Multiple Optimization Problems with Equilibrium Constraints (MOPEC)

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The problems

- MCP(F): $0 \leq F(x) \perp x \geq 0$
- VI(F, C): $x^* \in C$, $\langle F(x^*), x - x^* \rangle \geq 0, \forall x \in C$
- QVI: $x^* \in K(x^*)$, $\langle F(x^*), x - x^* \rangle \geq 0, \forall x \in K(x^*)$

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- MPEC:

$$\begin{aligned} \min_{x,y} \quad & \theta(x, y) \\ \text{s.t.} \quad & (x, y) \in D, \\ & y \text{ solves } VI(F(x, \cdot), C) \end{aligned}$$

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- GNE: x^*: x_i^* solves $\min_{x_i \in K_i(x_{-i}^*)} \theta(x_i, x_{-i}^*), \forall i$
- MOPEC: x^*, p:

$$x_i^* \text{ solves } \min_{x_i \in K_i(x_{-i}^*, p)} \theta(x_i, x_{-i}^*, p), \forall i$$

$$p \text{ solves } VI(F(x^*, \cdot), C)$$

(M)OPEC

$$\min_{x} \theta(x, p) \text{ s.t. } g(x, p) \leq 0$$

$$0 \leq p \perp h(x, p) \geq 0$$

```
equilibrium
min theta x g
vi h p
```

- Solved concurrently (in a Nash manner)

(M)OPEC

$$\frac{\min_{x} \theta(x, p) \text{ s.t. } g(x, p) \leq 0}{0 \leq p \perp h(x, p) \geq 0}$$

$$\begin{aligned} x \perp \nabla_x \theta(x, p) + \lambda^T \nabla_x g(x, p) \\ 0 \leq \lambda \perp -g(x, p) \geq 0 \\ 0 \leq p \perp h(x, p) \geq 0 \end{aligned}$$

```
equilibrium
min theta x g
vi h p
```

- Solved concurrently (in a Nash manner)
- Requires global solutions of agents problems (or theory to guarantee KKT are equivalent)
- Theory of existence, uniqueness and stability based in variational analysis

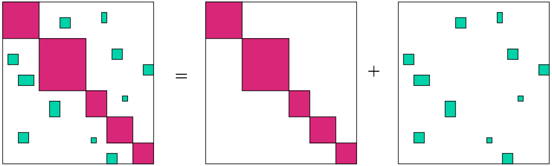
MOPEC

$$\min_{x_i} \theta_i(x_i, x_{-i}, p) \text{ s.t. } g_i(x_i, x_{-i}, p) \leq 0, \forall i$$

$$p \text{ solves } \mathrm{VI}(h(x, \cdot), C)$$

```
equilibrium
min theta(1) x(1) g(1)
...
min theta(m) x(m) g(m)
vi h p cons
```

- Reformulate optimization problem as first order conditions (complementarity)
- Use nonsmooth Newton methods to solve complementarity problem
- Solve overall problem using "individual optimizations"?



Iteration with Indefinite Splitting

$$Ax = b$$

Splitting $A = P - N$ naturally leads to a stationary iteration of the form

$$x_0 \text{ arbitrary}, \quad Px_{k+1} = Nx_k + b, \quad k = 0, 1, \ldots$$

- This iteration may or may not converge; simply applicable sufficient conditions for convergence are particularly valuable.
- Most well-known such conditions are diagonal dominance:
  - if the preconditioner is $P = \mathrm{diag}(A)$ (leading to Jacobi iteration) or
  - P is the lower triangular part of A (leading to Gauss-Seidel iteration),

  then convergence is guaranteed if the strict diagonal dominance condition

$$|a_{i,i}| > \sum_{j=1,\ldots,n, j\neq i} |a_{i,j}|, \quad i = 1, \ldots, n \qquad (1)$$

is satisfied by $A = \{a_{i,j}, i, j = 1, \ldots, n\}$.

Weaker diagonal dominance conditions

For irreducible matrices, it is well documented that the weaker condition

$$|a_{i,i}| \geq \sum_{j=1,\ldots,n, j \neq i} |a_{i,j}|, \quad i = 1, \ldots, n \tag{2}$$

is also sufficient provided strict inequality holds for at least one row index, i

The condition (1) or (2) also guarantees that $A \in \mathbb{R}^{n \times n}$ is invertible, so a unique solution exists.

