: ED, avg, max, weighted avg



Operational view: LMP, Demand, Response



MOPEC

$$\min_{x_i} \theta_i(x_i, x_{-i}, y) \text{ s.t. } g_i(x_i, x_{-i}, y) \leq 0, \forall i$$

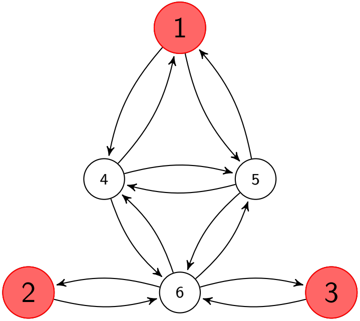
and

$$y \text{ solves } \mathrm{VI}(h(x, \cdot), C)$$

```
equilibrium
min theta(1) x(1) g(1)
...
min theta(m) x(m) g(m)
vi h y cons
```

is solved in a Nash manner

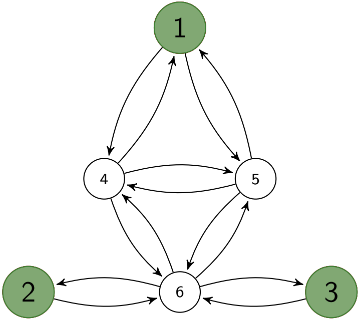
Spatial Price Equilibrium



$n \in \{1, 2, 3, 4, 5, 6\}$
$L \in \{1, 2, 3\}$

Supply quantity: S_L
Production cost: $\Psi(S_L) = ..$

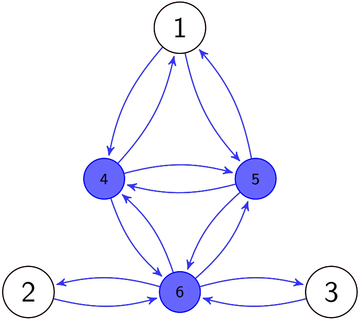
Spatial Price Equilibrium



$n \in \{1, 2, 3, 4, 5, 6\}$
$L \in \{1, 2, 3\}$

Supply quantity: S_L
Production cost: $\Psi(S_L) = ..$
Demand: D_L
Unit demand price: $\theta(D_L) = ..$

Spatial Price Equilibrium



$n \in \{1, 2, 3, 4, 5, 6\}$
$L \in \{1, 2, 3\}$

Supply quantity: S_L
Production cost: $\Psi(S_L) = ..$
Demand: D_L
Unit demand price: $\theta(D_L) = ..$
Transport: T_{ij}
Unit transport cost: $c_{ij}(T_{ij}) = ..$

One large system of equations and inequalities to describe this (GAMS).

$$\max_{(D,S,T)\in\mathcal{F}} \quad \sum_{l\in L} \pi_l D_l - \sum_{l\in L} \Psi_l(S_l) - \sum_{i,j} p_{ij} T_{ij}$$

$$\text{s.t.} \quad S_l + \sum_{i,l} T_{il} = D_l + \sum_{l,j} T_{lj}, \quad \forall l \in L$$

$$p_{ij} = c_{ij}(T_{ij}), \pi_l = \theta_l(D_l)$$

Cournot-Nash equilibrium (multiple agents)

Assumes that each agent (producer):

- Treats other agent decisions as fixed
- Is a price-taker in transport and demand

EMP info file

```
equilibrium
max obj('one') vars('one') eqns('one')
max obj('two') vars('two') eqns('two')
max obj('three') vars('three') eqns('three')
vi tcDef tc
vi pricedef price
```

EMP = MOPEC $\Longrightarrow$ MCP

Bilevel Program (Stackelberg)

- Assumes one leader firm, the rest follow
- Leader firm optimizes subject to expected follower behavior
- Follower firms act in a Nash manner
- All firms are price-takers in transport and demand

EMP info file

```
bilevel obj('one') vars('one') eqns('one')
max obj('two') vars('two') eqns('two')
max obj('three') vars('three') eqns('three')
vi tcDef tc
vi pricedef price
```

EMP = bilevel $\Longrightarrow$ MPEC $\Longrightarrow$ (via NLPEC) NLP(μ)

What is EMP?

