Initially, $\text{DefIn}(b_0)$ = false.

For basic block b:

```
DefOut(b) =
  If the definition being analyzed is
    the last definition to v in b
  Then True
  Elsif any other definition to v
occurs
     in b
  Then False
  Else DefIn(b)
```

The meet operation (to combine solutions) is:

$$\text{DefIn}(b) = \underset{p \in \text{Pred}(b)}{\text{OR}} \text{DefOut}(p)$$

To get all reaching definition, we do a series of single definition analyses.

Live Variable Analysis

This is a boolean-valued, backward flow data flow problem.

Initially, $\text{LiveOut}(b_{last})$ = false.

For basic block b:

```
LiveIn(b) =
  If the variable is used before it is
    defined in b
  Then True
  Elsif it is defined before it is used
     in b
  Then False
  Else LiveOut(b)
```

The meet operation (to combine solutions) is:

$$\text{LiveOut}(b) = \underset{s \in \text{Succ}(b)}{\text{OR}} \text{LiveIn}(s)$$

Bit Vectoring Data Flow Problems

The four data flow problems we have just reviewed all fit within a *single* framework.

Their solution values are Booleans (bits).

The meet operation is And or OR.

The transfer function is of the general form

$\text{Out}(b) = (\text{In}(b) - \text{Kill}_b) \cup \text{Gen}_b$

or

$\text{In}(b) = (\text{Out}(b) - \text{Kill}_b) \cup \text{Gen}_b$

where Kill_b is true if a value is “killed” within b and Gen_b is true if a value is “generated” within b.

In Boolean terms:

$\text{Out}(b) = (\text{In}(b) \text{ AND Not } \text{Kill}_b) \text{ OR } \text{Gen}_b$

or

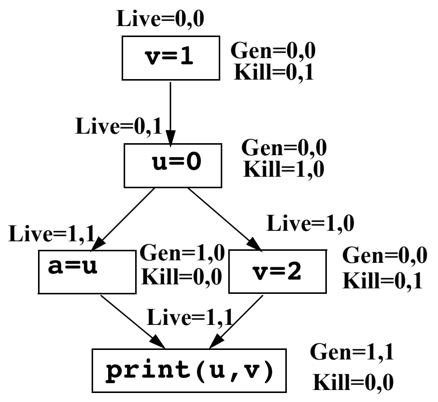
$\text{In}(b) = (\text{Out}(b) \text{ AND Not } \text{Kill}_b) \text{ OR } \text{Gen}_b$

An advantage of a bit vectoring data flow problem is that we can do a series of data flow problems “in parallel” using a bit vector.

Hence using ordinary word-level ANDs, ORs, and NOTs, we can solve 32 (or 64) problems simultaneously.

Example

Do live variable analysis for u and v, using a 2 bit vector:



We expect *no variable* to be live at the start of b_0. (Why?)

Depth-First Spanning Trees

Sometimes we want to “cover” the nodes of a control flow graph with an acyclic structure.

This allows us to visit nodes once, without worrying about cycles or infinite loops.

Also, a careful visitation order can approximate forward control flow (very useful in solving forward data flow problems).

A Depth-First Spanning Tree (DFST) is a tree structure that covers the nodes of a control flow graph, with the start node serving as root of the DFST.

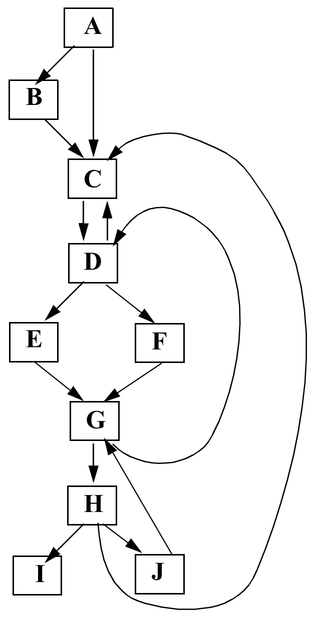
Building a DFST

We will visit CFG nodes in depth-first order, keeping arcs if the visited node hasn’t be reached before.

