Chaos in Fixed-Point Iteration

Introduction to Numerical Methods
cs412-1
2-10-2006

Overview

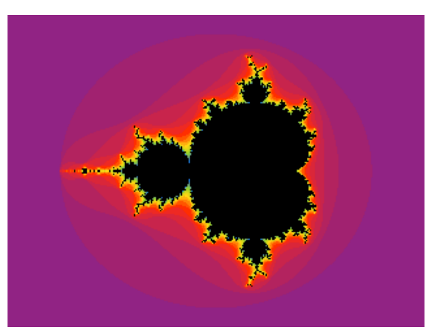
- Introduction to Chaos
- Life in the Complex plane
- Application of Newton-Raphson
- Bifurcation
- Bibliography

Introduction to Chaos

- "A butterfly flaps it's wings in one part of the world and creates a tornado in another."
- Can we predict how steam will rise from a cup of coffee?
- How can we predict the behavior of an iteration method?

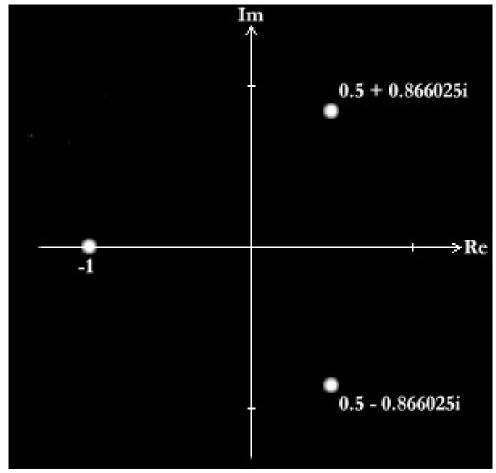
Background

- **Chaos**- apparently random behavior with purely deterministic causes
- Became more popular in the 80's with flashy fractal images generated by computers



Newton-Raphson

- Designed for real numbers (works with complex numbers when x_0 contains complex component)



- Eg. $x^3+1=0$

roots:

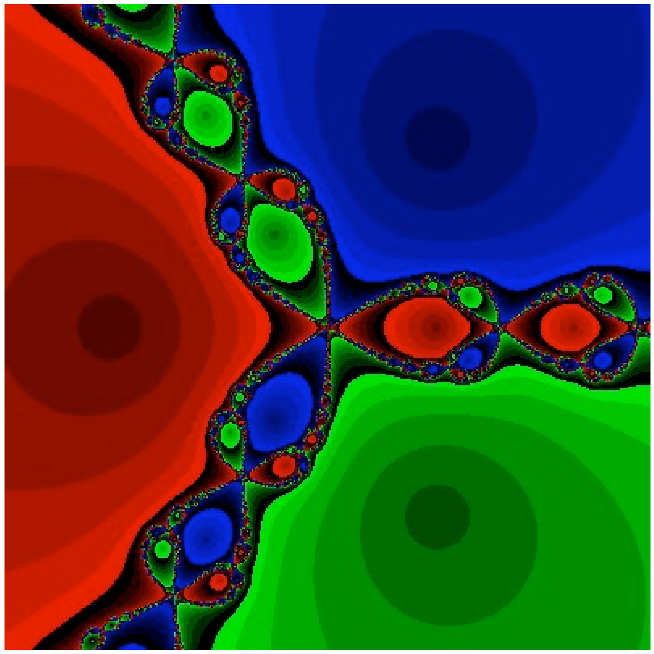
x= -1

x= 0.5 +/- 0.866025i

Newton-Raphson cont.

- Note the points are equally spaced on a unit circle (with radius=1)
  - **Symmetric**
- In a plane with real numbers, a good choice of x_0 will get closer to the true root with every iteration.
- Picking any point and iterating using Newton-Raphson for $f(x) = x^3+1$ will fall shows “basins of attraction”
  - **Easiest to see this visually**
  - **Each basin (root) is given it’s own color**
  - **The darker the color, the quicker the point chosen will iterate to the root**

Newton-Raphson cont.