Annotates existing equations/variables/models for modeler to provide/define additional structure

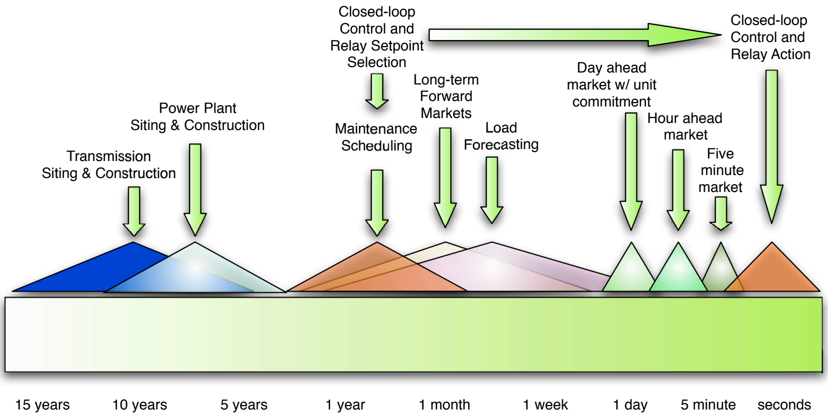
- equilibrium
- vi (agents can solve min/max/vi)
- bilevel (reformulate as MPEC, or as SOCP)
- disjunction (or other constraint logic primitives)
- randvar
- dualvar (use multipliers from one agent as variables for another)
- extended nonlinear programs (library of plq functions)

Currently available within GAMS

Extension: The smart grid

- The next generation electric grid will be more dynamic, flexible, constrained, and more complicated.
- Decision processes (in this environment) are predominantly hierarchical.
- Models to support such decision processes must also be layered or hierarchical.
- Optimization and computation facilitate adaptivity, control, treatment of uncertainties and understanding of interaction effects.
- Developing interfaces and exploiting hierarchical structure using computationally tractable algorithms will provide FLEXIBILITY, overall solution speed, understanding of localized effects, and value for the coupling of the system.

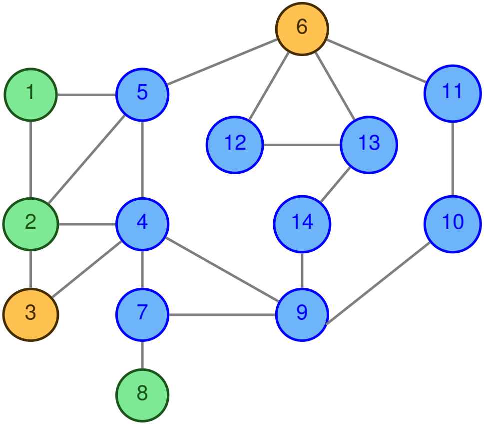
Representative decision-making timescales in electric power systems



A monster model is difficult to validate, inflexible, prone to errors.

Combine: Transmission Line Expansion Model (F./Tang)

$$\min_{x \in X} \sum_{\omega} \pi_\omega \sum_{i \in N} d_i^\omega p_i^\omega(x)$$



- Nonlinear system to describe power flows over (large) network
- Multiple time scales
- Dynamics (bidding, failures, ramping, etc)
- Uncertainty (demand, weather, expansion, etc)
- $p_i^\omega(x)$: Price (LMP) at i in scenario ω as a function of x
- Use other models to construct approximation of $p_i^\omega(x)$

Generator Expansion (2): $\forall f \in F$:

$$\min_{y_f} \sum_{\omega} \pi_\omega \sum_{j \in G_f} C_j(y_j, q_j^\omega) - r(h_f - \sum_{j \in G_f} y_j)$$

$$\text{s.t.} \sum_{j \in G_f} y_j \leq h_f, y_f \geq 0$$

G_f:	Generators of firm $f \in F$
y_j:	Investment in generator j
q_j^ω:	Power generated at bus j in scenario ω
C_j:	Cost function for generator j
r:	Interest rate

