Approximation via Correlation Decay when Strong Spatial Mixing Fails

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Abstract

Approximate counting via correlation decay is the core algorithmic technique used in the sharp delineation of the computational phase transition that arises in the approximation of the partition function of anti-ferromagnetic two-spin models.

Previous analyses of correlation-decay algorithms implicitly depended on the occurrence of strong spatial mixing. This, roughly, means that one uses worst-case analysis of the recursive procedure that creates the sub-instances. In this paper, we develop a new analysis method that is more refined than the worst-case analysis. We take the shape of instances in the computation tree into consideration and we amortise against certain “bad” instances that are created as the recursion proceeds. This enables us to show correlation decay and to obtain an FPTAS even when strong spatial mixing fails.

We apply our technique to the problem of approximately counting independent sets in hypergraphs with degree upper-bound ∆ and with a lower bound k on the arity of hyperedges. Liu and Lin gave an FPTAS for $k \geq 2$ and $\Delta \leq 5$ (lack of strong spatial mixing was the obstacle preventing this algorithm from being generalised to $\Delta = 6$). Our technique gives a tight result for $\Delta = 6$, showing that there is an FPTAS for $k \geq 3$ and $\Delta \leq 6$. The best previously-known approximation scheme for $\Delta = 6$ is the Markov-chain simulation based FPRAS of Bordewich, Dyer and Karpinski, which only works for $k \geq 8$.

Our technique also applies for larger values of $k$, giving an FPTAS for $k \geq 1.66\Delta$. This bound is not as strong as existing randomised results, for technical reasons that are discussed in the paper. Nevertheless, it gives the first deterministic approximation schemes in this regime. We further demonstrate that in the hypergraph independent set model, approximating the partition function is NP-hard even within the uniqueness regime.

1 Introduction

We develop a new method for analysing correlation decays in spin systems. In particular, we take the shape of instances in the computation tree into consideration and we amortise against

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certain “bad” instances that are created as the recursion proceeds. This enables us to show correlation decay and to obtain an FPTAS even when strong spatial mixing fails. To the best of our knowledge, strong spatial mixing is a requirement for all previous correlation-decay based algorithms. To illustrate our technique, we focus on the computational complexity of approximately counting independent sets in hypergraphs, or equivalently on counting the satisfying assignments of monotone CNF formulas.

The problem of counting independent sets in graphs (denoted \#IS) is extensively studied. A beautiful connection has been established, showing that approximately counting independent sets in graphs of maximum degree \(\Delta\) undergoes a computational transition which coincides with the uniqueness phase transition from statistical physics on the infinite \(\Delta\)-regular tree. The computational transition can be described as follows. Weitz [17] designed an FPTAS for counting independent sets on graphs with maximum degree at most \(\Delta = 5\). On the other hand, Sly [15] proved that there is no FPRAS for approximately counting independent sets on graphs with maximum degree at most \(\Delta = 6\) (unless NP = RP). The same connection has been established in the more general context of approximating the partition function of the hard-core model [17, 12, 15, 5, 6, 16] and in the even broader context of approximating the partition functions of generic antiferromagnetic 2-spin models [14, 6, 16, 9] (which includes, for example, the antiferromagnetic Ising model). As a consequence, the boundary for the existence of efficient approximation algorithms for these models has been mapped out\(^1\).

Approximate counting via correlation decay is the core technique in the algorithmic developments which enabled the sharp delineation of the computational phase transition. Another standard approach for approximate counting, namely Markov Chains Monte Carlo (MCMC) simulation, is also conjectured to work up to the uniqueness threshold, but the current analysis tools that we have do not seem to be powerful enough to show that. For example, sampling independent sets via MCMC simulation is known to have fast mixing only for graphs with degree at most 4 [3, 4], rather than obtaining the true threshold of 5.

In this work, we consider counting independent sets in hypergraphs with upper-bounded vertex degree, and lower-bounded hyperedge size. A hypergraph \(H = (V, F)\) consists of a vertex set \(V\) and a set \(F\) of hyperedges, each of which is a subset of \(V\). A hypergraph is said to be \(k\)-uniform if every hyperedge contains exactly \(k\) vertices. Thus, a 2-uniform hypergraph is the same as a graph. We will consider the more general case where each hyperedge has arity at least \(k\), rather than exactly \(k\).

An independent set in a hypergraph \(H\) is a subset of vertices that does not contain a hyperedge as a subset. We will be interested in computing \(Z_H\), which is the total number of independent sets in \(H\) (also referred to as the partition function of \(H\)). Formally, the problem of counting independent sets has two parameters — a degree upper bound \(\Delta\) and a lower bound \(k\) on the arity of hyperedges. The problem is defined as follows.\(^2\)

**Name** \#\text{HyperIndSet}(k, \Delta).

**Instance** A hypergraph \(H\) with maximum degree at most \(\Delta\) where each hyperedge has cardinality (arity) at least \(k\).

\(^1\) Note that there are non-monotonic examples of antiferromagnetic 2-spin systems where the boundary is more complicated because the uniqueness threshold fails to be monotonic in \(\Delta\) [8]. However, this can be cleared up by stating the uniqueness condition as uniqueness for all \(d \leq \Delta\). See [8, 9] for details.

\(^2\) Equivalently, one may think of this problem as the problem of counting satisfying assignments of a monotone CNF formulas, where vertices are variables and hyperedges are clauses. Being out of the independent set (as a vertex) corresponds to being true (as a variable).
Output The number $Z_H$ of independent sets in $H$.

Previously, $\#\text{HyperIndSet}(k, \Delta)$ has been studied using the MCMC technique by Borderwich, Dyer, and Karpinski [1, 2]. They give an FPRAS for all $k \geq \Delta + 2 \geq 5$ and for $k \geq 2$ and $\Delta = 3$. Despite equipping path coupling with optimized metrics obtained using linear programming, these bounds are not tight for small $k$. Liu and Lu [10] showed that there exists an FPTAS for all $k \geq 2$ and $\Delta \leq 5$ using the correlation decay technique.

Thus, the situation seems to be similar to the graph case — given the analysis tools that we have, correlation-decay brings us closer to the truth than the best-tuned analysis of MCMC simulation algorithms. On the other hand, the technique of Liu and Lu [10] does not extend beyond $\Delta = 5$. To explain the reason why it does not, we need to briefly describe the correlation-decay-based algorithm framework introduced by Weitz [17]. The main idea is to build a recursive procedure for computing the marginal probability that any given vertex is in the independent set. The recursion works by examining sub-instances with “boundary conditions” which require certain vertices to be in, or out, of the independent set. The recursion structure is called a “computation tree”. Nodes of the tree correspond to intermediate instances, and boundary conditions are different in different branches. The computation tree allows one to compute the marginal probability exactly but the time needed to do so may be exponentially large since, in general, the tree is exponentially large. Typically, an approximate marginal probability is obtained by truncating the computation tree to logarithmic depth so that the (approximation) algorithm runs in polynomial time. If the correlation between boundary conditions at the leaves of the (truncated) computation tree and the marginal probability at the root decays exponentially with respect to the depth, then the error incurred from the truncation is small and the algorithm succeeds in obtaining a close approximation.

All previous instantiations under this framework require a property called strong spatial mixing (SSM)\(^3\), which roughly states that, conditioned on any boundary condition on intermediate nodes, the correlation decays. In other words, SSM guards against the worst-case boundary conditions that might be created by the recursive procedure.

Let the $\Delta_1$-ary $k$-uniform hypertree $T_{k, \Delta}$ be the recursively-defined hypergraph in which each vertex has $\Delta_1$ “descending” hyperedges, each containing $k_1$ new vertices.

**Observation 1.** Let $k \geq 2$. For $\Delta \geq 6$, strong spatial mixing does not hold on $T_{k, \Delta}$.

Observation 1 follows from the fact that the infinite $(\Delta - 1)$-ary tree $T_{2, \Delta}$ can be embedded in the hypertree $T_{k, \Delta}$, and from well-known facts about the phase transition on $T_{2, \Delta}$.

Observation 1 prevents the generalisation of Liu and Lu’s algorithm [10] so that it applies for $\Delta \geq 6$, even with an edge-size lower bound $k$. The problem is that the construction of the computation tree involves constructing intermediate instances in which the arity of a hyperedge can be as small as 2. So, even if we start with a $k$-uniform hypergraph, the computation tree will contain instances with small hyperedges. Without strong spatial mixing, these small hyperedges cause problems in the analysis. Lu, Yang, and Zhang [11] discuss this problem and say “How to avoid this effect is a major open question whose solution may have applications in many other problems.” This question motivates our work.

To overcome this difficulty, we introduce a new amortisation technique in the analysis. Since lack of correlation decay is caused primarily by the presence of small-arity hyperedges within the intermediate instances, we keep track of such hyperedges. Thus, we track not

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\(^3\)See Section 2.1 for a definition.
Theorem 2. There is an FPTAS for \( \#\text{HyperIndSet}(3, 6) \).

Note that \( \#\text{HyperIndSet}(2, 6) \) is NP-hard to approximate due to [15], so our result is tight for \( \Delta = 6 \). This also shows that \( \Delta = 6 \) is the first case where the complexity of approximately counting independent sets differs on hypergraphs and graphs, as for \( \Delta \leq 5 \) both admit an FPTAS [10]. Moreover, Theorem 2 is stronger than the best MCMC algorithm [2] when \( \Delta = 6 \) as [2] only works for \( k \geq 8 \).

We also apply our technique to large \( k \).

Theorem 3. There exists a constant \( k_0 \) such that for all positive integers \( k \geq k_0 \) and \( \Delta \) satisfying \( k \geq 1.66\Delta \) there is an FPTAS for the problem \( \#\text{HyperIndSet}(k, \Delta) \).

In the large \( k \) case, our result is worse than that obtained by analysis of the MCMC algorithm [2] \( (k \geq 1.66\Delta \) rather than \( k \geq \Delta + 2) \) but it is incomparable since our algorithm is deterministic rather than randomised. The reason that our bound is worse is mainly due to technical difficulty in the analysis that we will explain shortly. In fact, we believe that, in the long run, analysis of the correlation-decay based algorithm is more likely to reveal the exact critical threshold than analysis of MCMC simulation, although other new ideas will probably be required in order to achieve sufficiently precise analysis.

The main technical difficulty in correlation-decay analysis is bounding a function that we call the “decay rate.” This boils down to solving an optimization problem with \( (k-1)(\Delta-1) \) variables. In previous work (e.g. [13]), this optimization has been solved using a so-called “symmetrization” argument, which reduces the problem to a univariate optimization via convexity. However, the many variables represent different branches in the computation tree. Since our analysis takes the shape of intermediate instances in the tree into consideration, the symmetrization argument does not work for us, and different branches take different values at the maximum. This problem is compounded by the fact that the shape of the sub-tree consisting of “bad” intermediate instances is heavily lopsided, and the assignment of variables achieving the maximum is far from uniform. Given these problems, there does not seem to be a clean solution to the optimization in our analysis. Instead of optimizing, we give an upper bound on the maximum decay rate. In Theorem 2, as \( k \) and \( \Delta \) are small, the number of variables is manageable, and our bounds are much sharper than those in Theorem 3. On the other hand, because of this, the proof of Theorem 3 is much more accessible, and we will use Theorem 3 as a running example to demonstrate our technique.

We also provide some insight on the hardness side. Recall that for graphs it is NP-hard to approximate \( \#\text{IS} \) beyond the uniqueness threshold \( (\Delta = 6) \) [15]. We prove that it is NP-hard to approximate \( \#\text{HyperIndSet}(6, 22) \) (Corollary 29). In contrast, we show that uniqueness holds on the 6-uniform \( \Delta \)-regular hypertree iff \( \Delta \leq 28 \) (Corollary 36). Thus, efficient approximation schemes cease to exist well below the uniqueness threshold on the hypertree. In fact, we show that this discrepancy grows exponentially in \( k \): for large \( k \), it is NP-hard to approximate \( \#\text{HyperIndSet}(k, \Delta) \) when \( \Delta \geq 20 \cdot 2^{k/2} \) (Theorem 28 and Corollary 30), despite the fact that uniqueness holds on the hypertree for all \( \Delta \leq 2^k/(2k) \) (Lemma 37). Theorem 28 follows from a rather standard reduction to the hard-core model

\footnote{For graphs the uniqueness and SSM thresholds coincide, but for hypergraphs they differ.}
on graphs. Nevertheless, it demonstrates that the computational-threshold phenomena in the hypergraph case \((k > 2)\) are substantially different from those in the graph case \((k = 2)\).

As mentioned earlier, there are models where efficient (randomised) approximation schemes exist (based on MCMC simulation) even though SSM does not hold. In fact, this can happen even when uniqueness does not hold. A striking example is the ferromagnetic Ising model (with external field). As [14] shows, there are parameter regimes where uniqueness holds but strong spatial mixing fails. It is easy to modify the parameters so that even uniqueness fails. Nevertheless, Jerrum and Sinclair [7] gave an MCMC-based FPRAS that applies for all parameters and for general graphs (with no degree bounds). It is still an open question to give a correlation decay based FPTAS for the ferromagnetic Ising model.

1.1 Outline of Paper

In Section 2, we first give some preliminaries. We give a formal definition of strong spatial mixing (Section 2.1) and a reformulation of \(\#\text{HyperIndSet}(k, \Delta)\) as the problem of counting satisfying assignments in monotone CNF formulas (Section 2.2). This will allow us to use the computation tree used by Liu and Lu [10]. A formal description of the computation tree of [10] is given in Section 2.3.

