How Good Is Iterative Data Flow Analysis?

A single execution of a program will follow some path
$b_0,b_{i_1},b_{i_2},...,b_{i_n}$.

The Data Flow solution along this path is

$f_{i_n}(...f_{i_2}(f_{i_1}(f_0(T)))...) \equiv f(b_0,b_1,...,b_{i_n})$

The best possible static data flow solution at some block b is computed over all possible paths from b_0 to b.

Let P_b = The set of all paths from b_0 to b.

$$MOP(b) = \bigwedge_{p \in P_b} f(p)$$

Any particular path p_i from b_0 to b is included in P_b.

Thus $MOP(b) \wedge f(p_i) = MOP(b) \le f(p_i)$.

This means MOP(b) is *always* a safe approximation to the "true" solution $f(p_i)$.

If we have the distributive property for transfer functions,

$f(a \wedge b) = f(a) \wedge f(b)$

then our iterative algorithm *always* computes the MOP solution, the best static solution possible.

To prove this, note that for trivial path of length 1, containing only the start block, b_0, the algorithm computes $f_0(T)$ which is $MOP(b_0)$ (trivially).

Now assume that the iterative algorithm for paths of length n or less to block c *does* compute MOP(c).

We'll show that for paths to block b of length n+1, MOP(b) is computed.

Let P be the set of all paths to b of length n+1 or less.

The paths in P end with b.

$MOP(b) = f_b(f(P_1)) \wedge f_b(f(P_2) \wedge ...$

where P_1, P_2, ... are the prefixes (of length n or less) of paths in P with b removed.

Using the distributive property,

$f_b(f(P_1)) \wedge f_b(f(P_2) \wedge ... =$

$f_b(f(P_1) \wedge f(P_2) \wedge ...)$.

But note that $f(P_1) \wedge f(P_2) \wedge ...$ is just the input to f_b in our iterative algorithm, which then applies f_b.

Thus MOP(b) for paths of length n+1 is computed.

For data flow problems that aren’t distributive (like constant propagation), the iterative solution is ≤ the MOP solution.

This means that the solution is a safe approximation, but perhaps not as “sharp” as we might wish.

Reading Assignment

Read “An Efficient Method of Computing Static Single Assignment Form.”
(Linked from the class Web page.)

Exploiting Structure in Data Flow Analysis

So far we haven’t utilized the fact that CFGs are constructed from standard programming language constructs like IFs, Fors, and Whiles.

Instead of iterating across a given CFG, we can isolate, and solve symbolically, subgraphs that correspond to “standard” programming language constructs.

We can then progressively simplify the CFG until we reach a single node, or until we reach a CFG structure that matches no standard pattern.

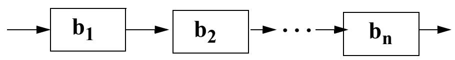
In the latter case, we can solve the residual graph using our iterative evaluator.

Three Program-Building Operations

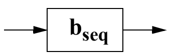
1. Sequential Execution (“;”)
2. Conditional Execution (If, Switch)
3. Iterative Execution
   (While, For, Repeat)

Sequential Execution

We can reduce a sequential "chain" of basic blocks:



into a single composite block:



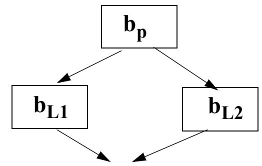
The transfer function of b_{seq} is

$f_{seq} = f_n \circ f_{n-1} \circ ... f_1$

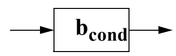
where $\circ$ is functional composition.

Conditional Execution

Given the basic blocks:



we create a single composite block:



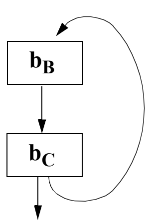
The transfer function of b_{cond} is

$f_{cond} = f_{L1} \circ f_p \wedge f_{L2} \circ f_p$

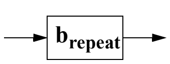
Iterative Execution

Repeat Loop

Given the basic blocks:



we create a single composite block:



Here b_B is the loop body, and b_C is the loop control.

If the loop iterates once, the transfer function is f_C o f_B.

If the loop iterates twice, the transfer function is $(f_C \circ f_B) \circ (f_C \circ f_B)$.