Market Clearing Model (3): $\forall \omega$:

$$\min_{z,\theta,q^\omega} \sum_f \sum_{j \in G_f} C_j(y_j, q_j^\omega) \quad \text{s.t.}$$

$$q_j^\omega - \sum_{i \in I(j)} z_{ij} = d_j^\omega \quad \forall j \in N (\perp p_j^\omega)$$

$$z_{ij} = \Omega_{ij}(\theta_i - \theta_j) \quad \forall (i,j) \in A$$

$$-b_{ij}(x) \leq z_{ij} \leq b_{ij}(x) \quad \forall (i,j) \in A$$

$$\underline{u}_j(y_j) \leq q_j^\omega \leq \overline{u}_j(y_j)$$

z_{ij}:	Real power flowing along line ij
q_j^ω:	Real power generated at bus j in scenario ω
θ_i:	Voltage phase angle at bus i
Ω_{ij}:	Susceptance of line ij
$b_{ij}(x)$:	Line capacity as a function of x
$\underline{u}_j(y)$, $\overline{u}_j(y)$:	Generator j limits as a function of y

Solution approach

- Use derivative free method for the upper level problem (1)
- Requires $p_i^\omega(x)$
- Construct these as multipliers on demand equation (per scenario) in an Economic Dispatch (market clearing) model
- But transmission line capacity expansion typically leads to generator expansion, which interacts directly with market clearing
- Interface blue and black models using Nash Equilibria (as EMP):

empinfo: equilibrium
forall f: min expcost(f) y(f) budget(f)
forall ω: min scencost(ω) q(ω) ...

Feasibility

$$\text{KKT of } \min_{y_f \in Y} \sum_{\omega} \pi_\omega \sum_{j \in G_f} C_j(y_j, q_j^\omega) - r(h_f - \sum_{j \in G_f} y_j) \qquad \forall f \in F \quad (2)$$

$$\text{KKT of } \min_{(z,\theta,q^\omega) \in Z(x,y)} \sum_{f} \sum_{j \in G_f} C_j(y_j, q_j^\omega) \qquad \forall \omega \quad (3)$$

- Models (2) and (3) form a complementarity problem (CP via EMP)
- Solve (3) as NLP using global solver (actual $C_j(y_j, q_j^\omega)$ are not convex), per scenario (SNLP) this provides starting point for CP
- Solve (KKT(2) + KKT(3)) using EMP and PATH, then repeat
- Identifies CP solution whose components solve the scenario NLP's (3) to global optimality

Scenario	ω_1	ω_2
Probability	0.5	0.5
Demand Multiplier	8	5.5

SNLP (1):

Scenario	q_1	q_2	q_3	q_6	q_8
ω_1	3.05	4.25	3.93	4.34	3.39
ω_2		4.41	4.07	4.55	

Scenario	ω_1	ω_2
Probability	0.5	0.5
Demand Multiplier	8	5.5

SNLP (1):

Scenario	q_1	q_2	q_3	q_6	q_8
ω_1	3.05	4.25	3.93	4.34	3.39
ω_2		4.41	4.07	4.55	

EMP (1):

Scenario	q_1	q_2	q_3	q_6	q_8
ω_1	2.86	4.60	4.00	4.12	3.38
ω_2		4.70	4.09	4.24	

Firm	y_1	y_2	y_3	y_6	y_8
f_1	167.83	565.31			266.86
f_2			292.11	207.89	

Scenario	ω_1	ω_2
Probability	0.5	0.5
Demand Multiplier	8	5.5

SNLP (2):

Scenario	q_1	q_2	q_3	q_6	q_8
ω_1	0.00	5.35	4.66	5.04	3.91
ω_2		4.70	4.09	4.24	

Scenario	ω_1	ω_2
Probability	0.5	0.5
Demand Multiplier	8	5.5

SNLP (2):

Scenario	q_1	q_2	q_3	q_6	q_8
ω_1	0.00	5.35	4.66	5.04	3.91
ω_2		4.70	4.09	4.24	

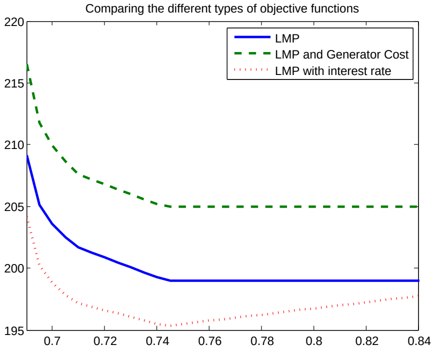
EMP (2):