In Section 3, we give an overview of our proof approach, i.e., the main idea behind our new amortisation technique. In Section 4, we give the main ingredients to prove Theorem 3 (large \(k\)). Sections 5 and 6 contain the proofs for the technical ingredients used to prove Theorem 2 \((k = 3)\). Section 7 concludes the proof of Theorems 2 and 3.

Section 8 gives the formal statements and proofs of the hardness results stated in the Introduction. Also, Section 9 studies the uniqueness threshold on the \(k\)-uniform \(\Delta\)-regular hypertree (and gives the proofs of the uniqueness statements made in the Introduction). Finally, Section 10 gives the proof for several technical inequalities used in Section 6.

2 Preliminaries

2.1 Strong Spatial Mixing

For the purposes of this section, it will be convenient to view the independent set model as a 2-spin model. Namely, if \(H = (V, \mathcal{F})\) is a hypergraph, each independent set \(I\) can be viewed as a \(\{0, 1\}\)-assignment \(\sigma\) to the vertices in \(V\), where a vertex \(v\) is assigned the spin 1 under \(\sigma\) if \(v \in I\) and 0 otherwise.

We denote by \(\Omega_H\) the set of all independent sets in \(H\). The Gibbs distribution \(\mu_H(\cdot)\) is the uniform distribution over \(\Omega_H\). The Gibbs distribution of \(H\) can clearly be viewed as the uniform distribution over those assignments \(\sigma: V \to \{0, 1\}\) which encode a valid independent set of \(H\). For an assignment \(\sigma : V \to \{0, 1\}\) and a subset \(\Lambda \subset V\), we denote by \(\sigma|_\Lambda\) the restriction of \(\sigma\) to the subset \(\Lambda\).

For a hypergraph \(H = (V, \mathcal{F})\) and a subset \(\Lambda \subset V\), we denote by \(H_\Lambda\) the subgraph of \(H\) induced by \(\Lambda\), i.e., \(H_\Lambda := (\Lambda, \bigcup_{e \in \mathcal{F}} (e \cap \Lambda))\). Also, for a vertex \(v \in V\) and \(\Lambda \subset V\), we denote by \(\text{dist}(v, \Lambda)\) the length of the shortest path\(^5\) between \(v\) and a vertex of \(\Lambda\).

\(^5\)A path in a hypergraph with hyperedge set \(\mathcal{F}\) is a sequence of edges \(e_0, \ldots, e_\ell \in \mathcal{F}\) such that \(e_i \cap e_{i+1} \neq \emptyset\) for all \(i = 0, \ldots, \ell - 1\).
Let $\delta : \mathbb{Z}_+ \rightarrow [0,1]$. The independent set model exhibits strong spatial mixing on a hypergraph $H = (V, \mathcal{F})$ with decay rate $\delta(\cdot)$ iff for every $v \in V$, for every $\Lambda \subset V$, for any two configurations $\eta, \eta' : \Lambda \rightarrow \{0,1\}$ encoding independent sets of $H_{\Lambda}$, it holds that
\[
\left| \mu_H(\sigma(v) = 1 \mid \sigma_{\Lambda} = \eta) - \mu_H(\sigma(v) = 1 \mid \sigma_{\Lambda} = \eta') \right| \leq \delta(\text{dist}(v, \Lambda')),
\]
where $\Lambda'$ denotes the set of vertices in $\Lambda$ such that $\eta$ and $\eta'$ differ.

2.2 Reformulation in terms of Monotone CNF formulas

The problem of counting the independent sets of a hypergraph has an equivalent formulation in terms of monotone CNF formulas. Given a hypergraph $H = (V, \mathcal{F})$, let $C$ be a Boolean formula with variable set $V$. For each hyperedge, construct a clause which is the conjunction of all variables corresponding to vertices in the hyperedge. Let $C$ be the conjunction of all such clauses. Note that $C$ is a monotone formula — no variable is negated. Also, independent sets of $H$ are in one-to-one correspondence with satisfying assignments of $C$ — a variable is assigned value “true” in an assignment if and only if it is out of the corresponding independent set. Going the other direction, any monotone CNF formula can be viewed as a hypergraph. In the technical sections of this paper, we use the monotone CNF formulation.

2.3 The Computation Tree

In this section, we set up relevant notation and give an exposition of the computation tree of Liu and Lu [10] which will also be used in our proof (though our analysis will be different). The computation tree of Liu and Lu is given in terms of the monotone CNF version of the problem. Below we give the relevant definitions and notation; our notation aligns as much as possible with that of [10].

Let $C$ be an instance of a monotone CNF formula. We will denote the set of variables in $C$ by $V$ and set $n := |V|$. Variables in $V$ will be denoted by $x_1, x_2, \ldots$ and clauses in $C$ by $c_1, c_2, \ldots$. The arity of a clause $c$ will be denoted by $|c|$, i.e., $|c|$ is the number of variables appearing in the clause $c$. We assume throughout that no variable appears twice in the same clause. For a variable $x \in V$, we denote by $d_x(C)$ the number of clauses where $x$ appears. When $x$ and $C$ are clear from context, we will simply use $d$ to denote $d_x(C)$. When $C$ is clear from context, we will use $\Delta$ to denote $\max_{x \in V} d_x(C)$ and we will say that $C$ is a formula with max degree $\Delta$. Let $C_{k,\Delta}$ be the set of all monotone CNF formulas which have max degree $\Delta$ and whose clauses have arity at least $k$. Note that some formulas in $C_{k,\Delta}$ may have some clauses with arbitrarily large arities.

Our goal is to approximately count the number of satisfying assignments of a formula $C \in C_{k,\Delta}$, which we denote by $Z(C)$. Since $C$ is monotone, an assignment $\sigma : V \rightarrow \{0,1\}$ is satisfying if, for every clause in $C$, there is at least one variable $x \in c$ with $\sigma(x) = 1$. Note that $Z(C) > 0$ since the all-1 assignment satisfies every monotone CNF formula. For convenience, we will use the simplified notation “$x = 1$” to denote (the set of) satisfying assignments of $C$ in which $x$ is set to 1, and we similarly use “$x = 0$”. We associate the formula $C$ with a probability distribution in which each satisfying assignment has probability mass $1/Z(C)$. We will denote probabilities with respect to this distribution by $\Pr_C(\cdot)$.

Let $x$ be a variable in $V$. Define $R(C, x) := \frac{\Pr_C(x=0)}{\Pr_C(x=1)}$, this is well-defined since $\Pr_C(x = 1) > 0$ by the monotonicity of $C$. In fact, the monotonicity of $C$ also implies that $0 \leq \Pr_C(x = 0) < 1$.
Let \( R(C, x) \leq 1 \), where the upper bound follows from the fact that, for every satisfying assignment with \( x = 0 \), flipping the assignment of \( x \) to 1 does not affect satisfiability. Our interest in the quantity \( R(C, x) \) stems from the following simple lemma (the proof follows the argument in [10, Appendix A] and is given for completeness in Section 7).

**Lemma 5.** Let \( k \) and \( \Delta \) be positive integers. Suppose that there is a polynomial-time algorithm (in \( n \) and \( 1/\varepsilon \)) that takes an \( n \)-variable formula \( C \in C_{k, \Delta} \), a variable \( x \) of \( C \), and an \( \varepsilon > 0 \) and computes a quantity \( \hat{R}(C, x) \) satisfying \(|\hat{R}(C, x) - R(C, x)| \leq \varepsilon\). Then, there exists an FPTAS which approximates \( Z(C) \) for every \( C \in C_{k, \Delta} \).

Liu and Lu [10] established that a computation tree approach gives a recursive procedure for exactly calculating \( R(C, x) \) for any monotone CNF formula \( C \) and any variable \( x \in C \). We next give the details of this recursive procedure (see [10, Lemma 5]). First, we introduce the following definitions.

**Definition 6.** Let \( C \) be a monotone CNF formula and let \( x \) be a variable in \( C \). We call the variable \( x \) forced (in \( C \)) if \( x \) appears in a clause of arity 1 in \( C \) (note that in every satisfying assignment of \( C \) it must be the case that \( x \) is 1 and hence \( R(C, x) = 0 \)). We call the variable \( x \) free if \( x \) does not appear in any clause of \( C \) (note that \( R(C, x) = 1 \) in this case).

**Definition 7.** Let \( C \) be a monotone CNF formula and let \( c \) be a clause in \( C \). We call the clause \( c \) redundant (in \( C \)) if there is a clause \( c' \) in \( C \) such that \( c \) is a (strict) superset of \( c' \) (note that removing \( c \) from \( C \) does not affect the set of satisfying assignments of \( C \)).

We next give the details of the computation tree. The nodes in the computation tree will be pairs \((C, x)\) such that

\[
C \text{ is a monotone CNF formula and } x \text{ is a variable which is not forced in } C. \tag{1}
\]

Let \( C, x \) satisfy (1). We first perform a pre-processing step on \( C \) which involves (i) initially removing all of the redundant clauses, (ii) then, removing all clauses of arity 1. Note that part (ii) of the pre-processing step removes all forced variables that were present in \( C \); at the time of the removal, forced variables appear only in clauses of arity 1 since part (i) of the pre-processing step has already removed all redundant clauses in \( C \) (and hence all clauses of arity greater than 1 that contain forced variables). Denote the formula after the completion of the pre-processing step by \( \tilde{C} \). Note that every clause in \( \tilde{C} \) is also a clause in the initial formula \( C \). It follows that \( x \) is not forced in \( \tilde{C} \). Further, since removing redundant clauses does not change the set of satisfying assignments of \( C \) and \( x \) is not forced in \( C \), we have that \( R(\tilde{C}, x) = R(C, x) \).

If \( x \) is free in \( \tilde{C} \) (the formula after the pre-processing step), then the start node \((C, x)\) is (declared) a leaf of the computation tree (note that in this case \( R(C, x) = 1 \)). In the sequel, we assume that \( x \) is not free in \( \tilde{C} \). Denote by \( \{c_i\}_{i \in [d]} \) the clauses where \( x \) occurs in \( \tilde{C} \) and let \( w_i = |c_i| - 1 \) (note that \( d \geq 1 \)). We will use \( \mathbf{w} \) to denote the vector \((w_1, \ldots, w_d)\). The variables in clause \( c_i \) other than \( x \) will be denoted by \( x_{i,1}, \ldots, x_{i,w_i} \). For the pair \((C, x)\), we next construct pairs \((C_{i,j}, x_{i,j})\) for \( i \in [d] \) and \( j \in [w_i] \), where \( C_{i,j} \) is an appropriate subformula obtained from \( \tilde{C} \), roughly, by hard-coding (some of) the occurrences of the variables in \( C \) to either 1 or 0 (this will be explained below and will henceforth be referred to as pinning)

\footnote{Note that our notation for \( i,j \) is different from the one in [10]; there, the roles of \( i,j \) are interchanged.}
Precisely, for \(i \in [d]\), let \(C_i\) be the formula obtained from \(\tilde{C}\) by removing clauses \(c_1, \ldots, c_{i-1}\) (note that this has the same effect as pinning the occurrences of \(x\) in these clauses to 1) and pinning the occurrences of \(x\) in \(c_{i+1}, \ldots, c_d\) to 0. For \(j \in [w_i]\), the formula \(C_{i,j}\) is obtained from \(C_i\) by further removing clause \(c_i\) and pinning all the occurrences of \(x_{i,1}, \ldots, x_{i,j-1}\) to 0.

Before proceeding, let us argue that the pairs \((C_{i,j}, x_{i,j})\) satisfy (1) for all \(i \in [d]\) and \(j \in [w_i]\). For such \(i, j\), we first prove that \(C_{i,j}\) is a (satisfiable) monotone CNF formula. That is, we prove that the various pinnings in the construction of \(C_{i,j}\) from \(\tilde{C}\) do not pin all variables of some clause of \(\tilde{C}\) to 0. For the sake of contradiction, assume otherwise. Observe that \(C_{i,j}\) is obtained from \(\tilde{C}\) by either removing some clauses or by pinning some occurrences of the variables to 0. Clearly, removal of clauses does not affect satisfiability, so we may focus on the effect of pinning. For \(i \in [d]\) and \(j \in [w_i]\), the only variables whose (some of the) occurrences in \(\tilde{C}\) get pinned to 0 are \(x, x_{i,1}, \ldots, x_{i,j-1}\). Since we assumed that \(C_{i,j}\) is unsatisfiable, it must be the case that there exists a clause \(c'\) in \(\tilde{C}\) all of whose variables are (a subset of) \(x, x_{i,1}, \ldots, x_{i,j-1}\). It follows that \(c_i\) is redundant in \(\tilde{C}\) since it is a strict superset of clause \(c'\), contradiction, since the pre-processing operation ensures that \(\tilde{C}\) has no redundant clauses. Thus, \(C_{i,j}\) is satisfiable as wanted. Next, we show that \(x_{i,j}\) is not forced in \(C_{i,j}\).

First, observe that \(x_{i,j}\) is not forced in \(\tilde{C}\) since the second part of the preprocessing step ensures that \(\tilde{C}\) does not contain forced variables. Thus, the only way that \(x_{i,j}\) can be forced in \(C_{i,j}\) is if there existed a clause \(c'\) in \(\tilde{C}\) whose variables were \(x_{i,j}\) together with a subset of \(x, x_{i,1}, \ldots, x_{i,j-1}\). Since \(\tilde{C}\) includes \(c_i\) and \(\tilde{C}\) does not have redundant clauses, it must be the case that \(c' = c_i\). It remains to observe that \(C_{i,j}\) does not include (any part of) \(c_i\), from which it follows that \(x_{i,j}\) is not forced in \(C_{i,j}\).