Considering all paths, the transfer function is $(f_C \circ f_B) \wedge (f_C \circ f_B)^2 \wedge ...$

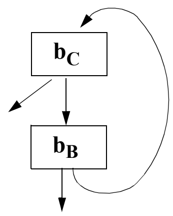
Define fix $f \equiv f \wedge f^2 \wedge f^3 \wedge ...$

The transfer function of repeat is then

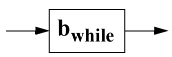
$f_{repeat} = fix(f_C \circ f_B)$

While Loop.

Given the basic blocks:



we create a single composite block:



Here again b_B is the loop body, and b_C is the loop control.

The loop always executes b_C at least once, and always executes b_C as the last block before exiting.

The transfer function of a while is therefore

$f_{while} = f_C \wedge \text{fix}(f_C \circ f_B) \circ f_C$

Evaluating Fixed Points

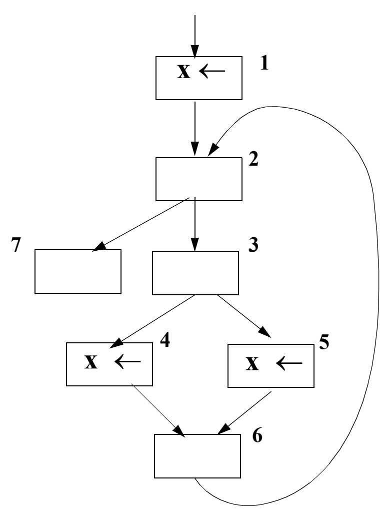
For lattices of height H, and monotone transfer functions, fix f needs to look at no more than H terms.

In practice, we can give fix f an operational definition, suitable for implementation:

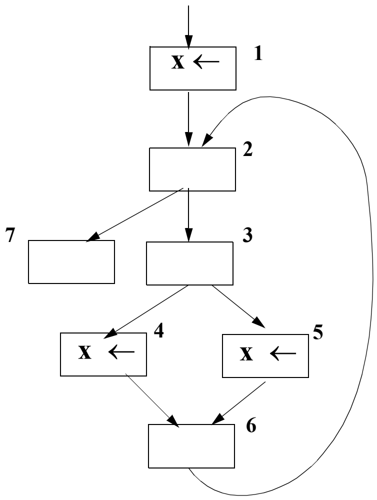
Evaluate

```
(fix f)(x) {
  prev = soln = f(x);
  while (prev ≠ new = f(prev)){
    prev = new;
    soln = soln ∧ new;
  }
  return soln;
}
```


Example—Reaching Definitions



The transfer functions are either constant-valued (f_1={b1}, f_4={b4}, f_5={b5}) or identity functions ($f_2=f_3=f_6=f_7$=Id).

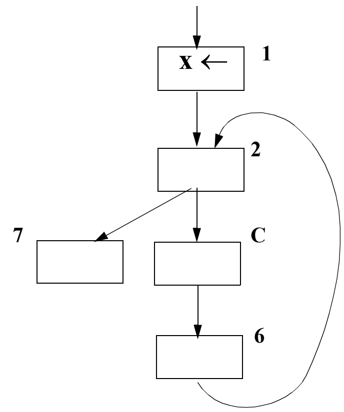


First we isolate and reduce the conditional:

$f_C = f_4 \circ f_3 \wedge f_5 \circ f_3 =$

$\{b4\} \circ Id \cup \{b5\} \circ Id = \{b4,b5\}$

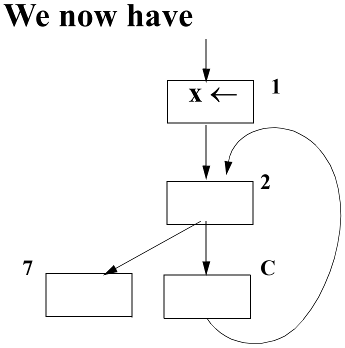
Substituting, we get



We can combine b_C and b_6, to get a block equivalent to b_C. That is,

$f_6 \circ f_C = Id \circ f_C = f_C$

We now have



We isolate and reduce the while loop formed by b_2 and b_C, creating b_W. The transfer function is

$f_W = f_2 \wedge (fix(f_2 \circ f_C)) \circ f_2 =$

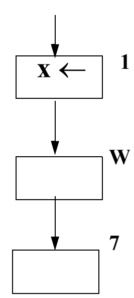
$Id \cup (fix(Id \circ f_C)) \circ Id =$

$Id \cup (fix(f_C)) =$

$Id \cup (f_C \wedge f_C^2 \wedge f_C^3 \wedge ...) =$

$Id \cup \{b4,b5\}$

We now have



We compose these three sequential blocks to get the whole solution, f_P.