Scenario	q_1	q_2	q_3	q_6	q_8
ω_1	0.00	5.34	4.62	5.01	3.99
ω_2		4.71	4.07	4.25	

Firm	y_1	y_2	y_3	y_6	y_8
f_1	0.00	622.02			377.98
f_2			283.22	216.79	

Observations

- But this is simply one function evaluation for the outer "transmission capacity expansion" problem
- Number of critical arcs typically very small
- But in this case, p_j^ω are very volatile
- Outer problem is small scale, objectives are open to debate, possibly ill conditioned

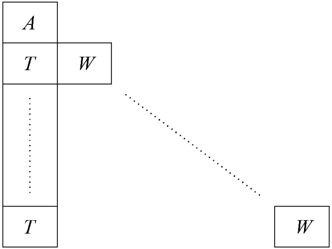


- Economic dispatch should use AC power flow model
- Structure of market open to debate
- Types of "generator expansion" also subject to debate
- Suite of tools is very effective in such situations

Agents have stochastic recourse?

- Two stage stochastic programming, x is here-and-now decision, recourse decisions y depend on realization of a random variable
- $\mathbb{R}$ is a risk measure (e.g. expectation, CVaR)

$$\begin{aligned} \text{SP:}\ \min\quad & c^\top x + \mathbb{R}[q^\top y] \\ \text{s.t.}\quad & Ax = b, \quad x \geq 0, \\ \forall \omega \in \Omega: \quad & T(\omega)x + W(\omega)y(\omega) \leq d(\omega), \\ & y(\omega) \geq 0. \end{aligned}$$



EMP/SP extensions to facilitate these models

PJM buy/sell model (2009)

- Storage transfers energy over time (horizon $= T$).
- PJM: given price path p_t, determine charge q_t^+ and discharge q_t^-:

$$\max_{h_t, q_t^+, q_t^-} \sum_{t=0}^{T} p_t (q_t^- - q_t^+)$$

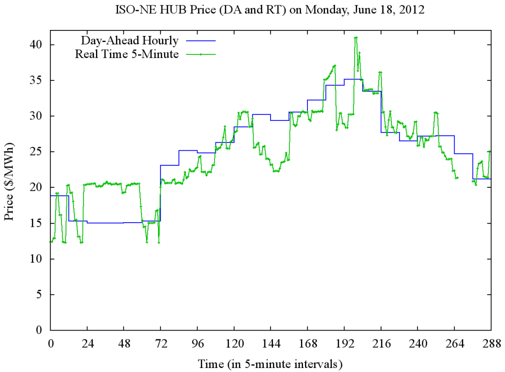
$$\text{s.t. } \partial h_t = e q_t^+ - q_t^-$$

$$0 \leq h_t \leq \mathcal{S}$$

$$0 \leq q_t^+ \leq \mathcal{Q}$$

$$0 \leq q_t^- \leq \mathcal{Q}$$

$$h_0, h_T \text{ fixed}$$



- Uses: price shaving, load shifting, transmission line deferral
- what about different storage technologies?

Stochastic price paths (day ahead market)

$$\min_{x,s,q^+,q^-} c^0(x) + \mathbb{E}_\omega \left[\sum_{t=0}^{T} p_{\omega t}(q^+_{\omega t} - q^-_{\omega t}) + c^1(q^+_{\omega t} + q^-_{\omega t}) \right]$$

$$\begin{aligned} \text{s.t. } & \partial h_{\omega t} = e q^+_{\omega t} - q^-_{\omega t} \\ & 0 \le h_{\omega t} \le \mathcal{S} x \\ & 0 \le q^+_{\omega t}, q^-_{\omega t} \le \mathcal{Q} x \\ & h_{\omega 0}, h_{\omega T} \text{ fixed} \end{aligned}$$

- First stage decision x: amount of storage to deploy.
- Second stage decision: charging strategy in face of uncertainty