Create a DFST, T, from a CFG, G:

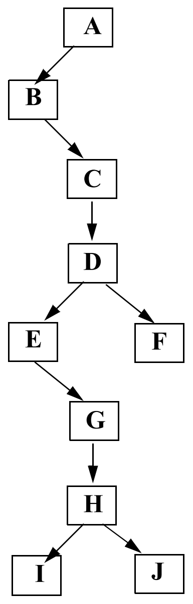
```
1. T ← empty tree
2. Mark all nodes in G as “unvisited.”
3. Call DF(start node)

DF (node) {
1. Mark node as visited.
2. For each successor, s, of node in G:
    If s is unvisited
      (a) Add node → s to T
      (b) Call DF(s)
```

Example



Visit order is A, B, C, D, E, G, H, I, J, I

The DFST is



Categorizing Arcs using a DFST

Arcs in a CFG can be categorized by examining the corresponding DFST.

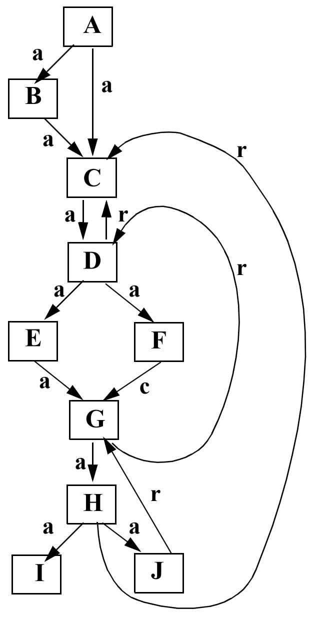
An arc A→B in a CFG is

(a) An *Advancing Edge* if B is a proper
descendent of A in the DFST.

(b) A *Retreating Edge* if B is an ancestor of A in the DFST. (This includes the A→A case.)

(c) A *Cross Edge* if B is neither a descendent nor an ancestor of A in the DFST.

Example



Depth-First Order

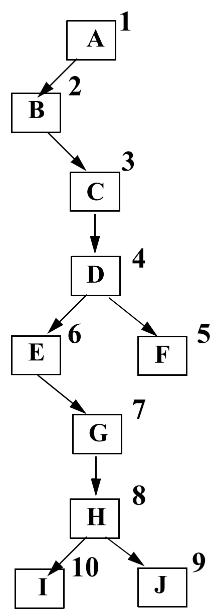
Once we have a DFST, we can label nodes with a *Depth-First Ordering* (DFO).

Let i = the number of nodes in a CFG (= the number of nodes in its DFST).

```
DFO(node) {
  For (each successor s of node) do
      DFO(s);
  Mark node with i;
  i--;
}
```

Example

The number of nodes = 10.



Application of Depth-First Ordering

- *Retreating edges* (a necessary component of loops) are easy to identify:
  a→b is a retreating edge if and only if dfo(b) ≤ dfo(a)
- A depth-first ordering in an excellent *visit order* for solving forward data flow problems. We want to visit nodes in essentially topological order, so that all predecessors of a node are visited (and evaluated) before the node itself is.

Dominators

A CFG node M *dominates* N (M dom N) if and only if *all* paths from the start node to N *must* pass through M.

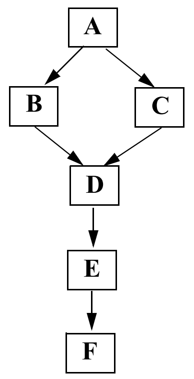
A node trivially dominates itself.

Thus (N dom N) is always true.

A CFG node M *strictly dominates* N (M sdom N) if and only if (M dom N) and M ≠ N.
A node can't strictly dominates itself.

Thus (N sdom N) is never true.

A CFG node may have many dominators.



Node F is dominated by F, E, D and A.