We are now ready to state the relation between \(R(C, x)\) and the quantities \(R(C_{i,j}, x_{i,j})\) with \(i \in [d]\) and \(j \in [w_i]\).

**Lemma 8.** [10, Lemma 5] It holds that

\[
R(C, x) = \prod_{i=1}^{d} \left( 1 - \prod_{j=1}^{w_i} \frac{R(C_{i,j}, x_{i,j})}{1 + R(C_{i,j}, x_{i,j})} \right). \tag{2}
\]

**Proof.** The proof is identical to the proof of [10, Lemma 5] (which in turn builds on the technique of [17]), we give the proof for completeness.

Recall that \(\tilde{C}\) is the formula after the preprocessing step and that \(R(\tilde{C}, x) = R(C, x)\). We may assume that \(x\) is not free in \(\tilde{C}\) (otherwise, it holds that \(R(\tilde{C}, x) = 1\), which coincides with the evaluation of the right hand side of (2) under the standard convention that the empty product evaluates to 1).

Equation (2) follows immediately from the following two equalities.

\[
R(\tilde{C}, x) = \prod_{i \in [d]} R(C_i, x), \quad R(C_i, x) = 1 - \prod_{j=1}^{w_i} \frac{R(C_{i,j}, x_{i,j})}{1 + R(C_{i,j}, x_{i,j})}. \tag{3}
\]

The first equality in (3) is a consequence of a telescoping expansion of \(\frac{Pr_p(x=0)}{Pr_p(x=1)}\). To see this, let \(\tilde{C}'\) be the formula obtained from \(\tilde{C}\) by replacing, for all \(i \in [d]\), the occurrence of the
variable $x$ in clause $c_i$ by a new variable $x'_i$. We have that

$$R(\tilde{C}, x) = \frac{\Pr_{\tilde{C}}(x = 0)}{\Pr_{\tilde{C}}(x = 1)} = \frac{\Pr_{\tilde{C}}(x'_1 = 0, \ldots, x'_d = 0)}{\Pr_{\tilde{C}}(x'_1 = 1, \ldots, x'_d = 1)} = \prod_{i \in [d]} \frac{\Pr_{\tilde{C}}(x'_1 = 1, \ldots, x'_{i-1} = 1, x'_i = 0, \ldots, x'_d = 0)}{\Pr_{\tilde{C}}(x'_1 = 1, \ldots, x'_{i-1} = 1, x'_{i+1} = 0, \ldots, x'_d = 0)} = \prod_{i \in [d]} \frac{\Pr_{C_i}(x = 0)}{\Pr_{C_i}(x = 1)},$$

which yields the first equality in (3) after substituting $R(C_i, x) = \frac{\Pr_{C_i}(x = 0)}{\Pr_{C_i}(x = 1)}$.

For the second equality in (3), observe that $x$ appears only in clause $c_i$ of the formula $C_i$, and thus (denoting by $C_i \setminus c_i$ the formula which is obtained from $C_i$ by deleting clause $c_i$)

$$R(C_i, x) = \frac{\Pr_{C_i}(x = 0)}{\Pr_{C_i}(x = 1)} = 1 - \Pr_{C_i \setminus c_i}(x_i = 0, \ldots, x_{i,j} = 0) = 1 - \prod_{j=1}^{w_i} \Pr_{C_{i,j}}(x_{i,j} = 0),$$

which proves the desired equality after substituting $\Pr_{C_{i,j}}(x_{i,j} = 0) = \frac{R(C_{i,j,x_{i,j}})}{1 + R(C_{i,j,x_{i,j}})}$. \hfill \Box

By applying (2) recursively, it is not hard to see that one can compute the quantity $R(C, x)$ exactly. Of course, exact computation using this scheme will typically require exponential time, so as in [10] we will stop the recursion at some (small) depth $L$ to keep the computations feasible within polynomial time. This will yield a quantity $R(C, x, L)$ and the hope is that, by choosing $L$ appropriately, the error $|R(C, x, L) - R(C, x)|$ will be sufficiently small.

In light of (2), a natural way to define $R(C, x, L)$ for integer $L \geq 0$ is as follows\(^7\).

$$R(C, x, L) = \begin{cases} 1, & \text{if } x \text{ is free in } \tilde{C} \text{ or } L = 0, \\ \prod_{i=1}^{d} \left(1 - \prod_{j=1}^{w_i} \frac{R(C_{i,j,x_{i,j},L-1})}{1 + R(C_{i,j,x_{i,j},L-1})}\right), & \text{otherwise.} \end{cases}$$

It is immediate that when the formula $C$ has max degree bounded by a constant and, further, every clause has arity also bounded above by a constant, one can compute $R(C, x, L)$ in time polynomial in $n$ whenever $L = O(\log n)$. To account for formulas where the arities of the clauses can be arbitrarily large (but still where the degrees of variables are bounded by a constant), one needs to be more careful with clauses of large arity (i.e., their arity as a function of $n$ is $\omega(1)$, say $\log n$). As in [10], we will account for this more general setting by pruning the recursion earlier whenever we encounter clauses with large arity. Namely, for integer $L$ we set

$$R(C, x, L) = \begin{cases} 1, & \text{if } x \text{ is free in } \tilde{C} \text{ or } L \leq 0, \\ \prod_{i=1}^{d} \left(1 - \prod_{j=1}^{w_i} \frac{R(C_{i,j,x_{i,j},L-I_{w_i}})}{1 + R(C_{i,j,x_{i,j},L-I_{w_i}})}\right), & \text{otherwise,} \end{cases}$$

(4)

where $l_{w_i} := \lceil \log_6 (w_i + 1) \rceil$ (note that $l_1 = \ldots = l_5 = 1$). We remark here that the particular choice of the logarithm base in the definition of $l_{w_i}$ is not very important as long as it is a big enough constant.

\(^7\)Note that the value 1 of $R(C, x, L)$ when $L \leq 0$ is somewhat arbitrary since $L \leq 0$ corresponds to stopping the recursion. Our choice of the value 1 will be convenient for technical reasons that will become apparent in the proof of the upcoming Lemma 13.
Remark 9. For formulas $C$ with a variable $x$ which is not forced in $C$, we have the lower bound $R(C, x) \geq (1/2)^{d_x(C)}$. The bound is simple to see using (2) and the fact that $R(C_{i,j}, x_{i,j}) \leq 1$ for all $i \in [d_x(C)]$ and $j \in [w_i]$. Similarly, for all integer $L$ and all nodes $(C, x)$ in the computation tree we have the bound $R(C, x, L) \geq (1/2)^{d_x(C)}$.

Using correlation decay techniques together with a new method to account for the shape of the computation tree, we will show the following key lemma (proved in Section 5).

Lemma 10. Let $\Delta = 6$. There exist constants $\alpha, \tau$ with $0 < \alpha < 1$ and $\tau > 0$ such that the following holds for all integers $L$. Let $C$ be a monotone CNF formula whose clauses all have arity greater than or equal to 3 and, further, each variable occurs in at most $\Delta$ clauses. Then,

$$|R(C, x, L) - R(C, x, \infty)| \leq \tau \alpha^L,$$

where the quantity $R(C, x, L)$ is defined recursively from (4).

We prove an analogous lemma in the case where $k, \Delta$ are large, see Lemma 11 in Section 4. In the following section, we give an overview of our approach to proving Lemma 10.

3 Proof Approach

To prove Lemma 10, the standard approach so far in the literature has been to show that, for a node $(C, x)$ in the computation tree, the quantity $|R(C, x, L) - R(C, x, \infty)|$ is bounded by $\alpha \max_{i,j} |R(C_{i,j}, x_{i,j}, L - 1) - R(C_{i,j}, x_{i,j}, \infty)|$ for some constant $0 < \alpha < 1$ and then, inductively, to deduce that $|R(C, x, L) - R(C, x, \infty)|$ decays exponentially in $L$. This approach has been extremely successful when strong spatial mixing holds [14, 9, 10, 13, 18, 11].

In our setting, this inductive approach is problematic since, inside the computation tree, we are faced with the possibility that the formula at the root of a subtree has many arity-2 clauses. For $\Delta \geq 6$, these subtrees prohibit the application of the above proof scheme since they are in non-uniqueness and hence the desired step-by-step decay is no longer present.

While arity-2 clauses are problematic, clauses with larger arity do at least lead to good decay of correlation in a single step. Thus, our approach is to do an amortised analysis. In a single step, we track both the one-step decay of correlation and the immediate creation of decay of correlation in a single step. Thus, our approach is to do an amortised analysis. In our setting, this inductive approach is problematic since, inside the computation tree, we will show the following key lemma (proved in Section 5).

Crucially, note that the root formula $C$ satisfies

$$|m(C, x, L) - m(C, x, \infty)| = |R(C, x, L) - R(C, x, \infty)|,$$

since by the assumption in Lemma 10 we have that $b(C) = 0$. Thus, the key step in the proof of Lemma 10 is to show that the quantity $|m(C, x, L) - m(C, x, \infty)|$ decays exponentially with $L$; we will show that, for an arbitrary node $(C, x)$ in the computation tree, it holds that

$$|m(C, x, L) - m(C, x, \infty)| \leq \alpha \max_{i,j} |m(C_{i,j}, x_{i,j}, L - 1) - m(C_{i,j}, x_{i,j}, \infty)|,$$  \hspace{1cm} (5)

\footnote{In fact, in Section 5, we define $m(C, x, L)$ as $\delta(C) \Phi(R(C, x, L))$ for an appropriate function $\Phi$, see (28).}
where $\alpha \in (0, 1)$ is a constant.

Since arity-2 clauses are the source of problems it is important not to create too many of them. Therefore, we also have to be careful in the construction of the computation tree. We achieve this by carefully ordering the clauses that we process at each step to avoid creating arity-2 clauses as much as possible.

Unfortunately, the quantity $m(C, x, L)$ is more complicated than the plain message $R(C, x, L)$ which has been studied in the past since it incorporates combinatorial information about the formula $C$ and thus it does not satisfy a simple recursion (unlike $R(C, x, L)$). Nevertheless, we are able to define a multi-variable quantity $\kappa$ (see (32)) and to show that when $\kappa \leq 1$, inequality (5) holds.

Details are given in Sections 5 and 6 which make up the bulk of this paper. First, in Section 4, we apply the approach to the case of large $k$, where the proof is (significantly) shorter. There, instead of tracking the number of clauses of arity 2, we track (roughly) the aggregate arities of the clauses in $C$. Other than that, the high-level proof approach is similar to what is described above.

4 The case of large $k$

In this section we will assume that the instance has an arity lower bound $k$ for all clauses and a degree upper bound $\Delta$ for all variables. We show that when $k$ is linear in $\Delta$ and $\Delta$ is large enough, the correlation decays exponentially. More precisely, let $c := 0.565$ and $\beta \sim 1.65115$ be the solution to $2^{0.0001c^3} = 2 \times 0.9997(1 - c^3)c^2$. Then $k \geq \beta \Delta + 3$ is sufficient to imply correlation decay.

Lemma 11. Let $\beta \sim 1.65$ be defined as above. Let $k$ be a sufficiently large positive integer and let $\Delta$ be a positive integer satisfying $k \geq \beta \Delta + 3$. There are real numbers $\alpha$ and $\tau$ satisfying $0 < \alpha < 1$ and $\tau > 0$ such that for every $C \in \mathcal{C}_k, \Delta$ and every integer $L$,

$$|R(C, x, L) - R(C, x, \infty)| \leq \tau \alpha^L,$$

where the quantity $R(C, x, L)$ is defined recursively from (4).

To estimate the error $|R(C, x, L) - R(C, x, \infty)|$, we will track a specific quantity $m(C, x, L)$ which is assigned to each node in the computation tree. Let $C$ be the original monotone CNF formula and let $(N, x)$ be a node in the computation tree. As explained earlier, each clause $c'$ of $N$ is obtained from a clause $c$ of $C$ by pinning a certain number of variables to 0 (possibly none), which effectively is the same as removing those variables. We call these 0-pinnings deficits and let $\max\{0, k - |c'|\}$ be number of deficits of $c'$. Note that a clause of arity larger than $k$ is considered to have no deficit, although some variables of it may have been pinned to 0. Let $D(N) = \sum_{c' \in N} \max\{0, k - |c'|\}$ denote the total deficits of $N$. Also observe that if a clause $c$ of $C$ does not show up in $N$, it does not contribute any deficits. For any node $(N, x)$ in the computation tree, let

$$m(N, x, L) := \delta^{D(N)} R(N, x, L)$$

where $\delta \in (0, 1)$ is a constant that we will choose later.

Now let us calculate how the number of deficits changes in one step of the recursion. Let $(N, x)$ be a node in the computation tree. Note that pinning any variable to 1 will remove
all clauses containing it, and will therefore eliminate all deficits of these clauses. Moreover,
for a clause of arity 2, pinning any of its variables to either 0 or 1 will eliminate the whole
clause due to the preprocessing step. Let \( b_2(N, x) \) (or simply \( b_2 \) when \( N \) and \( x \) are clear from
the context) denote the number of arity-2 clauses containing \( x \) in \( N \). In the recursion, we
will always order the clauses so that arity-2 clauses come last. Thus, clauses \( c_1, \ldots, c_{d-b_2} \)
each have arity at least 3 and clauses \( c_{d-b_2+1}, \ldots, c_d \) each have arity 2. Due to these arity-2
clauses, the deficits in every branch below \( (N, x) \) will decrease by at least \( b_2(k - 2) \).