Strongly Convex (Generalized) Nash Equilibria

$$\min_{x_1 \geq 0} \; \frac{1}{2}x_1^2 - \theta x_1 x_2 - 4x_1 \text{ s.t. } x_1 + x_2 \geq 1$$
$$\min_{x_2 \geq 0} \; \frac{1}{2}x_2^2 - x_1 x_2 - 3x_2$$

- No solution for $\theta \geq 1$:

$$x_1(x_2) = (\theta x_2 + 4)_+, \;\; x_2(x_1) = (x_1 + 3)_+$$

- Solution $-\frac{4}{3} \leq \theta < 1$: $x_1 = \frac{4+3\theta}{1-\theta}$, $x_2 = x_1 + 3$
- Solution $\theta \leq -\frac{4}{3}$: $x_1 = 0$, $x_1 = 3$
- Jacobi works provided $\theta < 1$, but theory fails

The Issues

This is not the optimality conditions of a single optimization problem:

$$0 \leq \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 1 & & 1 \\ \hline -1 & & 1 \end{array}\right] \begin{bmatrix} x_1 \\ -p_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \quad \perp \quad \begin{bmatrix} x_1 \\ -p_1 \\ x_2 \end{bmatrix} \geq 0$$

- The matrix $\mathcal{A}$ in general is never diagonally dominant except in trivial cases
- Iterations based on succesive inversion of local blocks (or successive optimization of local strategies) can converge.
- We establish sufficient conditions which guarantee convergence of block Jacobi and block Gauss-Seidel iterations for such matrices.

The Setting

We focus on matrices of the form

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_1 & A_{1,2} & \cdots & A_{1,p} & E_1 \\ A_{2,1} & \mathcal{A}_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & A_{p-1,p} & E_{p-1} \\ A_{p,1} & \cdots & A_{p,p-1} & \mathcal{A}_p & E_p \\ F_1 & \cdots & F_{p-1} & F_p & D \end{bmatrix} \tag{3}$$

where

$$\mathcal{A}_i = \begin{bmatrix} Q_i & B_i^T \\ B_i & 0 \end{bmatrix}, i = 1, \ldots, p \tag{4}$$

with $Q_i = Q_i^T \in \mathbb{R}^{n_i \times n_i}$ positive definite and $B_i \in \mathbb{R}^{m_i \times n_i}$ of full rank $m_i < n_i$ for each i $(m_i > 0)$. These conditions guarantee that each $\mathcal{A}_i$ is invertible. The submatrix $D \in \mathbb{R}^{s \times s}, s \geq 0$ must be symmetric and invertible (unless $s = 0$).

Old Block Theory

For the blocked matrix (3) a result of Feingold and Varga (1962) applies: If $\mathcal{A}$ is block irreducible and

$$
\begin{aligned}
(\|\mathcal{A}_i^{-1}\|_2)^{-1} &\geq \|E_i\|_2 + \sum_{j=1,\ldots,p,j\neq i} \|A_{i,j}\|_2, \quad i = 1,\ldots,p && (5)\\
\text{and} \quad (\|D^{-1}\|_2)^{-1} &\geq \sum_{j=1,\ldots,p,j\neq i} \|F_i\|_2 && (6)
\end{aligned}
$$

with strict inequality in (6) or for at least one index, i, in (5), then $\mathcal{A}$ is invertible (existence and uniqueness)

Relation to Iteration

Before considering these conditions in more detail, consider a block Jacobi or block Gauss-Seidel iteration based on the splitting with

$$P = \begin{bmatrix} \mathcal{A}_1 & 0 & \cdots & 0 & 0 \\ 0 & \mathcal{A}_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & \mathcal{A}_p & 0 \\ 0 & \cdots & 0 & 0 & D \end{bmatrix} \text{ or } P = \begin{bmatrix} \mathcal{A}_1 & 0 & \cdots & 0 & 0 \\ A_{2,1} & \mathcal{A}_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ A_{p,1} & \cdots & A_{p,p-1} & \mathcal{A}_p & 0 \\ F_1 & \cdots & F_{p-1} & F_p & D \end{bmatrix}$$

Asymptotic convergence of the corresponding stationary (or simple) iteration will be guaranteed for any starting vector if all of the eigenvalues, λ, of $I - P^{-1}\mathcal{A}$ lie strictly inside the unit disc.