Immediate Dominators

If a CFG node has more than one dominator (which is common), there is always a unique "closest" dominator called its *immediate dominator*.

(M idom N) if and only if
 (M sdom N) and
 (P sdom N) $\Rightarrow$ (P dom M)

To see that an immediate dominator always exists (except for the start node) and is unique, assume that node N is strictly dominated by M_1, M_2, ..., M_p, $P \geq 2$.

By definition, M_1, ..., M_p must appear on *all* paths to N, including acyclic paths.

Look at the relative ordering among M_1 to M_p on some arbitrary acyclic path from the start node to N.

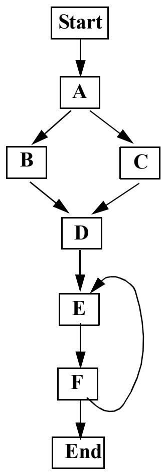
Assume that M_i is "last" on that path (and hence "nearest" to N).

If, on some other acyclic path, $M_j \neq M_i$ is last, then we can shorten this second path by going directly from M_i to N without touching any more of the M_1 to M_p nodes.

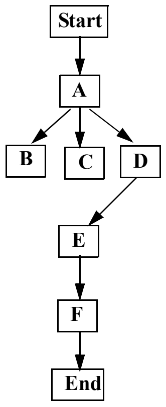
But, this totally removes M_j from the path, contradicting the assumption that (M_j sdom N).

Dominator Trees

Using immediate dominators, we can create a *dominator tree* in which A$\rightarrow$B in the dominator tree if and only if (A idom B).

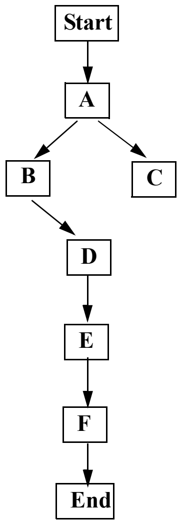


Control Flow Graph

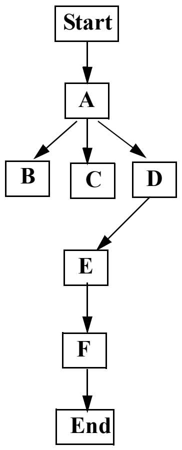


Dominator Tree

Note that the Dominator Tree of a CFG and its DFST are distinct trees (though they have the same nodes).



Depth-First Spanning Tree



Dominator Tree

A Dominator Tree is a compact and convenient representation of both the dom and idom relations.

A node in a Dominator Tree dominates all its descendents in the tree, and immediately dominates all its children.

Computing Dominators

Dominators can be computed as a Set-valued Forward Data Flow Problem.

If a node N dominates all of node M's predecessors, then N appears on all paths to M. Hence (N dom M).

Similarly, if M *doesn't* dominate all of M's predecessors, then there is a path to M that doesn't include M. Hence $\neg$(N dom M).

These observations give us a "data flow equation" for dominator sets:

$$\mathrm{dom}(N) = \{N\} \cup \bigcap_{M \in \mathrm{Pred}(N)} \mathrm{dom}(M)$$

The analysis domain is the lattice of all subsets of nodes. Top is the set of all nodes; bottom is the empty set. The ordering relation is subset.

The meet operation is intersection.

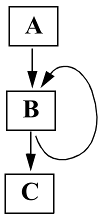
The Initial Condition is that $\mathrm{DomIn}(b_0) = \phi$

$$\mathrm{DomIn}(b) = \bigcap_{c \in \mathrm{Pred}(b)} \mathrm{DomOut}(c)$$

$$\mathrm{DomOut}(b) = \mathrm{DomIn}(b) \cup \{b\}$$

Loops Require Care

Loops in the Control Flow Graph induce circularities in the Data Flow equations for Dominators. In



we have the rule dom(B) =
DomOut(B) =
DomIn(B) U {B} =
{B} U ($\underline{\mathrm{DomOut(B)}}$ $\cap$ DomOut(A))

If we choose DomOut(B) = ϕ initially, we get DomOut(B) = {B} U ($\phi \cap$ DomOut(A)) = {B} which is *wrong*.