Now consider an integer \( i \) in the range \( 1 \leq i \leq d - b_2 \). Recall that in \( N_i \), we pin appearances
of \( x \) prior to \( i \) to 1, thus eliminating deficits in clauses \( c_1, \ldots, c_{i-1} \). Together with the removal
of clause \( c_i \), these removals decrease the total deficits by \( \sum_{i=1}^{t} \max\{0, k - 1 - w_t\} \), where
\( w_t = |c_t| - 1 \). Let \( s_i = \sum_{i=1}^{t} \max\{0, k - 1 - w_t\} \). We also pin all appearances of \( x \) after \( i \)
to 0, increasing the total number of deficits by at most \( d - i - b_2 \). Here we have a "\(-b_2""
because pinning a variable to 0 in a arity-2 clause does not increase the deficits. Moreover,
in \( N_{i;j} \), we pin all appearances of \( j - 1 \) other variables to 0. Each of these pinnings may
contribute at most \( \Delta - 1 \) deficits, giving a total increase of at most \( (\Delta - 1)(j - 1) \). Next
consider \( i \geq d - b_2 + 1 \). For such an \( i \), clause \( c_i \) has arity 2, and it is easy to see that deficits
do not increase in the branch corresponding to \( N_{i;j} \). Also note that the preprocessing steps
do not increase deficits. Hence we have the following upper bounds on \( D(N_{i;j}) \):

\[
D(N_{i;j}) \leq \begin{cases} 
D(N) - b_2(k - 2) - s_i + d - i - b_2 + (\Delta - 1)(j - 1) & \text{if } 1 \leq i \leq d - b_2, \\
D(N) - b_2(k - 2) - s_{d-b_2} & \text{if } d - b_2 + 1 \leq i \leq d.
\end{cases}
\]

(7)

The key to our analysis is to bound the correlation decay of \( m(C, x, L) \). We will analyse
the recursion for \( m(C, x, L) \) based on that of \( R(C, x, L) \). Recall that the recursion for \( R(C, x, L) \)
depends on the function \( F^{d,w}(r) \) implicitly defined by (4), i.e.,

\[
F^{d,w}(r) := \prod_{i=1}^{d} \left( 1 - \prod_{j=1}^{w_i} \frac{r_{i,j}}{1 + r_{i,j}} \right),
\]

(8)

where \( r_{i,j} \in [0, 1] \) for all \( i \in [d], j \in [w_i] \) (we have that \( R(C, x, L) = F^{d,w}(\{R(C_{i,j}, x_{i,j}, L - l_{w_i})\}) \)).

Hence,

\[
\frac{m(C, x, L)}{\delta^{D(C)}} = F^{d,w} \left( \left\{ \frac{m(C_{i,j}, x_{i,j}, L - l_{w_i})}{\delta^{D(C_{i,j})}} \right\} \right)
\]

(9)

For \( i \in [d] \), the following quantity \( \rho^{w,i}_{d,\alpha} \) will roughly upper bound the sensitivity of \( |m(C, x, L) - m(C, x, \infty)| \) to the \( i \)-th clause in which \( x \) appears in \( C \), or more precisely, to the quantities

\[
|m(C_{i,j}, x_{i,j}, L - l_{w_i}) - m(C_{i,j}, x_{i,j}, \infty)| \] for \( j \in [w_i] \):

\[
\rho^{w,i}_{d,\alpha}(r) := \begin{cases} 
\alpha^{-l_{w_i}} \delta^{s_{i-1} + d - b_2} \sum_{j=1}^{w_i} \delta^{-(j-1)(\Delta - 1)} \left| \frac{\partial F^{d,w}}{\partial r_{i,j}} \right| & \text{if } 1 \leq i \leq d - b_2, \\
\alpha^{-l_{w_i}} \delta^{s_{d-b_2}} \left| \frac{\partial F^{d,w}}{\partial r_{i,1}} \right| & \text{if } d - b_2 + 1 \leq i \leq d.
\end{cases}
\]

(10)

Note that \( \rho^{w,i}_{d,\alpha}(r) \) depends also on \( r_{i',j} \) with \( i' \neq i \). The decay rate of \( |m(C, x, L) - m(C, x, \infty)| \)
in terms of \( L_i \) for all formulas \( C \) with max degree \( \Delta \), will be captured by the aggregation of
\( \rho_{\delta, \alpha}^{w,i}(r) \)'s, namely,
\[
\rho_{\delta, \alpha}^{d_2, w}(r) := \delta^{d_2(k-2)} \sum_{i=1}^{d} \rho_{\delta, \alpha}^{w,i}(r)
\]  
(11)

The main technical lemma of the section is the following.

**Lemma 12.** For all sufficiently large \( \Delta \), for any \( k \geq \beta \Delta + 3 \) where \( \beta \sim 1.65115 \) is defined earlier, there exists constants \( 0 < \delta < 1 \), \( 0 < \alpha < 1 \), and \( U > 0 \) such that
\[
\rho_{\delta, \alpha}^{d_2, w}(r) \leq \begin{cases} 1, & \text{when } d \leq \Delta - 1, \\ U, & \text{when } d = \Delta, \end{cases}
\]  
(12)

for all \( 0 \leq r \leq 1 \).

Before proving Lemma 12, we show that it is sufficient to imply Lemma 11.

**Proof of Lemma 11.** We will show that for some constant \( \tilde{\tau} > 0 \), it holds that
\[
|m(C, x, L) - m(C, x, \infty)| \leq \tilde{\tau} \alpha^L.
\]  
(13)

It is obvious that (6) follows from (13) as \( D(C) = 0 \) and
\[
|m(C, x, L) - m(C, x, \infty)| = |R(C, x, L) - R(C, x, \infty)|.
\]

To prove (13), we will show that
\[
|m(C, x, L) - m(C, x, \infty)| \leq U \max_{i,j} \{\alpha^{w_i} |m(C_{i,j}, x_{i,j}, L - l_{w_i}) - m(C_{i,j}, x_{i,j}, \infty)|\},
\]  
(14)

and for any node \((N, x)\) where \( N \neq C \),
\[
|m(N, x, L) - m(N, x, \infty)| \leq \max_{i,j} \{\alpha^{w_i} |m(N_{i,j}, x_{i,j}, L - l_{w_i}) - m(N_{i,j}, x_{i,j}, \infty)|\}.
\]  
(15)

The two equations (14) and (15) imply (13) by induction, where the base case is the fact that for any node \((N, x)\) and \( L \leq 0 \),
\[
|m(N, x, L) - m(N, x, \infty)| = \delta^{D(N)} |R(N, x, L) - R(N, x, \infty)| \leq 1,
\]
as \( \delta \in (0, 1) \), \( D(N) \geq 0 \), and \( 0 \leq R(N, x, L), R(N, x, \infty) \leq 1 \).

The only difference between (14) and (15) is that the degree of \( x \) may be \( \Delta \) in \( C \) but would never exceed \( \Delta - 1 \) in \( N \). This is because any intermediate node has one degree consumed in the previous step. To unify the reasoning, in the following we will assume that \((N, x)\) is a general node in the computation tree.

For \( i \in [d] \) and \( j \in [w_i] \), it holds that
\[
m(N_{i,j}, x_{i,j}, L - l_{w_i}) = \delta^{D(N_{i,j})} R(N_{i,j}, x_{i,j}, L - l_{w_i}),
m(N_{i,j}, x_{i,j}, \infty) = \delta^{D(N_{i,j})} R(N_{i,j}, x_{i,j}, \infty).
\]
Denote by \( r^{(1)} \) the vector whose coordinates are given by \( R(N_{i,j}, x_{i,j}, L - l_{w_i}) \) for \( i \in [d] \) and \( j \in [w_i] \), and by \( r^{(2)} \) the vector whose coordinates are given by \( R(N_{i,j}, x_{i,j}, \infty) \) for \( i \in [d] \) and \( j \in [w_i] \). Observe that

\[
R(N, x, L) = F_{d,w}^{d,w}(r^{(1)}) \quad \text{and} \quad R(N, x, \infty) = F_{d,w}^{d,w}(r^{(2)}).
\]

We next bound \(|F_{d,w}^{d,w}(r^{(1)}) - F_{d,w}^{d,w}(r^{(2)})|\) in terms of \( \max_{i,j} \{ |r^{(1)}_{i,j} - r^{(2)}_{i,j}| \} \). For \( \theta \in [0,1] \), let \( r_{i,j}(\theta) = \theta r^{(1)}_{i,j} + (1 - \theta) r^{(2)}_{i,j} \). Denote by \( r(\theta) \) the vector whose entries are \( r_{i,j}(\theta) \). Let \( h(\theta) := F(r(\theta)) \). By applying the Mean Value theorem to the function \( h(\theta) \), there exists \( \theta_0 \in (0,1) \) such that

\[
F(r^{(1)}) - F(r^{(2)}) = \frac{\partial h}{\partial \theta} \bigg|_{\theta = \theta_0}.
\]

We have that

\[
\frac{\partial h}{\partial \theta} = \sum_{i=1}^{d} \sum_{j=1}^{w_i} \frac{\partial F(r(\theta))}{\partial r_{i,j}(\theta)} \left( r^{(1)}_{i,j} - r^{(2)}_{i,j} \right).
\]

It follows that

\[
|m(N, x, L) - m(N, x, \infty)| = \delta^{D(N)} |R(N, x, L) - R(N, x, \infty)|
\]

\[
= \delta^{D(N)} \left| \sum_{i=1}^{d} \sum_{j=1}^{w_i} \frac{\partial F(r(\theta_0))}{\partial r_{i,j}(\theta_0)} \left( r^{(1)}_{i,j} - r^{(2)}_{i,j} \right) \right|
\]

\[
\leq \delta^{D(N)} \sum_{i=1}^{d} \sum_{j=1}^{w_i} \delta^{D(N)} |\alpha^{-l_{w_i}}| \left( \frac{\partial F(r(\theta_0))}{\partial r_{i,j}(\theta_0)} \right) \left( \alpha^{l_{w_i}} |m(N_{i,j}, x_{i,j}, L - l_{w_i}) - m(N_{i,j}, x_{i,j}, \infty)| \right)
\]

\[
\leq \left( \sum_{i=1}^{d} \sum_{j=1}^{w_i} \delta^{D(N)-D(N_{i,j})} |\alpha^{-l_{w_i}}| \left( \frac{\partial F(r(\theta_0))}{\partial r_{i,j}(\theta_0)} \right) \right) \max_{i,j} \left\{ \alpha^{l_{w_i}} |m(N_{i,j}, x_{i,j}, L - l_{w_i}) - m(N_{i,j}, x_{i,j}, \infty)| \right\},
\]

(16)

where we applied the triangle inequality in the third line. To get (14) and (15), it remains to bound \( D(N) - D(N_{i,j}) \) for \( i \in [d] \) and \( j \in [w_i] \). Due to (7) and \( \delta \in (0,1) \), we have that

\[
\sum_{i=1}^{d} \sum_{j=1}^{w_i} \delta^{D(N)-D(N_{i,j})} |\alpha^{-l_{w_i}}| \left( \frac{\partial F(r(\theta_0))}{\partial r_{i,j}(\theta_0)} \right) \leq \kappa^{d,b_2,w} F(r(\theta_0)).
\]

(17)

Combining (16), (17), and Lemma 12 yields (14) and (15). This concludes the proof of Lemma 11.

We prove Lemma 12 next.

Proof of Lemma 12. We will choose \( \alpha = 0.9999 \) and \( \delta \) such that \( \delta^\Delta = c \), where \( c = 0.565 \). First note that

\[
\left| \frac{\partial F}{\partial r_{i,j}} \right| = \frac{F(r)}{r_{i,j}(1 + r_{i,j})} \cdot \prod_{t=1}^{w_i} \frac{r_{i,t}}{1 + r_{i,t}} \cdot \frac{\prod_{t=1}^{w_i} \frac{r_{i,t}}{1 + r_{i,t}}}{1 - \prod_{t=1}^{w_i} \frac{r_{i,t}}{1 + r_{i,t}}}.
\]

(18)
Hence,

$$
\kappa_{d,\alpha}^{b,\omega}(r) = \delta_{b^2(k-2)} F(r) \left( \sum_{i=1}^{d-b_2} \alpha^{-l_{w_i}} \delta_{s_i+i-d-b_2} \prod_{j=1}^{w_i} \frac{r_{i,j}}{1+\prod_{j=1}^{w_i} r_{i,j}} \sum_{j=1}^{w_i} \delta^{-(j-1)(\Delta-1)} \frac{1}{r_{i,j}(1+r_{i,j})} \right) + \sum_{i=d-b_2+1}^{d} \alpha^{-1} \delta^{s_i-d-b_2} \frac{1}{1+r_{i,1}},
$$

where we used the fact that $l_{w_i} = 1$ if $i \geq d - b_2 + 1$. Clearly $F(r) \leq 1$. Hence,

$$
\kappa_{d,\alpha}^{b,\omega}(r) \leq \delta_{b^2(k-2)} \left( \sum_{i=1}^{d-b_2} \alpha^{-l_{w_i}} \delta_{s_i+i-d-b_2-(w_i-1)(\Delta-1)} \prod_{j=1}^{w_i} \frac{r_{i,j}}{1+\prod_{j=1}^{w_i} r_{i,j}} \sum_{j=1}^{w_i} \frac{1}{r_{i,j}(1+r_{i,j})} + \alpha^{-1} b_2 \delta^{s_i-d-b_2} \right).
$$