Distribution of (multiple) storage types

Determine storage facilities x_k to build, given distribution of price paths: no entry barriers into market, etc. MOPEC: for all k solve a two stage stochastic program

$$\forall k : \min_{x_k, h_k, q_k^+, q_k^-} c_k^0(x_k) + \mathbb{E}_\omega \left[\sum_{t=0}^{T} p_{\omega t}(q_{\omega kt}^+ - q_{\omega kt}^-) + c_k^1(q_{\omega kt}^+ + q_{\omega kt}^-) \right]$$

$$\begin{aligned} \text{s.t. } & \partial h_{\omega kt} = e q_{\omega kt}^+ - q_{\omega kt}^- \\ & 0 \leq h_{\omega kt} \leq \mathcal{S} x_k \\ & 0 \leq q_{\omega kt}^+, q_{\omega kt}^- \leq \mathcal{Q} x_k \\ & h_{\omega k0}, h_{\omega kT} \text{ fixed} \end{aligned}$$

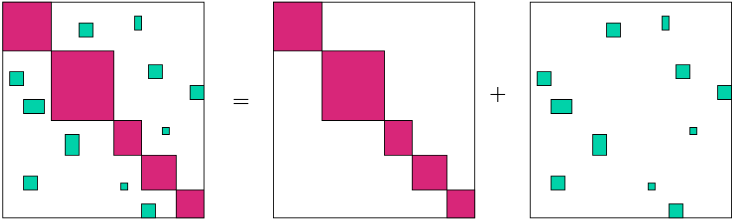
and

$$p_{\omega t} = f\left(\theta, \mathcal{D}_{\omega t} + \sum_k (q_{\omega kt}^+ - q_{\omega kt}^-)\right)$$

Parametric function (θ) determined by regression. Storage operators react to shift in demand.

Model and solve

- Can model financial instruments such as "financial transmission rights", "spot markets", "reactive power markets"
- Reduce effects of uncertainty, not simply quantify
- Use structure in preconditioners
  - Use nonsmooth Newton methods to formulate complementarity problem
  - Solve each "Newton" system using GMRES
  - Precondition using "individual optimization" with fixed externalities



Additional techniques requiring extensive computation

- Continuous distributions, sampling functions, density estimation
- Chance constraints: $Prob(T_i x + W_i y_i \geq h_i) \geq 1 - \alpha$ - can reformulate as MIP and adapt cuts (Luedtke) empinfo: chance E1 E2 0.95
- Use of discrete variables (in submodels) to capture logical or discrete choices (logmip - Grossmann et al)
- Robust or stochastic programming
- Decomposition approaches to exploit underlying structure identified by EMP
- Nonsmooth penalties and reformulation approaches to recast problems for existing or new solution methods (ENLP)
- Conic or semidefinite programs - alternative reformulations that capture features in a manner amenable to global computation

Conclusions

- Optimization helps understand what drives a system
- Modern optimization within applications requires multiple model formats, computational tools and sophisticated solvers
- EMP model type is clear and extensible, additional structure available to solver
- Extended Mathematical Programming available within the GAMS modeling system
- Uncertainty is present everywhere (the world is not "normal")
- We need not only to quantify it, but we need to hedge/control/ameliorate it
- Modeling, optimization, and computation embedded within the application domain is critical

Stochastic competing agent models (F./Wets)

- Competing agents (consumers, or generators in energy market)
- Each agent minimizes objective independently (cost)
- Market prices are function of all agents activities
- Additional twist: model must "hedge" against uncertainty
- Facilitated by allowing contracts bought now, for goods delivered later
- Conceptually allows to transfer goods from one period to another (provides wealth retention or pricing of ancillliary services in energy market)
- Can investigate new instruments to mitigate risk, or move to system optimal solutions from equilibrium (or market) solutions

Example as MOPEC: agents solve a Stochastic Program

Each agent minimizes:

$$u_a = (\kappa - f(q_{a,0,*}))^2 + \sum_s \pi_s (\kappa - f(q_{a,s,*}))^2$$

Budget time 0: $\sum_i p_{0,i} q_{a,0,i} + \sum_j v_j y_{a,j} \leq \sum_i p_{0,i} e_{a,0,i}$

Budget time 1: $\sum_i p_{s,i} q_{a,s,i} \leq \sum_i p_{s,i} \sum_j D_{s,i,j} y_{a,j} + \sum_i p_{s,i} e_{a,s,i}$

Additional constraints (complementarity) outside of control of agents:

$$\text{(contract) } 0 \leq -\sum_a y_{a,j} \perp v_j \geq 0$$

$$\text{(walras) } 0 \leq -\sum_a d_{a,s,i} \perp p_{s,i} \geq 0$$