The convergence of the secant method is superlinear

The purpose of this document is to show the following theorem:

Theorem 1.1 *Let $\{x_k\}_k^\infty$ be the sequence produced by the secant method. Assume the sequence converges to a root of $f(x) = 0$, i.e., $x_k \rightarrow x_\infty$, $f(x_\infty) = 0$. Moreover, assume the root x_∞ is regular: $f'(x_\infty) \neq 0$. Denote the error in the kth step by $E_k = x_k - x_\infty$. Under these assumptions, we have*

$$E_{k+1} \approx C E_k^{(1+\sqrt{5})/2} \approx C E_k^{1.618}, \quad \textit{for some constant } C. \tag{1}$$

The theorem is implied by three lemmas.

Lemma 1.2 *Under the assumptions and notations of the theorem:*

$$E_{k+1} \approx \frac{1}{2} \frac{f''(x_\infty)}{f'(x_\infty)} E_{k-1} E_k. \tag{2}$$

Proof. Using the definition of x_{k+1}, we find

$$E_{k+1} = x_{k+1} - x_\infty = x_k - f(x_k) \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} - x_\infty. \tag{3}$$

We can replace x_{x+1} by $x_k + E_k$ and x_k by $x_{k-1} + E_{k-1}$, so that

$$E_{k+1} = x_\infty + E_k - f(x_\infty + E_k) \frac{x_\infty + E_k - x_\infty - E_{k-1}}{f(x_\infty + E_k) - f(x_\infty + E_{k-1})} - x_\infty. \tag{4}$$

To simplify this expression, we apply the Taylor expansion of $f(x_\infty + E_k)$ and $f(x_\infty + E_{k-1})$ about x_∞:

$$f(x_\infty + E_k) = f(x_\infty) + f'(x_\infty) E_k + \frac{1}{2} f''(x_\infty) E_k^2 + O(E_k^3), \tag{5}$$

$$f(x_\infty + E_{k-1}) = f(x_\infty) + f'(x_\infty) E_{k-1} + \frac{1}{2} f''(x_\infty) E_{k-1}^2 + O(E_{k-1}^3). \tag{6}$$

Subtracting $f(x_\infty + E_{k-1})$ from $f(x_\infty + E_k)$:

$$f(x_\infty + E_k) - f(x_\infty + E_{k-1}) = f'(x_\infty)(E_k - E_{k-1}) + \frac{1}{2} f''(x_\infty)(E_k^2 - E_{k-1}^2) + O(E_k^3) - O(E_{k-1}^3). \tag{7}$$

Since $O(E_k^3) - O(E_{k-1}^3)$ is of a smaller order than E_k and E_{k-1} we omit this term. Using $E_k^2 - E_{k-1}^2 = (E_k - E_{k-1})(E_k + E_{k-1})$, we organize the above expression as

$$f(x_\infty + E_k) - f(x_\infty + E_{k-1}) \approx (E_k - E_{k-1})(f'(x_\infty) + f''(x_\infty)(E_k + E_{k-1})). \tag{8}$$

The left of (8) appears at the right of (4), so we derive the following expression

$$E_{k+1} \approx E_k - f(x_\infty + E_k) \frac{E_k - E_{k-1}}{(E_k - E_{k-1})(f'(x_\infty) + f''(x_\infty)(E_k + E_{k-1}))}. \tag{9}$$

Using a Taylor expansion for $f(x_\infty + E_k)$ about x_∞ (recall $f(x_\infty) = 0$) we have

$$E_{k+1} \approx E_k - E_k \frac{f'(x_\infty) + \frac{1}{2} f''(x_\infty) E_k}{f'(x_\infty) + \frac{1}{2} f''(x_\infty)(E_k + E_{k-1})}. \tag{10}$$

Now we put everything on the same denominator:

$$E_{k+1} \approx E_k \frac{f'(x_\infty) + \frac{1}{2} f''(x_\infty)(E_k + E_{k-1}) - f'(x_\infty) - \frac{1}{2} f''(x_\infty) E_k}{f'(x_\infty) + \frac{1}{2} f''(x_\infty)(E_k + E_{k-1})}, \tag{11}$$

which can be simplified as

$$E_{k+1} \approx E_k \frac{\frac{1}{2} f''(x_\infty) E_{k-1}}{f'(x_\infty) + \frac{1}{2} f''(x_\infty)(E_k + E_{k-1})}. \tag{12}$$

Because $E_k \to 0$ as $k \to \infty$, $\frac{1}{2} f''(x_\infty)(E_k + E_{k-1})$ is negligible compared to $f'(x_\infty)$, so we omit the second term in the denominator, to find the estimate

$$E_{k+1} \approx \frac{1}{2} \frac{f''(x_\infty)}{f'(x_\infty)} E_k E_{k-1}. \tag{13}$$

Q.E.D.

Lemma 2.1 *There exists a positive real number r such that:*

$$E_{k+1} \approx C E_{k-1} E_k \quad \Rightarrow \quad E_k^{1+1/r} \approx K E_k^r, \quad \textit{for some constants } C \textit{ and } K. \tag{14}$$

Proof. Assuming the convergence rate is r, there exists some constant A, so we can write

$$E_{k+1} \approx A E_k^r \quad \text{and} \quad E_k \approx A E_{k-1}^r \text{ or } \left(\frac{1}{A} E_k \right)^{1/r} \approx E_{k-1}. \tag{15}$$

Now we can replace the expressions for E_k and E_{k-1} in the left hand side of (14):

$$E_{k+1} \approx C \left(\frac{1}{A} \right)^{1/r} E_k^{1/r} E_k \approx B E_k^{1+1/r}. \tag{16}$$

Together with the assumption that $E_{k+1} \approx A E_k^r$, we obtain $E_k^{1+1/r} \approx \frac{A}{B} E_k^r$. So, we set $K = \frac{A}{B}$ and the lemma is proven. Q.E.D.

Lemma 2.2 *For the r of Lemma 2.1, we have*

$$E_k^{1+1/r} \approx C E_k^r \quad \Rightarrow \quad r = \frac{1+\sqrt{5}}{2}. \tag{17}$$

Proof. r satisfies the following equation

$$1 + \frac{1}{r} = r \Rightarrow r + 1 = r^2 \Rightarrow r^2 - r - 1 = 0. \tag{18}$$

The roots of $r^2 - r - 1 = 0$ are $r = \frac{1 \pm \sqrt{5}}{2}$. We take the positive value for r. Q.E.D.

The constant $r = \frac{1+\sqrt{5}}{2} \approx 1.618$ is the golden ratio.

References

[1] Floyd Hanson. MCS 471 Class Notes: Secant Method Error and Convergence Rate. Available at http://www.math.uic.edu/∼hanson/mcs471/classnotes.html.