Let $r_i$ be the sub-vector $\{r_{i,j} \mid 1 \leq j \leq w_i\}$ of $r$. A key observation is that the function

$$
G_{w_i}(r_i) := \frac{\prod_{j=1}^{w_i} \frac{r_{i,j}}{1+\prod_{j=1}^{w_i} r_{i,j}} \sum_{j=1}^{w_i} \frac{1}{r_{i,j}(1+r_{i,j})}} {1 - \prod_{j=1}^{w_i} \frac{r_{i,j}}{1+\prod_{j=1}^{w_i} r_{i,j}}} \sum_{j=1}^{w_i} \frac{1}{r_{i,j}(1+r_{i,j})}
$$

is increasing with respect to each $r_{i,j}$ when $w_i \geq 5$, as for any $1 \leq t \leq w_i$,

$$
\frac{\partial G_{w_i}(r_i)} {\partial r_{i,t}} = G_{w_i}(r_i) \left( \frac{1}{r_{i,t}(1+r_{i,t})} \cdot \frac{1}{1 - \prod_{j=1}^{w_i} \frac{r_{i,j}}{1+\prod_{j=1}^{w_i} r_{i,j}} \sum_{j=1}^{w_i} \frac{1}{r_{i,j}(1+r_{i,j})}} - \left( \sum_{j=1}^{w_i} \frac{1}{r_{i,j}(1+r_{i,j})} \right)^{-1} \frac{2r_{i,t} + 1} {r_{i,t}^2(1+r_{i,t})^2} \right) \geq \frac{\prod_{j=1}^{w_i} \frac{r_{i,j}}{1+\prod_{j=1}^{w_i} r_{i,j}} \sum_{j=1}^{w_i} \frac{1}{r_{i,j}(1+r_{i,j})}} {1 - \prod_{j=1}^{w_i} \frac{r_{i,j}}{1+\prod_{j=1}^{w_i} r_{i,j}}} \sum_{j=1}^{w_i} \frac{1}{r_{i,j}(1+r_{i,j})} \left( \sum_{j=1}^{w_i} \frac{1}{r_{i,j}(1+r_{i,j})} - \frac{2r_{i,t} + 1} {r_{i,t}^2(1+r_{i,t})^2} \right) \geq \frac{\prod_{j=1}^{w_i} \frac{r_{i,j}}{1+\prod_{j=1}^{w_i} r_{i,j}} \sum_{j=1}^{w_i} \frac{1}{r_{i,j}(1+r_{i,j})}} {1 - \prod_{j=1}^{w_i} \frac{r_{i,j}}{1+\prod_{j=1}^{w_i} r_{i,j}}} \sum_{j=1}^{w_i} \frac{1}{r_{i,j}(1+r_{i,j})} \left( \frac{w_i - 1}{2} - \frac{2}{1+r_{i,t}} \right) \geq 0.
$$

Hence, when $w_i \geq 5$, $G_{w_i}(r_i) \leq G_{w_i}(1) = w_i 2^{-(w_i+1)} (1 - 2^{-w_i})^{-1}$. It is also not hard to verify that this bound also holds for $w_i = 4$, and $G_{w_i}(r_i) \leq 2^{1-w_i}$ when $w_i = 2, 3$. Combining the two cases together, we have that $G_{w_i}(r_i) \leq 2^{-w_i} \mu(w_i)$ for all $w_i \geq 1$, where

$$
\mu(w_i) := \max\{2, w_i (2 - 2^{1-w_i})^{-1}\}.
$$

Going back to (20), we get that

$$
\kappa_{d,\alpha}^{b,\omega}(r) \leq \delta_{b^2(k-2)} \left( \sum_{i=1}^{d-b_2} \alpha^{-l_{w_i}} \delta_{s_i+i-d-b_2-(w_i-1)(\Delta-1)} 2^{-w_i} \mu(w_i) + \alpha^{-1} b_2 \right).
$$

It is easy to see that $\kappa_{d,\alpha}^{b,\omega}(r) \leq k \Delta \delta^{-k} \alpha^{-k}$. Let $U = k \Delta \delta^{-k} \alpha^{-k}$ and the part of the claim regarding $d = \Delta$ is proved. We will assume $d \leq \Delta - 1$ in the sequel. On the right hand side of (21) we have completely eliminated $r$. 

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As \( l_{w_i} = \lceil \log_6 (w_i + 1) \rceil \leq \log_6 (w_i + 1) + 1 \), \( \alpha^{-l_{w_i}} \leq \alpha^{-1}(w_i + 1)^{-\log_6 \alpha} \). Recall that \( \alpha = 0.9999 \), so \( -\log_6 \alpha < 0.00006 \). It is easy to verify that \( \alpha^{-l_{w_i}} \leq \alpha^{-1} w_i^{0.0001} \) for any \( w_i \geq 0 \). As a consequence, for each \( w_i \), if \( w_i > k - 1 \), then the right hand side of (21) increases if we replace \( w_i \) with \( k - 1 \). Henceforth we may assume that \( w_i \leq k - 1 \), which allows us to rewrite \( s_i = s_{i+1} + k - 1 - w_i \). It implies that

\[
\kappa_{d,b_2,w}^2(r) \leq \delta^{b_2(k-2)} \left( \sum_{i=1}^{d-b_2} \alpha^{-1} w_i^{0.0001} \delta^{s_{i,-1} + k + i + b_2 - 1 - w_i} \Delta_2 - w_i \mu(w_i) + \alpha^{-1} b_2 \right).
\]

Recall that \( \delta^\Delta = c \), where \( c = 0.565 \). Then we have that

\[
\alpha \kappa_{d,b_2,w}^2(r) \leq \delta^{b_2(k-2)} \left( \sum_{i=1}^{d-b_2} \delta^{s_{i,-1} + k + i + b_2 - 1 - w_i} w_i^{0.0001} (2c)^{-w_i} \mu(w_i) + b_2 \right) \\
\leq \delta^{b_2(k-1)} \sum_{i=1}^{d-b_2} \delta^{s_{i,-1} w_i^{0.0001} (2c)^{-w_i} \mu(w_i) + b_2} \delta^{b_2(k-2)}.
\]

Define the following function of \( w \):

\[
\eta_d(w) := \sum_{i=1}^{d-b_2} \delta^{s_{i,-1} w_i^{0.0001} (2c)^{-w_i} \mu(w_i)},
\]

where \( 2 \leq w_i \leq k - 1 \) for all \( 1 \leq i \leq d - b_2 \).

We claim that the maximum of \( \eta_d(w) \) is taken at some \( w' \) such that \( w'_i = k - 1 \) for \( 1 \leq i \leq t \) and \( w'_i = 2 \) for \( t + 1 \leq i \leq d - b_2 \) for some \( 0 \leq t \leq d - b_2 - 1 \). First observe that once \( w_j \) is fixed for all \( j > i \), we may maximize \( \eta_d(w) \) with respect to \( w_i \), regardless of any \( w_j \) where \( j < i \).

We will show the claim inductively by finding the value of \( w'_i \) one at a time from \( i = d - b_2 \) to \( i = 1 \). It is easy to verify that the function

\[
\xi(w_i) := w_i^{0.0001} (2c)^{-w_i} \mu(w_i)
\]

takes its maximum at \( w_i = 2 \) with our choice of \( c \). Hence the base case holds for \( i = d - b_2 \) with \( w'_{d-b_2} = 2 \). Moreover, \( \xi(2) - \xi(x) > 0.05 \) for all \( x \geq 3 \).

Now assume the claim holds for all \( j > i \). We will show that \( w'_i = 2 \) or \( w'_i = k - 1 \). We have that

\[
\eta_d(w) \leq \eta_d(w_1, \ldots, w_i, w'_i+1, \ldots, w'_{d-b_2}) \\
= \sum_{j=1}^{i-1} \delta^{s_j} \xi(w_j) + \delta^{s_{i-1}} \left( \xi(w_i) + \delta^{k-1 - w_i} \eta_i \right),
\]

where the term

\[
\eta_i := \sum_{j=i+1}^{d-b_2} \delta^{s_{j-1} + \sum_{k=i+1}^{j-1} (k-1 - w'_k)} \xi(w'_j)
\]
depends on $w'_j$ for $j > i$ but not $w_i$. By the induction hypothesis, there exists some $t$ such that $w'_j = k - 1$ for $i < j \leq \max \{t, i\}$, and $w'_j = 2$ for $\max \{t, i\} < j \leq d - b_2$. It implies that

$$
\eta_i \leq t(k-1)^{1.0001}(2c)^{-(k-1)} \left(2 - 2^{2-k}\right)^{-1} + \frac{2^{0.0001}}{2c^2} \sum_{j=t+1}^{d-b_2} \delta^{(k-3)(j-t-1)}
\leq \Delta(k-1)^{1.0001}(2c)^{-(k-1)} \left(2 - 2^{2-k}\right)^{-1} + \frac{2^{0.0001}}{2c^2(1 - \delta^{k-3})}, \quad (23)
$$

Recall that $k \geq \beta \Delta + 3$ where $\beta \sim 1.65115$ is the solution to $2^{0.0001}c^\beta = 2 \times 0.9997(1 - c^\beta)c^2$. With sufficiently large $\Delta$ and $k$, we have that $\Delta(k-1)^{1.0001}(2c)^{-(k-1)} \left(2 - 2^{2-k}\right)^{-1} < 0.0001$ as $2c > 1$. (In fact, $\Delta \geq 90$ suffices.) Moreover we have that

$$
\frac{2^{0.0001}}{2c^2(1 - \delta^{k-3})} \leq \frac{2^{0.0001}}{2c^2} \cdot \frac{1}{1 - c^\beta} = 0.9997c^{-\beta}.
$$

Then,

$$
\Delta(k-1)^{1.0001}(2c)^{-(k-1)} \left(2 - 2^{2-k}\right)^{-1} + \frac{2^{0.0001}}{2c^2(1 - \delta^{k-3})} \leq 0.9998c^{-\beta}, \quad (24)
$$

and combining (24) and (23) we get that

$$
\eta_i < c^{-\beta}. \quad (25)
$$

Let

$$
\sigma_i(w_i) := \xi(w_i) + \delta^{k-1-w_i} \eta_i.
$$

Since $w'_i$ maximizes $\eta_d(w')$, $w'_i$ maximizes $\sigma_i(x)$ among integers $2 \leq x \leq k - 1$. The function $\sigma_i(w_i)$ is a convex function when $w_i \geq 17$ as a sum of convex functions. This is because that $\delta^{k-1-w_i} \eta_i$ is always convex and $\xi(w_i)$ is convex when $w_i$ is sufficiently large (17 suffices in this case). To see the latter, notice that $\xi(w_i) = w_i^{1.0001}(2 - 2^{1-w_i})^{-1}(2c)^{-w_i}$ when $w_i \geq 4$. Straightforward calculation shows that $w_i^{1.0001}(2c)^{-w_i}$ is convex when $w_i \geq 17$ and $(2 - 2^{1-w_i})^{-1}$ is convex for all $w_i \geq 1$. Moreover, both functions are decreasing when $w_i \geq 17$. Then $\xi(w_i)$, as the product of the two, is convex when $w_i \geq 17$.

Hence $\sigma_i(w_i)$ takes its maximum at either some $w_i \leq 17$ or $w_i = k - 1$. For any integer $3 \leq x \leq 17$, we have that

$$
\sigma_i(2) - \sigma_i(x) = \xi(2) - \xi(x) - \eta_i \delta^{k-3}(\delta^{2-x} - 1)
> 0.05 - \eta_i(\delta^{-15} - 1) \geq 0,
$$

where we used (25) and the fact that $\delta$ goes to 1 for sufficiently large $\Delta$. (In fact $\Delta \geq 500$ suffices.) It implies that the maximum of $\sigma_i$ is either $\sigma_i(2)$ or $\sigma_i(k - 1)$, namely either $w'_i = 2$ or $w'_i = k - 1$.

For the rest of the claim, we need to show that if $w'_i = k - 1$ for some $i$, then $w'_j = k - 1$ for all $j < i$. Let $i$ be the first $i$ such that $w'_i = k - 1$, then $\sigma_i(k - 1) > \sigma_i(2)$. It implies that

$$
\xi(k-1) + \eta_i > \xi(2) + \delta^{k-3} \eta_i.
$$
Notice that $\eta_{i-1} > \eta_i$ if $w'_i = k - 1$. Hence,

$$\xi(k - 1) + \eta_{i-1} > \xi(2) + \delta^{k-3}\eta_{i-1}.$$ 

It means that $\sigma_{i-1}(k - 1) > \sigma_{i-1}(2)$ and $w'_i = k - 1$. Induct on $i$ and the claim holds. Similar to (23), we have that

$$\eta_d(w) \leq \eta_d(w') = l(k - 1)^{1.0001}(2c)^{-(k-1)} \left(2 - 2^{2^k-1} + \frac{9.0001}{2c} \sum_{i=t+1}^{d-b_2} \delta^{(k-3)(i-t-1)} \right) \leq \Delta(k - 1)^{1.0001}(2c)^{-(k-1)} \left(2 - 2^{2^k-1} + \frac{9.0001}{2c^2(1-\delta^{k-3})} \right) \leq 0.9998c^{-\beta},$$

where in the last line we used (24). Plug (26) into (22):

$$\alpha \kappa_{\delta,\alpha}^{d,b_2,w}(r) \leq \delta^{k+b_2(k-1)} \eta_d(w) + b_2 \delta^{b_2(k-2)} \leq \delta^{b_2(k-2)} \left(0.9998 \delta^k c^{-\beta} + b_2 \right) \leq c^{b_2} \left(0.9998 + b_2 \right).$$

It is easy to verify that $c^\beta < 0.5$ and for all integers $b_2 \geq 0$,

$$c^{b_2} \left(0.9998 + b_2 \right) < 2^{-b_2} \left(0.9998 + b_2 \right) \leq 0.9999 = \alpha.$$

Hence $\kappa_{\delta,\alpha}^{d,b_2,w}(r) \leq 1$ by (27) and the claim holds. \hfill \Box

5 A finer analysis to treat $k \geq 3$, $\Delta = 6$

In this section, we prove Lemma 10. Recall, our goal is to show that, for monotone CNF formulas whose clauses have arity greater or equal than $3$ and, further, each variable occurs in at most $\Delta = 6$ clauses and no variable appears in the same clause twice, the sequence $R(C, x, L)$ (defined recursively in (4)) converges to $R(C, x)$ exponentially fast in $L$.