$f_P = Id \circ (Id \cup \{b4,b5\}) \circ \{b1\} =$

$\{b1,b4,b5\}$.

These are the definitions that reach the end of the program.

We can expand subgraphs to get the solutions at interior blocks.

Thus at the beginning of the while, the solution is {b1}.

At the head if the If, the solution is

$$(Id \cup (Id \circ f_C \circ Id) \cup (Id \circ f_C \circ Id \circ f_C \circ Id) \cup \ldots)\circ(\{b1\}) = \{b1\} \cup \{b4,b5\} \cup \{b4,b5\} \cup \ldots = \{b1,b4,b5\}$$

At the head of the then part of the If, the solution is Id({b1,b4,b5}) = {b1,b4,b5}.

Static Single Assignment Form

Many of the complexities of optimization and code generation arise from the fact that a given variable may be assigned to in *many* different places.

Thus reaching definition analysis gives us the *set* of assignments that *may* reach a given use of a variable.

Live range analysis must track *all* assignments that may reach a use of a variable and merge them into the same live range.

Available expression analysis must look at *all* places a variable may be assigned to and decide if any kill an already computed expression.

What If

each variable is assigned to in only one place?
(Much like a named constant).

Then for a given use, we can find a single *unique* definition point.

But this seems *impossible* for most programs—or is it?

In *Static Single Assignment* (SSA) Form each assignment to a variable, v, is changed into a unique assignment to new variable, v_i.

If variable v has n assignments to it throughout the program, then (at least) n new variables, v_1 to v_n, are created to replace v. All uses of v are replaced by a use of some v_i.

Phi Functions

Control flow can't be predicted in advance, so we can't always know which definition of a variable reached a particular use.

To handle this uncertainty, we create *phi functions*.

As illustrated below, if v_i and v_j both reach the top of the same block, we add the assignment

$$v_k \leftarrow \phi(v_i,v_j)$$

to the top of the block.

Within the block, all uses of v become uses of v_k (until the next assignment to v).

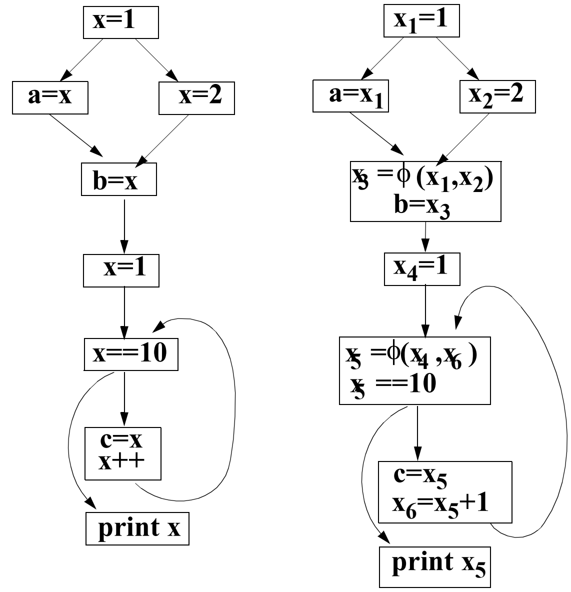
What does $\phi(v_i,v_j)$ Mean?

One way to read $\phi(v_i,v_j)$ is that if control reaches the phi function via the path on which v_i is defined, ϕ "selects" v_i; otherwise it "selects" v_j.

Phi functions may take more than 2 arguments if more than 2 definitions might reach the same block.

Through phi functions we have simple links to all the places where v receives a value, directly or indirectly.

Example



Original CFG **CFG in SSA Form**

In SSA form computing live ranges is almost trivial. For each x_i include all x_j variables involved in phi functions that define x_i.

Initially, assume x_1 to x_6 (in our example) are independent. We then union into equivalence classes x_i values involved in the same phi function or assignment.

Thus x_1 to x_3 are unioned together (forming a live range). Similarly, x_4 to x_6 are unioned to form a live range.