Instead, we should use the Universal Set (of all nodes) which is the *identity* for ∩.

Then we get DomOut(B) = {B} U ({all nodes} ∩ DomOut(A)) = {B} U DomOut(A) which is correct.

A Worklist Algorithm for Dominators

The data flow equations we have developed for dominators can be evaluated using a simple Worklist Algorithm.

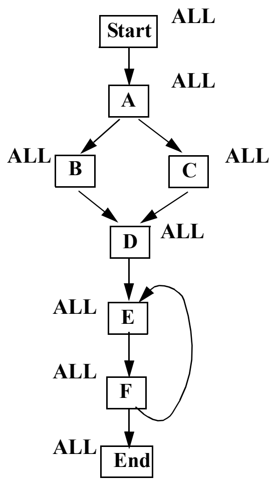
Initially, each node's dominator set is set to the set of all nodes. We add the start node to our worklist.

For each node on the worklist, we reevaluate its dominator set. If the set changes, the updated dominator set is used, and all the node's successors are added to the worklist (so that the updated dominator set can be propagated).

The algorithm terminates when the worklist is empty, indicating that a stable solution has been found.

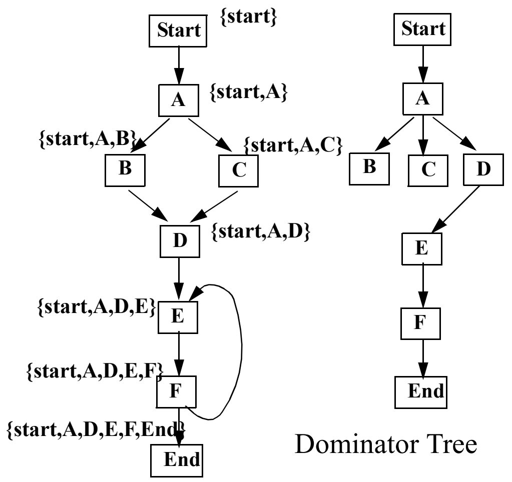
```
Compute Dominators(){
  For (each n ∈ NodeSet)
      Dom(n) = NodeSet
  WorkList = {StartNode}
  While (WorkList ≠ ϕ) {
    Remove any node Y from WorkList
     New = {Y} U ∩_{X ∈ Pred(Y)} Dom(X)
    If New ≠ Dom(Y) {
       Dom(Y) = New
       For (each Z ∈ Succ(Y))
          WorkList = WorkList U {Z}
}}}
```

Example



Initially the WorkList = {Start}.

Be careful when Pred(Node) = ϕ.



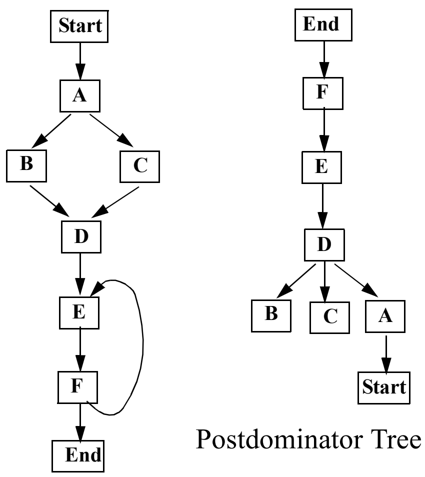
Dominator Tree

Control Flow Graph

Postdominance

A block Z *postdominates* a block Y (Z pdom Y) if and only if all paths from Y to an exit block must pass through Z. Notions of immediate postdominance and a postdominator tree carry over.

Note that if a CFG has a single exit node, then postdominance is equivalent to dominance if flow is reversed (going from the exit node to the start node).



Postdominator Tree

Control Flow Graph