The first point that will be important for us is the ordering of the clauses $c_i$ in the computation tree. More precisely, recall that for a node $(C, x)$ of the computation tree, the clauses where $x$ appears in $C$ are denoted by $c_i$ for $i \in [d]$, where $d$ denotes the degree of the variable $x$ in $C$. Namely, in the construction of the computation tree, clauses with $|c_i| = 3$ are processed first. In particular, using $b_3$ to denote the number of clauses such that $|c_i| = 3$, we will order the clauses so that $w_1 = \cdots = w_{b_3} = 2$ and for $i \geq b_3 + 1$, $w_i \neq 2$ and $w_i \geq 1$ (the latter follows by our assumption that the variable $x$ is not forced in $C$, cf. (1)).

The second point that will also be important for us is to pay special attention to clauses of arity $2$, i.e., clauses with arity $|c_i| = 2$ or, equivalently, $w_i = 1$.

To estimate the error $|R(C, x, L) - R(C, x, \infty)|$ for formulas with max degree $\Delta = 6$, we will track a specific quantity $m(C, x, L)$ which is assigned to each node in the computation tree. Let

$$\eta := \eta(\Delta) = (1/2)^\Delta.$$

Denote by $b(C)$ the total number of clauses of arity $2$ in $C$. Then, for a node $(C, x)$ in the computation tree, let

$$m(C, x, L) := \delta^{b(C)} \Phi(R(C, x, L))$$

(28)
where $\delta \in (0, 1]$, and $\Phi : [\eta, 1] \to \mathbb{R}$ satisfies:

$\Phi$ is continuously differentiable on $[\eta, 1]$, and $\varphi := \Phi'$ satisfies $\varphi(z) > 0$ for $z \in [\eta, 1]$. \hfill (29)

For $\Delta = 6$, the value of $\delta$ will be later chosen to be $\delta = 9789/10000$ and $\Phi$ will be specified in the upcoming equation (46) (in particular, $\delta$ and $\Phi$ depend only on $\Delta$ but not on $C, x$). Also, note that for all nodes $(C, x)$ in the computation tree the quantity $m(C, x, L)$ is well-defined; this is because of the property (1) and the lower bound on $R(C, x, L)$ given in Remark 9.

We will focus on upper bounding $|m(C, x, L) - m(C, x, \infty)|$. More precisely, we will show that this quantity decays exponentially with $L$. In turn, this will imply that $|R(C, x, L) - R(C, x, \infty)|$ decays exponentially with $L$. Before presenting the main lemma for this section, we will need to set up some quantities which will capture the decay rate of $|m(C, x, L) - m(C, x, \infty)|$. Let $\alpha := 1 - 10^{-4}$. We will show that $\alpha$ upper bounds the decay rate of $|m(C, x, L) - m(C, x, \infty)|$ and thus the decay rate of $|R(C, x, L) - R(C, x, \infty)|$ as well for all formulas $C$ with max degree $\Delta = 6$ and all variables $x$ in $C$.

Let $d, b_2, b_3 \in \mathbb{Z}$ and $w = (w_1, \ldots, w_d) \in \mathbb{Z}^d$ be such that

$$1 \leq d, \quad 0 \leq b_2, b_3 \leq d, \quad b_2 + b_3 \leq d$$

$$w_i = 2 \text{ for } i = 1, \ldots, b_3, \quad w_i \geq 3 \text{ or } w_i = 1 \text{ for } i = b_4 + 1, \ldots, d.$$ \hfill (30)

When we use these quantities, we will have a formula $C$ in mind which will have no redundant clauses. Then, $d$ will be the number of clauses in $C$ where $x$ appears and $b_2, b_3$ will be the number of clauses in $C$ of arity exactly 2 and 3 where $x$ appears, respectively. Consider also the function $F^{d,w}(r)$ implicitly defined by (4), i.e.,

$$F^{d,w}(r) := \prod_{i=1}^{d} \left( 1 - \frac{w_i}{1 + r_{i,j}} \right),$$ \hfill (8)

where $r_{i,j} \in [0, 1]$ for all $i \in [d], j \in [w_i]$. Denote by $r$ the vector with components $r_{i,j}$ for $i \in [d]$ and $j \in [w_i]$. For $i \in [d]$, the following quantity $\rho^{w,i}_{\delta, \Phi, \alpha}(r)$ will roughly upper bound the sensitivity of $|m(C, x, L) - m(C, x, \infty)|$ to the quantities $|m(C_{i,j}, x_{i,j}, L) - m(C_{i,j}, x_{i,j}, \infty)|$ for $j \in [w_i]$ (for all formulas $C$ with max degree $\Delta$):

$$\rho^{w,i}_{\delta, \Phi, \alpha}(r) := \alpha^{-w_i} \sum_{j=1}^{w_i} \left( \frac{1}{\delta} \right) (j-1)(\Delta-1) \frac{1}{\varphi(r_{i,j})} \left| \frac{\partial F^{d,w}}{\partial r_{i,j}} \right|.$$ \hfill (31)

Note that $\rho^{w,i}_{\delta, \Phi, \alpha}(r)$ depends also on $r_{i,j}$ with $i' \neq i$. The decay rate of $|m(C, x, L) - m(C, x, \infty)|$ in terms of $L$, for all formulas $C$ with max degree $\Delta = 6$, will be captured by the following quantity:

$$\kappa^{d,b_2,b_3,w}_{\delta, \Phi, \alpha}(r) := \varphi(F^{d,w}(r)) \delta^{b_2} \left( \sum_{i=1}^{b_3} \left( \frac{1}{\delta} \right)^{b_3-i} \rho^{w,i}_{\delta, \Phi, \alpha}(r) \right) + \sum_{i=b_3+1}^{d} \rho^{w,i}_{\delta, \Phi, \alpha}(r).$$ \hfill (32)

The main lemma we prove in this section is the following.

**Lemma 13.** Let $\Delta \geq 2$ be an integer. Suppose there exist constants $0 < \delta, \alpha < 1$, $U > 0$ and a function $\Phi$ satisfying (29) such that for all $d, b_2, b_3, w$ satisfying (30) it holds that

$$\kappa^{d,b_2,b_3,w}_{\delta, \Phi, \alpha}(r) \leq \begin{cases} 1, & \text{when } d \leq \Delta - 1, \\ U, & \text{when } d = \Delta, \end{cases}$$ \hfill (33)
for all $(1/2)^{\Delta - 1} \leq r \leq 1$.

Then, there exists a constant $\tau > 0$ such that for all monotone CNF formulas $C$ with max-degree $\Delta$ and no clauses of arity $\leq 2$, for every variable $x \in C$, for all integer $L$, it holds that

$$|R(C, x, L) - R(C, x, \infty)| \leq \tau \alpha^L.$$  \hfill (34)

Proof. We will show that for some constant $\hat{\tau} > 0$, for all monotone CNF formulas $C$ with max-degree $\Delta$, for all variables $x \in C$, for all $L \geq 0$, it holds that

$$|m(C, x, L) - m(C, x, \infty)| \leq \hat{\tau} \alpha^L.$$  \hfill (35)

Assuming (35) for the moment, let us conclude (34). Let $C$ be a monotone CNF formula with max-degree $\Delta$ and no clauses of arity $\leq 2$. We may assume that $L > 0$ since for $L = 0$ the inequality (34) holds for all sufficiently large $\tau$ (any $\tau \geq 1$ works). By the definition of $m(\cdot, \cdot, \cdot)$ and since by assumption $b(C) = 0$, we have that

$$|m(C, x, L) - m(C, x, \infty)| = |\Phi(R(C, x, L)) - \Phi(R(C, x, \infty))|.$$  \hfill (36)

Let

$$K_{\Phi}^{\text{min}} := \min_{x \in [\eta, 1]} \Phi'(x) = \min_{x \in [\eta, 1]} \varphi(x), \quad K_{\Phi}^{\text{max}} := \max_{x \in [\eta, 1]} \Phi'(x) = \max_{x \in [\eta, 1]} \varphi(x).$$

Since $\varphi$ is continuous and $\varphi(x) > 0$ for all $x \in [\eta, 1]$, we have that $K_{\Phi}^{\text{min}}, K_{\Phi}^{\text{max}} > 0$. We have that

$$|R(C, x, L) - R(C, x, \infty)| \leq \frac{1}{K_{\Phi}^{\text{min}}} |\Phi(R(C, x, L)) - \Phi(R(C, x, \infty))|.$$  \hfill (37)

To see (37), we may assume that $R(C, x, L) \neq R(C, x, \infty)$ (otherwise the inequality holds at equality), in which case the inequality follows by an immediate application of the Mean Value Theorem to the function $\Phi$. Combining (36), (37) with (35) yields (34) with $\tau = \hat{\tau}/K_{\Phi}^{\text{min}}$, as desired.

To prove (35), we will first show a slightly weaker claim. Namely, for all nodes $(C, x)$ in the computation tree where $C$ is a monotone CNF formula with max-degree $\Delta$ and $x \in C$ is a variable with degree $\leq \Delta - 1$, for all integer $L$, it holds that

$$|m(C, x, L) - m(C, x, \infty)| \leq K_{\Phi}^{\text{max}} \alpha^L.$$  \hfill (38)

For $L \leq 0$, we have that

$$|m(C, x, L) - m(C, x, \infty)| = \delta^{b(C)} |\Phi(R(C, x, L)) - \Phi(R(C, x, \infty))|$$

$$\leq K_{\Phi}^{\text{max}} |R(C, x, L) - R(C, x, \infty)| \leq K_{\Phi}^{\text{max}},$$

where in the first inequality we used that $\delta \in (0, 1]$, $b(C) \geq 0$ and an application of the Mean Value theorem analogous to the one used in (37), while in the second inequality we used that for $L \leq 0$ it holds that $R(C, x, L) = 1$ (by definition) and $0 \leq R(C, x, \infty) \leq 1$. Since $0 < \alpha < 1$, this proves (38) for $L \leq 0$.

To prove (38) for integer $L > 0$ we proceed by induction on $L$. Namely, we assume that $L > 0$ and that (38) holds for all smaller values than $L$ (the base cases $L \leq 0$ have already been shown).

Recall from (1) that $x$ is not forced in $C$ and that $\tilde{C}$ is the formula obtained by removing the redundant clauses in $C$. We may assume that $x$ is not free in $\tilde{C}$ — otherwise, observe
that $R(C, x, L) = R(C, x, \infty) = 1$ and thus $m(C, x, L) = m(C, x, \infty)$, so that (38) holds. We will thus focus on $x$ which appear only in (a non-zero number of) clauses in $C$ of arity $\geq 2$.

For $i \in [d]$ and $j \in [w_i]$, it holds that

$$m(C_{i,j}, x_{i,j}, L - l_{w_i}) = \delta(C_{i,j}) \Phi(R(C_{i,j}, x_{i,j}, L - l_{w_i}))$$
$$m(C_{i,j}, x_{i,j}, \infty) = \delta(C_{i,j}) \Phi(R(C_{i,j}, x_{i,j}, \infty)).$$

(39)

Denote by $r^{(1)}$ the vector whose coordinates are given by $R(C_{i,j}, x_{i,j}, L - l_{w_i})$ for $i \in [d]$ and $j \in [w_i]$. Denote also by $r^{(2)}$ the vector whose coordinates are given by $R(C_{i,j}, x_{i,j}, \infty)$ for $i \in [d]$ and $j \in [w_i]$. Observe that

$$R(C, x, L) = F^{d,w}(r^{(1)}) \text{ and } R(C, x, \infty) = F^{d,w}(r^{(2)}).$$

Note also that $(1/2)^{\Delta-1} \mathbf{1} \leq r^{(1)}, r^{(2)} \leq \mathbf{1}$ (cf. property (1) for the nodes of the computation tree, Remark 9 and footnote 7).