Constant Propagation in SSA

In SSA form, constant propagation is simplified since values flow directly from assignments to uses, and phi functions represent natural "meet points" where values are combined (into a constant or $\perp$).

Even conditional constant propagation fits in. As long as a path is considered unreachable, it variables are set to T (and therefore ignored at phi functions, which meet values together).

Example

```
i=6
j=1
k=1
repeat
  if (i==6)
     k=0
  else
     i=i+1
  i=i+k
  j=j+1
until (i==j)
```

```
i_1=6
j_1=1
k_1=1
repeat
  i_2=ϕ(i_1,i_5)
  j_2=ϕ(j_1,j_3)
  k_2=ϕ(k_1,k_4)
  if (i_2==6)
     k_3=0
  else
     i_3=i_2+1
  i_4=ϕ(i_2,i_3)
  k_4=ϕ(k_3,k_2)
  i_5=i_4+k_4
  j_3=j_2+1
until (i_5==j_3)
```

	i_1	j_1	k_1	i_2	j_2	k_2	k_3	i_3	i_4	k_4	i_5	j_3
Pass1	6	1	1	$6\wedge$ T	$1\wedge$ T	$1\wedge$ T	0	T	$6\wedge$ T	0	6	2
Pass2	6	1	1	$6\wedge6$	$\perp$	$\perp$	0	T	6	0	6	$\perp$

We have determined that i=6 everywhere.

Putting Programs into SSA Form

Assume we have the CFG for a program, which we want to put into SSA form. We must:

- **Rename all definitions and uses of variables**
- **Decide where to add phi functions**

Renaming variable definitions is trivial—each assignment is to a new, unique variable.

After phi functions are added (at the heads of selected basic blocks), only one variable definition (the most recent in the block) can reach any use. Thus renaming uses of variables is easy.

Placing Phi Functions

Let b be a block with a definition to some variable, v. If b contains more than one definition to v, the last (or most recent) applies.

What is the first basic block following b where some other definition to v *as well as* b's definition can reach?

In blocks dominated by b, b's definition *must* have been executed, though other later definitions may have overwritten b's definition.

Domination Frontiers (Again)

Recall that the Domination Frontier of a block b, is defined as

DF(N) =
{Z | M→Z & (N dom M) & ¬(N sdom Z)}

The Dominance Frontier of a basic block N, DF(N), is the set of all blocks that are immediate successors to blocks dominated by N, but which aren't themselves strictly dominated by N.

Assume that an initial assignment to all variables occurs in b_0 (possibly of some special "uninitialized value.")

We will need to place a phi function at the start of all blocks in b's Domination Frontier.

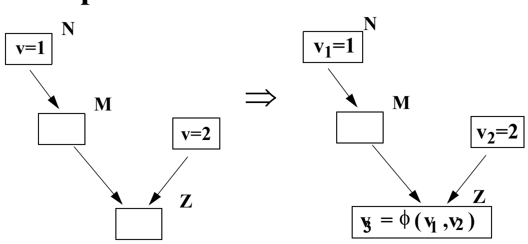
The phi functions will join the definition to v that occurred in b (or in a block dominated by b) with definitions occurring on paths that don't include b.

After phi functions are added to blocks in DF(b), the domination frontier of blocks with newly added phi's will need to be computed (since phi functions imply assignment to a new v_i variable).

Examples of How Domination Frontiers Guide Phi Placement

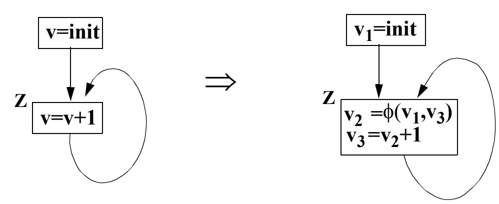
DF(N) =
{Z | M→Z & (N dom M) & ¬(N sdom Z)}

Simple Case:



Here, (N dom M) but ¬(N sdom Z),
so a phi function is needed in Z.

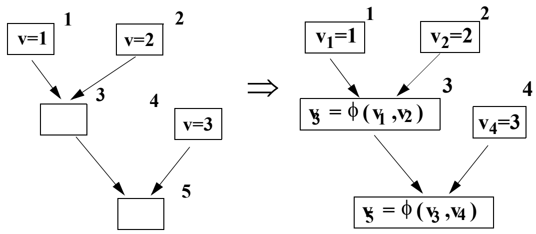
Loop:



Here, let M = Z = N. M→Z,
(N dom M) but ¬(N sdom Z),
so a phi function *is* needed in Z.