The link

Such eigenvalues satisfy $(I - P^{-1}\mathcal{A})x = \lambda x, x \neq 0$ or equivalently $(\mathcal{A} + (\lambda - 1)P)x = 0, x \neq 0$. In the case of block Jacobi, asymptotic convergence will be guaranteed if there does not exist any λ with $|\lambda| \geq 1$ such that the matrix

$$\mathcal{A}(\lambda) = \mathcal{A} + (\lambda - 1)P = \begin{bmatrix} \lambda\mathcal{A}_1 & A_{1,2} & \cdots & A_{1,p} & E_1 \\ A_{2,1} & \lambda\mathcal{A}_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & A_{p-1,p} & E_{p-1} \\ A_{p,1} & \cdots & A_{p,p-1} & \lambda\mathcal{A}_p & E_p \\ F_1 & \cdots & F_{p-1} & F_p & \lambda D \end{bmatrix}$$

is singular. But

$$(\|(\lambda\mathcal{A}_i)^{-1}\|_2)^{-1} = |\lambda|(\|\mathcal{A}_i^{-1}\|_2)^{-1} \geq (\|\mathcal{A}_i^{-1}\|_2)^{-1}$$

whenever $|\lambda| \geq 1$ with a identical argument holding for D. Hence satisfaction of the conditions (5),(6) not only guarantees invertibility of $\mathcal{A}$, but also guarantees covergence of the block Jacobi iteration.

Another piece

Let μ_i denote the smallest eigenvalue of the positive definite matrix Q_i and γ_i denote the smallest eigenvalue of the positive definite (Schur complement) matrix $B_i Q_i^{-1} B_i^T$, then there are no eigenvalues of

$$\mathcal{A}_i = \begin{bmatrix} Q_i & B_i^T \\ B_i & 0 \end{bmatrix}$$

in the interval

$$\left(\frac{1}{2} \left(\mu_i - \sqrt{\mu_i^2 + 4\gamma_i \mu_i} \right), \mu_i \right)$$

which contains the origin.

Finally...

If the matrix $\mathcal{A}$ given by (3),(4) is block irreducible, then it is invertible and the block Jacobi and block Gauss-Seidel iterations for a linear system $\mathcal{A}x = b$ converge to x for any starting vector if

$$\min\left\{\frac{1}{2}\left(\sqrt{\mu_i^2 + 4\gamma_i\mu_i} - \mu_i\right), \mu_i\right\} \geq \|E_i\|_2 + \sum_{j=1,\ldots,p, j\neq i} \|A_{i,j}\|_2, \quad i = 1, \ldots, p \tag{7}$$

$$\text{and} \quad d \geq \sum_{j=1,\ldots,p, j\neq i} \|F_i\|_2 \tag{8}$$

with strict inequality in (8)[1] or for at least one index, i, in (7).

[1] d is the absolute value of eighenvalue of D closest to origin

A Simplification

If for each $i = 1, \ldots, p$, $\gamma_i \geq 2\mu_i$ then $\mathcal{A}$ is invertible and the block Jacobi and block Gauss-Seidel iterations for a linear system $\mathcal{A}x = b$ converge to x for any starting vector if

$$\mu_i \geq \|E_i\|_2 + \sum_{j=1,\ldots,p,j\neq i} \|A_{i,j}\|_2, \quad i = 1, \ldots, p \tag{9}$$

$$\text{and} \quad d \geq \sum_{j=1,\ldots,p,j\neq i} \|F_i\|_2 \tag{10}$$

with strict inequality in (10) or for at least one index, i, in (9).

Extensions

- Can also prove same result for SOR schemes
- Can apply regularization (proximal iterations) on the constraints: for $\epsilon_i > 0$

$$\mathcal{A}_i = \begin{bmatrix} Q_i & B_i^T \\ B_i & -\epsilon_i I \end{bmatrix},$$

  can be used for some subset (or indeed all) of the indices $i = 1, \ldots, p$.
- This simply increases the value of γ_i to $\gamma_i + \epsilon_i$ and strengthens the above theory
- Can apply when block solves are smaller scale Quadratic Programs (with inequalities, etc) rather than systems of equations

Strongly convex optimization

$$\min_{x_1} \frac{1}{2}x_1^2 - x_1x_2 - 4x_1 \text{ s.t. } x_1 + x_2 = 1$$

$$\min_{x_2} \frac{1}{2}x_2^2 - x_1x_2 - 3x_2$$

$$\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 1 & & 1 \\ \hline -1 & & 1 \end{array}\right] \begin{bmatrix} x_1 \\ -p_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

- Solution: $x_1 = -1, x_2 = 2,\ p_1 = -7$
- Jacobi fails: after 4 steps back at $(1,1)^T$
- Modified Jacobi $\epsilon_1 = 0.1$ solves in 20 steps

Extensions

- Replace systems of equations by (normal map formulation of) complementarity problems $\forall i$:

$$
\begin{aligned}
&Q_i(\pi_{K_i}(x_i)) + B_i^T p + c_i + x_i - \pi_{K_i}(x_i) = 0 \\
&B_i(\pi_{K_i}(x_i)) = b_i
\end{aligned}
$$

- Note this is a natural extension of the case considered above
- Choose active set at each iteration based on prediction from previous iteration
- Apply theory to all selections of the resulting linear systems

Does it work at realistic scales: GTAP?