We next bound $|\Phi(F^{d,w}(r^{(1)})) - \Phi(F^{d,w}(r^{(2)}))|$ in terms of $\max_{i,j} |\Phi(r_{i,j}^{(1)}) - \Phi(r_{i,j}^{(2)})|$. For convenience, denote $F := F^{d,w}$ and for $i \in [d]$ and $j \in [w_i]$, let

$$z_{i,j}^{(1)} = \Phi(r_{i,j}^{(1)}), \quad z_{i,j}^{(2)} = \Phi(r_{i,j}^{(2)}).$$

For $\theta \in [0, 1]$, let $z_{i,j}(\theta) = \theta z_{i,j}^{(1)} + (1 - \theta) z_{i,j}^{(2)}$ and let $r_{i,j}(\theta) = \Phi^{-1}(z_{i,j}(\theta))$ (note that the inverse $\Phi^{-1}$ exists and is uniquely defined in the interval $\Phi([\eta, 1])$, cf. (29)). Denote by $r(\theta)$ the vector whose coordinates are $r_{i,j}(\theta)$ and note that $(1/2)^{\Delta-1} \mathbf{1} \leq r(\theta) \leq \mathbf{1}$. Finally, let $h(\theta) := \Phi(F(r(\theta)))$. Observe that $h$ is differentiable for all values of $\theta \in [0, 1]$. By applying the Mean Value theorem to the function $h(\theta)$, we obtain that there exists $\theta_0 \in (0, 1)$ such that

$$\Phi(F(r^{(1)})) - \Phi(F(r^{(2)})) = \frac{\partial h}{\partial \theta} \bigg|_{\theta = \theta_0}.$$

We have that

$$\frac{\partial h}{\partial \theta} = \frac{\partial \Phi(F(r(\theta)))}{\partial \theta} = \Phi'(F(r(\theta))) \frac{\partial F(r(\theta))}{\partial \theta} = \Phi'(F(r(\theta))) \left( \sum_{i=1}^{d} \sum_{j=1}^{w_i} \frac{\partial F(r(\theta))}{\partial r_{i,j}(\theta)} \frac{\partial r_{i,j}(\theta)}{\partial \theta} \right)$$
$$= \Phi'(F(r(\theta))) \left( \sum_{i=1}^{d} \sum_{j=1}^{w_i} \frac{1}{\Phi(r_{i,j}(\theta))} \frac{\partial F(r(\theta))}{\partial r_{i,j}(\theta)} \frac{\partial z_{i,j}(\theta)}{\partial \theta} \right)$$
$$= \Phi'(F(r(\theta))) \left( \sum_{i=1}^{d} \sum_{j=1}^{w_i} \frac{1}{\Phi(r_{i,j}(\theta))} \frac{\partial F(r(\theta))}{\partial r_{i,j}(\theta)} \left( z_{i,j}^{(1)} - z_{i,j}^{(2)} \right) \right).$$
It follows that
\[
|m(C, x, L) - m(C, x, \infty)| = \delta^{b(C)}|\Phi(R(C, x, L)) - \Phi(R(C, x, \infty))| \\
\leq \max_r \left\{ \delta^{b(C)} \left( \sum_{i=1}^{d} \sum_{j=1}^{w_i} \varphi(F(r)) \left| \frac{\partial F(r)}{\partial r_{i,j}} \right| |z_{i,j}^{(1)} - z_{i,j}^{(2)}| \right) \right\} \\
= \max_r \left( \sum_{i=1}^{d} \sum_{j=1}^{w_i} \delta^{b(C)-b(C_{i,j})} \varphi(F(r)) \varphi(r_{i,j}) \left| \frac{\partial F(r)}{\partial r_{i,j}} \right| |m(C_{i,j}, x_{i,j}, L - l_{w_i}) - m(C_{i,j}, x_{i,j}, \infty)| \right),
\]
where the inequality follows by the triangle inequality and the assumption that \( \varphi = \Phi' \) satisfies (29). Now note that for all \( i \in [d] \) and \( j \in [w_i] \), we have by induction that
\[
|m(C_{i,j}, x_{i,j}, L - l_{w_i}) - m(C_{i,j}, x_{i,j}, \infty)| \leq K_{\Phi}^{\max} \alpha^{L-l_{w_i}}.
\]
From (40) and the bounds (41), we obtain
\[
|m(C, x, L) - m(C, x, \infty)| \leq K_{\Phi}^{\max} \max_r \left( \sum_{i=1}^{d} \sum_{j=1}^{w_i} \delta^{b(C)-b(C_{i,j})} \varphi(F(r)) \varphi(r_{i,j}) \left| \frac{\partial F(r)}{\partial r_{i,j}} \right| \alpha^{-l_{w_i}} \right) \alpha^L.
\]
It remains to bound \( b(C) - b(C_{i,j}) \) for \( i \in [d] \) and \( j \in [w_i] \). Recall that \( \widetilde{C} \) is the formula obtained from \( C \) by removing redundant clauses. Clearly, we have that \( b(\widetilde{C}) \leq b(C) \). We will use \( b_2, b_3 \) to denote the number of clauses where \( x \) appears in the formula \( \widetilde{C} \) with arity exactly 2 and arity exactly 3, respectively. Note that \( d, b_2, b_3, w \) satisfy (30). For the \( i \)-th clause \( c_i \) that \( x \) appears in \( \widetilde{C} \), recall that \( C_i \) is the formula obtained by removing clauses \( c_1, \ldots, c_{i-1} \) and pinning the occurrences of \( x \) in \( c_{i+1}, \ldots, c_d \) to 0. We will denote by \( C_i \setminus c_i \) the formula which is obtained from \( C_i \) by deleting clause \( c_i \). Using that clauses of arity 3 appear first in the computation tree, we thus have the bounds
\[
b(C_i \setminus c_i) = b(\widetilde{C}) + (b_3 - i) - b_2 \leq b(C) + (b_3 - i) - b_2 \text{ for } i = 1, \ldots, b_3, \\
b(C_i \setminus c_i) = b(\widetilde{C}) - b_2 \leq b(C) - b_2 \text{ for } i = b_3 + 1, \ldots, d.
\]
For \( j \in [w_i] \), denote by \( t_{i,j} \) the number of clauses in \( C_i \) containing any variable from \( x_{i,1}, \ldots, x_{i,j-1} \); here, we exclude from consideration the clause \( c_i \) in \( C_i \) (since in the formula \( C_{i,j} \) the clause \( c_i \) has been removed). We have the bound
\[
b(C_{i,j}) - b(C_i \setminus c_i) \leq t_{i,j} \leq (\Delta - 1)(j - 1),
\]
where the last inequality follows from the observation that every variable in \( C \) has degree at most \( \Delta \) and \( x_{i,1}, \ldots, x_{i,j-1} \) all appear in the clause \( c_i \) (so they have degree at most \( \Delta - 1 \) in \( C_i \setminus c_i \)). Using (43) and (44), we can now bound the number of clauses of arity 2 in the formula \( C_{i,j} \) as follows:

1. for \( i = 1, \ldots, b_3 \) and \( j \in [w_i] \), \( b(C_{i,j}) \leq b(C) + (b_3 - i) + (\Delta - 1)(j - 1) - b_2 \).
2. for \( i = b_3 + 1, \ldots, d \) and \( j \in [w_i] \), \( b(C_{i,j}) \leq b(C) + (\Delta - 1)(j - 1) - b_2 \).
It follows that
\[
\sum_{i=1}^{d} \sum_{j=1}^{w_i} \frac{\partial^2 F}{\partial r_{i,j}} \alpha^{-l_{w_i}} \leq \kappa_{\delta, b, w}(r),
\]
(45)
where in the inequality we used that \( \delta \in (0, 1] \) and the lower bounds on \( b(C) - b(C_{i,j}) \) given in Items 1 and 2.

Combining (42), (45) and the assumption (33) that \( \kappa_{\delta, b, w}(r) \leq 1 \) for \( d \leq \Delta - 1 \) yields (38), as wanted.

Finally, we prove (35) with \( \tilde{\tau} := K_{\Phi}^{\text{max}} \cdot \max\{U, 1\} \), where \( U \) is the constant in assumption (33). Suppose that \( C, x \) are such that \( x \) has degree \( \Delta \) in \( C \). Note, for a node \((C', x')\) in the computation tree, \( x' \) may have degree \( d = \Delta \) in \( C' \) only if \((C', x')\) is the root of the tree. It follows that the children of the node \((C, x)\), say \((C_{i,j}, x_{i,j})\) with \( i \in [d] \) and \( j \in [w_i] \), are such that \( x_{i,j} \) has degree at most \( \Delta - 1 \) in \( C_{i,j} \). Hence, by applying (38), we obtain that (41) holds for all \( i, j \) and hence (as before) we deduce that (42), (45) hold as well. Inequality (35) now follows since by assumption (33) we have that \( \kappa_{\delta, b, w}(r) \leq \max\{U, 1\} \) for \( d \leq \Delta \).

This concludes the proof of Lemma 13. \( \square \)

By Lemma 13, our goal is to bound the quantity \( \kappa_{\delta, b, w}(r) \) for some appropriate choice of \( 0 < \delta, \alpha < 1 \) and \( \Phi \) satisfying (29). At this point, the most important step turns out to be the choice of \( \Phi \). For \( z \in (0, 1] \), let
\[
\Phi(z) := \frac{1}{\chi \psi} \log \left( \frac{z^\chi}{\psi - z^\chi} \right), \text{ so that } \varphi(z) = \Phi'(z) = \frac{1}{z(\psi - z^\chi)},
\]
(46)
where \( \chi = 1/2, \psi = 13/10 \). Note that \( \chi, \psi \) satisfy \( 0 < \chi \leq 1, \psi > 1 \).

Our main technical lemma is the following (proved in Section 6).

**Lemma 14.** Let \( \Delta = 6, \chi = 1/2, \psi = 13/10, \delta = 9789/10000, \alpha = 1 - 10^{-4} \) and \( \Phi \) be defined from (46). There exists a constant \( U > 0 \) such that, for all \( d, b, w \) satisfying (30), it holds that
\[
\kappa_{\delta, b, w}(r) \leq \begin{cases} 1, & \text{when } d \leq \Delta - 1, \\ U, & \text{when } d = \Delta, \end{cases}
\]
for all \((1/2)^{\Delta - 1} \leq r \leq 1\).

Modulo the upcoming proof of Lemma 14, Lemma 10 is now immediate to derive.

**Lemma 10.** Let \( \Delta = 6 \). There exist constants \( \alpha, \tau \) with \( 0 < \alpha < 1 \) and \( \tau > 0 \) such that the following holds for all integers \( L \). Let \( C \) be a monotone CNF formula whose clauses all have arity greater than or equal to 3 and, further, each variable occurs in at most \( \Delta \) clauses. Then,
\[
|R(C, x, L) - R(C, x, \infty)| \leq \tau \alpha^L,
\]
where the quantity \( R(C, x, L) \) is defined recursively from (4).

**Proof of Lemma 10.** Just invoke Lemmas 13 and 14. \( \square \)
6 Bounding the decay rate for $k \geq 3$, $\Delta = 6$

In this section, we give the proof of Lemma 14, which is the key technical ingredient that we used to prove Lemma 10. Recall from Section 5 that to show the step-wise correlation decay of the quantities $m(\cdot, \cdot, \cdot)$ along the computation tree, it suffices to give an upper bound on a certain multivariate quantity $\kappa$. This section carries out this final step and, unavoidably, it is technically more involved. So, before delving into the details, we reiterate and setup some convenient notation and then give a roadmap of the argument.

6.1 The optimisation problem: recap

Let $d, b_2, b_3$ and $w = \{w_i\}_{i \in [d]}$ satisfy (30) and let $r = \{r_{i,j}\}_{i \in [d], j \in [w_i]}$ be such that $\frac{1}{2^{\Delta-1}} \mathbf{1} \leq r \leq \mathbf{1}$. We briefly remind the reader the meaning of these variables. For a node $(C, x)$ in the computation tree, where $C$ is a monotone CNF formula and $x$ is a variable in $C$, we denote (as in Section 5):

- $d$ is the number of clauses in $C$ that contain $x$ (we have $d \leq \Delta - 1 = 5$ for all the internal nodes of the computation tree; the case $d = \Delta = 6$ can only arise at the root of the computation tree).
- $b_2$, resp. $b_3$, is the number of clauses in $C$ that contain $x$ and are of arity exactly 2, resp. 3.

- The vector $w$ has an entry for each clause in $C$ that contains $x$. The entry equals the arity of the corresponding clause minus one, i.e., when $x$ is deleted. Due to the ordering of the clauses (clauses of arity 3 come first, see also (30)), we have $w_i = 2$ for $i = 1, \ldots, b_3$ (for $i \in [d] \setminus [b_3]$, it holds that either $w_i \geq 3$ or $w_i = 1$).

- The $(i, j)$-th entry of the vector $r$ corresponds to our recursive estimate of the quantity $R(C_{i,j}, x_{i,j}) = \frac{\Pr_{C_{i,j}}(x_{i,j} = 0)}{\Pr_{C_{i,j}}(x_{i,j} = 1)}$, where $(C_{i,j}, x_{i,j})$ denotes the corresponding child of the node $(C, x)$ (see Section 2.3 for details).

- The function $F_{d,w}(r) := \prod_{i=1}^{d} \left(1 - \prod_{j=1}^{w_i} \frac{r_{i,j}}{1 + r_{i,j}}\right)$ gives the estimate for $R(C, x) = \frac{\Pr_C(x = 0)}{\Pr_C(x = 1)}$ given the estimates $r$.

For convenience, let $F := F_{d,w}$, $\kappa(r) := \kappa^d b_2 b_3 w(r)$ and for $i \in [d]$, let $\rho_i(r) := \rho^w_{i, \alpha}(r)$. Recall that

$$\rho_i(r) := \alpha^{-l_i} \sum_{j=1}^{w_i} \left(1 - \frac{j-1}{\delta}(\Delta-1)\right) \frac{1}{\varphi(r_{i,j})} \left| \frac{\partial F}{\partial r_{i,j}} \right|,$$  \hspace{1cm} (31)

and

$$\kappa(r) := \varphi(F(r)) \delta^{\beta_2} \left( \sum_{i=1}^{b_3} \left(1 - \frac{1}{\delta}\right) \rho_i(r) + \sum_{i=b_3+1}^{d} \rho_i(r) \right).$$  \hspace{1cm} (32)

Our goal is to show that, for all $r$ such that $\frac{1}{2^{\Delta-1}} \mathbf{1} \leq r \leq \mathbf{1}$, it holds that $\kappa(r) \leq 1$ when $d \leq \Delta - 1$ and that $\kappa(r)$ is bounded by a constant when $d = \Delta$. 

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6.2 Outline of the proof

Here is a rough outline of our analysis.