DF(N) =
{Z | M→Z & (N dom M) & ¬(N sdom Z)}

Sometimes Phi's must be Placed Iteratively



Now, DF(b1) = {b3}, so we add a phi function in b3. This adds an assignment into b3. We then look at DF(b3) = {b5}, so another phi function must be added to b5.

Phi Placement Algorithm

To decide what blocks require a phi function to join a definition to a variable v in block b:

1. Compute D_1 = DF(b).
Place Phi functions at the head of all members of D_1.

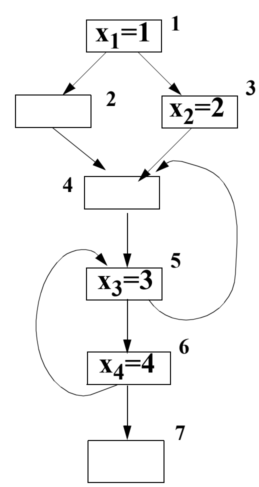
2. Compute D_2 = DF(D_1).
Place Phi functions at the head of all members of D_2-D_1.

3. Compute D_3 = DF(D_2).
Place Phi functions at the head of all members of D_3-D_2-D_1.

4. Repeat until no additional Phi functions can be added.


```
PlacePhi{
 For (each variable v ∈ program) {
   For (each block b ∈ CFG ){
     PhiInserted(b) = false
     Added(b) = false }
   List = ϕ
   For (each b ∈ CFG that assigns to V ){
     Added(b) = true
     List = List U {b}   }
   While (List ≠ ϕ) {
     Remove any b from List
     For (each d ∈ DF(b)){
       If (! PhiInserted(d)) {
         Add a Phi Function to d
         PhiInserted(d) = true
         If (! Added(d)) {
            Added(d) = true
            List = List U {d}
         }
       }
     }
   }
 }
}
```


Example



Initially, List={1,3,5,6}

Process 1: DF(1) = ϕ

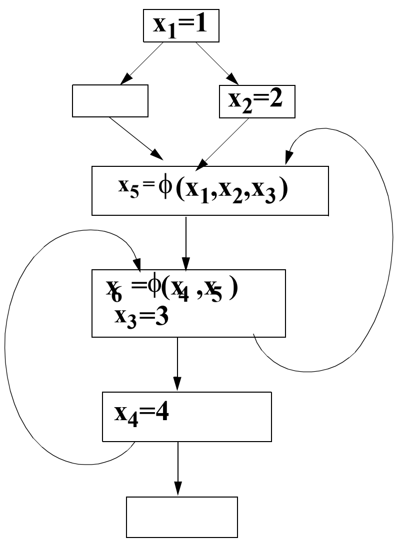
Process 3: DF(3) = 4, so add 4 to List and add phi fct to 4.

Process 5: DF(5)={4,5} so add phi fct to 5.

Process 5: DF(6) = {5}

Process 4: DF(4) = {4}

We will add Phi's into blocks 4 and 5. The arity of each phi is the number of in-arcs to its block. To find the args to a phi, follow each arc "backwards" to the sole reaching def on that path.



SSA and Value Numbering

We already know how to do available expression analysis to determine if a previous computation of an expression can be reused.

A limitation of this analysis is that it can't recognize that two expressions that aren't syntactically identical may actually still be equivalent.

For example, given

```
t1 = a + b
c = a
t2 = c + b
```

Available expression analysis won't recognize that `t1` and `t2` must be equivalent, since it doesn't track the fact that `a` = `c` at `t2`.

Value Numbering

An early expression analysis technique called *value numbering* worked only at the level of basic blocks. The analysis was in terms of "values" rather than variable or temporary names.

Each non-trivial (non-copy) computation is given a number, called its *value number*.

Two expressions, using the same operators and operands with the same value numbers, must be equivalent.

For example,

```
t1 = a + b
c = a
t2 = c + b
```

is analyzed as

```
v1 = a
v2 = b
t1 = v1 + v2
c = v1
t2 = v1 + v2
```

Clearly `t2` is equivalent to `t1` (and hence need not be computed).