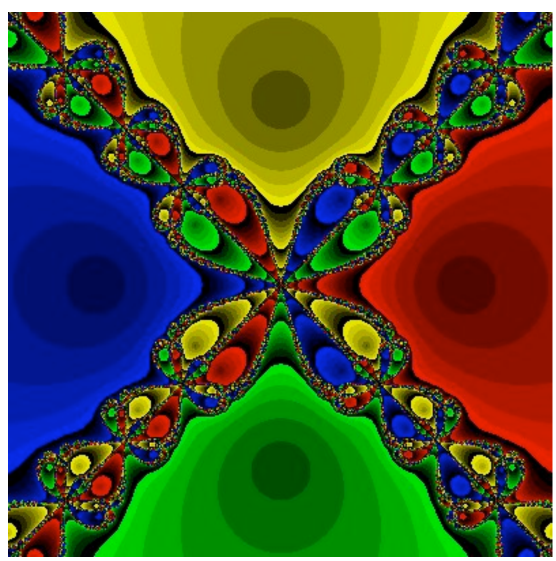
Blue:
x=0.5 + 0.866i

Red:
x= -1

Green:
x=0.5 - 0.866i

Newton-Raphson cont.

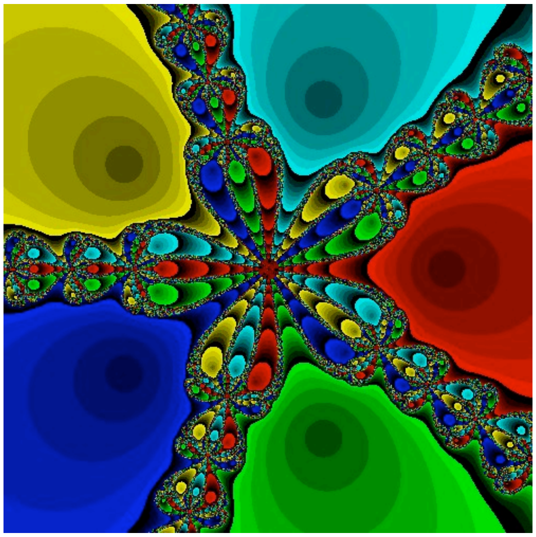
- Where any 2 colors meet, a 3rd color separates them.
- This pattern continues with self similarity *ad infinitum*
- Eg. $x^4-1=0$



Newton-Raphson cont.

- In general, for x^n +/- 1 = 0 has n roots
- Each root lies equally spaced on the unit circle
- Graphically, any two colors are separated by all other colors.

$x^5 - 1 = 0$



Logistic Equations

- Population change from year *n* to *n+1*

$$x_{n+1}=kx_n(1-x_n)$$

  - ***kx_n* indicates reproductive tendency proportional to the present population**
  - ***1-x* is the inhibiting term and takes into account the need to coexist and share resources**
- http://www.cut-the-knot.org/blue/chaos.shtml
  - **represent *k* as *4a* and plot *a* on the horizontal axis**

Bifurcation Summary

- Summary of results from the graph
  - **As *a* approaches 0.75 (k approaches 3), the rate of convergence decreases**
  - **At *a* = 0.75 (k=3), the graph bifurcates and splits cycles between 2 fixed points.**
  - **At *a* = 0.86237…, the graph has 4 fixed points**
  - **This process continues as *a* increases**
    - The next four points are replaced by 8 and 8 by 16 …
    - The horizontal distance between the split points (points of bifurcation) grows shorter and shorter.
  - **At *a* = 0.892 the bifurcation becomes so fast that the iterates race all over a segment instead of alternating between a few fixed points.**
    - The behavior is chaotic in the sense that it's absolutely impossible to predict where the next iterate will appear
  - **At *a*= 0.96…, the graph has 3 fixed points**

Bibliography

- Gleick, James, Chaos: Making a New Science, Viking, 1987.
- "Emergence in Chaos: There is order in Chaos" http://www.cut-the-knot.org/blue/chaos.shtml
- "Newton-Raphson method pictures" http://www.tardis.ed.ac.uk/~lard/fc/newton/index.html