- The latest GTAP database represents global production and trade for 113 country/regions, 57 commodities and 5 primary factors.
- Data characterizes intermediate demand and bilateral trade in 2007, including tax rates on imports/exports and other indirect taxes.
- The core GTAP model is a static, multi-regional model which tracks the production and distribution of goods in the global economy.
- In GTAP the world is divided into regions (typically representing individual countries), and each region's final demand structure is composed of public and private expenditure across goods.

The Model

The GTAP model may be posed as a system of *nonlinear equations*:

$$F(w, z; t) = 0$$

in which: where

- w_r is a vector of regional *welfare* levels
- $z \in \mathrm{R}^N$ represents a vector of endogenous economic variables, e.g. prices and quantities, $z = \begin{pmatrix} P \\ Q \end{pmatrix}$.
- t represents matrices of trade tax instruments – import tariffs (t_{irs}^M) and export taxes (t_{irs}^X) for each commodity i and region r

Optimal Sanctions

Coalition member states strategically choose trade taxes which *minimize* Russian welfare:

$$\min_{t_r : r \in \mathcal{C}} w_{rus}$$

s.t.

$$F(w, z; t) = 0$$

$$t_r = \bar{t}_r \quad \forall r \notin \mathcal{C}$$

$$t^M_{i,rus,r} \leq \bar{t}^M_{i,r,rus} \quad \forall r \in \mathcal{C}$$

$$t^X_{i,r,rus} \leq \bar{t}^X_{i,rus,r} \quad \forall r \in \mathcal{C}$$

Optimal Retaliation

Russia choose trade taxes which *maximize* Russian welfare in response to the coalition actions:

$$\max_{t_{rus}} w_{rus}$$

s.t.

$$F(w, z; t) = 0$$

$$t_r = \begin{cases} \hat{t}_r & r \in \mathcal{C} \\ \bar{t}_r & r \notin \mathcal{C} \end{cases}$$

where $\hat{t}_r$ represents trade taxes for coalition countries ($r \in \mathcal{C}$) from the optimal sanction calculation.

Coalition Member States for Illustrative Calculation

USA	United States
ANZ	Australia and New Zealand
CAN	Canada
FRA	France
DEU	Germany
ITA	Italy
JPN	Japan
GBR	United Kingdom
REU	Rest of the European Union

Welfare Changes (% Hicksian EV)

	sanction	retaliation	tradewar
RUS	-4.4	-3.5	-9.8
$\mathcal{C}$ AVERAGE	0.03	0.05	0.03
CAN	0.021	0.033	0.032
USA	0.007	-0.017	0.032
FRA	0.042	0.020	0.032
DEU	0.119	-0.047	0.032
ITA	0.069	0.050	0.032
GBR	0.045	-0.002	0.032
REU	0.058	0.365	0.032
ANZ	0.011	0.003	0.032
JPN	0.012	-0.020	0.032
CHN	0.115	0.057	0.290
SAU	0.240	1.865	-0.892

Scenarios and Key Insights

SANCTION If coalition states were to increases tariffs and export taxes on Russia to the same level which is currently applied by Russia on bilateral trade flows with the coalition, Russian welfare could be substantially impacted with no economic cost for any coalition members.

RETALIATION Russia could respond to such sanctions by changing it's own trade taxes, but optimal "retaliation" largely results in a *reduction* rather than an increase in trade taxes on trade flows to and from coalition states. These tariff changes can only partially offset the adverse impact of the sanctions.

TRADEWAR If sanctions and retaliation were to result in an unconstrained trade war, Russia faces a drastic economic cost while the coalition countries could even be slight better off.

What is EMP?

Annotates existing equations/variables/models for modeler to provide/define additional structure

- equilibrium
- vi (agents can solve min/max/vi)
- bilevel (reformulate as MPEC, or as SOCP)
- disjunction (or other constraint logic primitives)
- randvar
- dualvar (use multipliers from one agent as variables for another)
- extended nonlinear programs (library of plq functions)

Currently available within GAMS

Conclusions

- MOPEC problems capture complex interactions between optimizing agents
- Policy implications addressable using MOPEC
- MOPEC available to use within the GAMS modeling system
- New sufficient conditions for existence, uniqueness and convergence shown in special cases
- Many new settings available for deployment; need for more theoretic and algorithmic enhancements