1. The first part of the proof will be to bound the quantities \( \rho_i(r) \) appropriately for each \( i \in [d] \). Namely, the main goal here will be to replace the \( \{r_{i,j}\}_{j \in [w_i]} \) by a suitable quantity \( \tilde{r}_i \). In fact, for this part of the proof, rather than working with the \( r_{i,j} \)'s, it will be easier to work with \( t_{i,j} = \frac{r_{i,j}}{1 + r_{i,j}} \) (which corresponds to the probability that a variable is assigned the truth value 0 in the corresponding monotone CNF formula). See Lemma 15.

2. After the first part, we will have reduced significantly the dimensionality of the optimisation problem: from the initial number of \( \sum_{i \in [d]} w_i \) variables \( \{r_{i,j}\}_{i \in [d], j \in [w_i]} \), we will be left with just \( d \) “representative” variables (one for each clause).

3. Despite having reduced the number of variables quite a lot, the \( w_i \)'s so far can be arbitrarily large integers. It will be convenient for us to restrict the range of the \( w_i \)'s. Using a rather crude argument, we will be able to restrict our attention to \( i \)'s such that \( 1 \leq w_i \leq 5 \). Intuitively, it should be clear that larger arity clauses have a smaller effect on the correlation decay and we will quantify this in an appropriate way for our analysis. (See Lemma 16.)

4. The next step is a further reduction of the number of variables. In particular, recall from Item 2 that we have reduced to the case where the number of variables is \( d \) (one for each clause) and, from Item 3, for each clause \( i \in [d] \) it holds that \( 1 \leq w_i \leq 5 \). We will further group together these variables according to the arity of the corresponding clause, i.e., for integer \( 1 \leq w \leq 5 \) we will be able to use a single variable (indexed by \( w \)) to capture the aggregate contribution of clauses \( i \in [d] \) with \( w_i = w \). (See Equations (71) and (72), and the relevant Lemmas 18 and 19.)

5. At this point, we are able to do the final steps of the optimisation. The most important case turns out to be when all of the \( w_i \)'s are either 1 or 2 (Lemma 20). In this case, we obtain quite sharp bounds for the correlation decay for each value of the total number of clauses (i.e, \( d = 1, 2, \ldots, \Delta - 1 = 5 \)). This facilitates the application of cruder arguments to handle the cases where there exist \( i \in [d] \) with \( w_i \neq 1, 2 \) (Lemmas 21, 22 and 23).

6.3 The details of the argument

In this section, we expand in detail the outline of Section 6.2 and give the necessary technical ingredients needed to complete the proof of Lemma 14. Later subsections contain the left-over technical proofs which would significantly interrupt the flow.

The first part of the proof will be to bound the quantities \( \rho_i(r) \) appropriately. For \( i \in [d] \) and \( j \in [w_i] \), we have (by differentiating \( F(r) \)):

\[
\frac{\partial F}{\partial r_{i,j}} = -F(r) \left( \prod_{j'=1}^{w_i} \frac{r_{i,j'}}{1 + r_{i,j'}} \right) \frac{1}{1 - \prod_{j'=1}^{w_i} \frac{r_{i,j'}}{1 + r_{i,j'}}} \frac{1}{r_{i,j}(1 + r_{i,j})}
\]
Let \( g(z) := \frac{1}{\psi(z)} \frac{1}{z(z+1)} = \frac{\psi - z}{1+z} \). For \( i \in [d] \), the quantity \( \rho_i(r) \) can then be written as

\[
\rho_i(r) := F(r) \frac{\prod_{j=1}^{w_i} \frac{r_{i,j}}{1+r_{i,j}} \alpha^{-l_{w_i}}}{1 - \prod_{j=1}^{w_i} \frac{r_{i,j}}{1+r_{i,j}}} \sum_{j=1}^{w_i} \left( \frac{1}{\delta} \right)^{(j-1)(\Delta-1)} g(r_{i,j}).
\]

(48)

Let \( t_{i,j} := \frac{r_{i,j}}{1+r_{i,j}} \) and let \( \hat{t}_i \) be the geometric mean of the \( t_{i,j} \)'s, i.e., \( \hat{t}_i := \prod_{j=1}^{w_i} t_{i,j} = \prod_{j=1}^{w_i} \frac{r_{i,j}}{1+r_{i,j}} \). As we shall see soon, \( \hat{t}_i \) will be used to capture the “aggregate” effect of the \( i \)-th clause. Let \( t \) be the vector whose entries are given by \( t_{i,j} \) with \( i \in [d] \) and \( j \in [w_i] \). Note that \( \frac{1}{2^{\frac{1}{\Delta-1}}} 1 \leq t \leq (1/2)1 \).

We will view the quantities \( \kappa(r) \) and \( \rho_i(r) \) as a function of \( t \). For that, it will be convenient to consider the function

\[
h(t) := g \left( \frac{t}{1-t} \right) = (1-t) \left[ \psi - \left( \frac{t}{1-t} \right)^{\chi} \right]
\]

(49)

for \( t \in [1/(2\Delta-1)+1, 1/2] \). With this preprocessing, for \( i \in [d] \), the quantities \( \rho_i(r) \), \( \kappa(r) \) as a function of \( t \) become

\[
\rho_i(t) = F(t) \frac{\left( \hat{t}_i \right)^{w_i}}{1 - \left( \hat{t}_i \right)^{w_i}} \alpha^{-l_{w_i}} \sum_{j=1}^{w_i} \left( \frac{1}{\delta} \right)^{(j-1)(\Delta-1)} h(t_{i,j}),
\]

\[
\kappa(t) = \varphi(F(t)) \delta^{b_3} \left( \sum_{i=1}^{b_3} \left( \frac{1}{\delta} \right)^{b_3-i} \rho_i(t) + \sum_{i=b_3+1}^{d} \rho_i(t) \right),
\]

(50)

where

\[
F(t) := \prod_{i=1}^{d} \left( 1 - \prod_{j=1}^{w_i} t_{i,j} \right).
\]

After this preliminary step, for \( i \in [d] \), we will now pursue the task of substituting the variables \( t_{i,j} \) with \( j \in [w_i] \) with a single variable \( \hat{t}_i \). Let \( \hat{t} = \{ \hat{t}_i \}_{i=1,\ldots,d} \) and note that

\[
\frac{1}{2^{\frac{1}{\Delta-1}}} 1 \leq \hat{t} \leq (1/2)1.
\]

As a starting point, observe that

\[
F(t) = \hat{F}**(t), \text{ where } \hat{F}**(t) = \prod_{i=1}^{d} \left( 1 - \prod_{j=1}^{w_i} \hat{t}_{i,j} \right).
\]

(51)

The following technical lemma, proved in Section 6.4, will be crucial in reducing the number of the variables \( t_{i,j} \).

**Lemma 15.** Let \( \Delta = 6 \) and \( \delta \in (0, 1) \). For \( w = 1, 2, \ldots \), there exists a constant \( K_{d,w}^{(w)} \geq 1 \) so that the following inequality holds for all \( t_1, \ldots, t_w \in [0, 1/2] \):

\[
\sum_{j=1}^{w} \frac{1}{\delta^{(j-1)(\Delta-1)}} h(t_j) \leq w K_{d,w}^{(w)} h(t),
\]

(52)

where \( t \) is the geometric mean of the \( t_i \)'s, i.e., \( t = (t_1 \cdot \cdots \cdot t_w)^{1/w} \).

In particular, for \( \delta = 9789/10000 \), \( \chi = 1/2 \), \( \psi = 13/10 \), and \( \Delta = 6 \), the inequalities (52) hold with the following values of the constants \( K_{d,w}^{(w)} \):

\[
K_{d,w}^{(1)} = 1, \quad K_{d,w}^{(2)} = 1069/1000, \quad K_{d,w}^{(3)} = 1160/1000, \quad K_{d,w}^{(4)} = 1225/1000,
\]

\[
K_{d,w}^{(w)} = \left( \frac{1}{\delta} \right)^{(w-1)(\Delta-1)} \quad \text{for } w \geq 5.
\]

(53)
Applying Lemma 15 to the quantity $\rho_i(t)$ in (50) yields that for all $i \in [d]$ it holds that

$$\rho_i(t) \leq \rho_i^{(1)}(\hat{t}),$$

where $\rho_i^{(1)}(\hat{t}) = \hat{F}(\hat{t}) w_i K_i^{(w_i)} \alpha^{-b_{w_i}} \frac{(\hat{t}_i)^{w_i}}{1 - (\hat{t}_i)^{w_i}} h(\hat{t}_i)$, \hspace{1cm} (54)

where $\hat{F}(\hat{t})$ is given by (51).

The next part of the proof will be to bound the contribution of clauses with $w_i \geq 6$ by small quantities so that we will eliminate those $i$ with $w_i \geq 6$ (and hence the respective variables $\hat{t}_i$) from consideration. This will be accomplished by the following lemma (proved in Section 6.5) and the upcoming inequality (58).

**Lemma 16.** Let $i$ be such that $w_i \geq 6$. Then, for all $\hat{t}$ such that $0 \leq \hat{t} \leq (1/2)1$, it holds that

$$\rho_i^{(1)}(\hat{t}) \leq \frac{1}{\alpha} \hat{F}(\hat{t}) M, \text{ where } M = 25/1000,$$

and $\hat{F}(\hat{t})$ is given by (51).

Denote by $b_j$ the number of clauses of arity $j$, or equivalently the number of $i$ such that $w_i = j - 1$. Using Lemma 16 we will now be able to eliminate those $i$ such that $w_i \geq 6$. More precisely, let $B = b_6 + b_3 + b_4 + b_5 + b_6$ and note that $0 \leq B \leq d$. W.l.o.g. we may assume that $w_i \geq 6$ implies that $i \geq B + 1$, i.e., clauses with $w_i \geq 6$ have larger index $i$ than clauses with $w_i \leq 5$ (more generally, in the context of (50) the ordering of clauses with $w_i \neq 2$ does not matter as long as we maintain that their index $i$ satisfies $i \geq b_3 + 1$).

Using (50), (54), (55) and $l_1 = \cdots = l_5 = 1$, we thus obtain

$$\bar{\kappa}(t) \leq \frac{1}{\alpha} \kappa^{(1)}(\hat{t}),$$

where

$$\kappa^{(1)}(\hat{t}) := \varphi(\hat{F}(\hat{t})) \hat{F}(\hat{t}) s^{b_2}
\left(2K_2^{(2)} \sum_{i=1}^{b_6} \left(\frac{1}{\delta}\right)^{b_{6-i}-1} \frac{(\hat{t}_i)^2}{1 - (\hat{t}_i)^2} h(\hat{t}_i) + \sum_{i=b_3+1}^{B} w_i K_i^{(w_i)} \frac{(\hat{t}_i)^{w_i}}{1 - (\hat{t}_i)^{w_i}} h(\hat{t}_i) + (d - B)M\right),$$

where $\hat{F}(\hat{t})$ is given by (51) and $M = 25/1000$ (cf. Lemma 16).

To complete the program of eliminating those variables $\hat{t}_i$ where $i$ is such that $w_i \geq 6$, observe that $z\varphi(z) = 1/(\psi - z\chi)$, so

$$\varphi(\hat{F}(\hat{t})) \hat{F}(\hat{t}) = \frac{1}{\psi - (\hat{F}(\hat{t}))^\chi} = \frac{1}{\psi - \prod_{i=1}^{d} (1 - (\hat{t}_i)^{w_i})^\chi} \leq \frac{1}{\psi - \prod_{i=1}^{B} (1 - (\hat{t}_i)^{w_i})^\chi}.$$ \hspace{1cm} (58)

Using (58), we thus obtain that

$$\kappa^{(1)}(\hat{t}) \leq \kappa^{(2)}(\hat{t}),$$

where

$$\kappa^{(2)}(\hat{t}) := s^{b_2}
\left(2K_2^{(2)} \sum_{i=1}^{b_6} \left(\frac{1}{\delta}\right)^{b_{6-i}-1} \frac{(\hat{t}_i)^2}{1 - (\hat{t}_i)^2} h(\hat{t}_i) + \sum_{i=b_3+1}^{B} w_i K_i^{(w_i)} \frac{(\hat{t}_i)^{w_i}}{1 - (\hat{t}_i)^{w_i}} h(\hat{t}_i) + (d - B)M\right),$$

$$\psi - \prod_{i=1}^{B} (1 - (\hat{t}_i)^{w_i})^\chi \hspace{1cm} (60)$$
The following quantity \( \kappa^{(3)}(\hat{t}) \) is similar to \( \kappa^{(2)}(\hat{t}) \):
\[
\kappa^{(3)}(\hat{t}) = \delta_b^2 \frac{2K^{(2)}_\delta \sum_{i=1}^{b_3} \left( \frac{1}{\delta} \right)^{b_3-i} \frac{h(\hat{t}_i)}{1-\hat{t}_i} + \sum_{i=b_3+1}^{B} w_i K^{(w_i)}(\hat{t}_i) \psi - \frac{\delta^2 b_2 (d-B) M}{\prod_{i=1}^{B} (1-\hat{t}_i)^w_i} \psi - \frac{1}{\psi} \right)^{\epsilon}}. \tag{61}
\]

The only difference between \( \kappa^{(2)}(\hat{t}) \) and \( \kappa^{(3)}(\hat{t}) \) is that the term \( (d-B)M \) is not present in the numerator of the latter. We therefore have
\[
\kappa^{(2)}(\hat{t}) = \kappa^{(3)}(\hat{t}) + \frac{\delta^2 b_2 (d-B) M}{\psi - \prod_{i=1}^{B} (1-\hat{t}_i)^w_i} \leq \kappa^{(3)}(\hat{t}) + \frac{(d-B) M}{\psi - 1}, \tag{62}
\]
where the last inequality follows from \( b_2 \geq 0, \delta \in (0,1] \) and the fact that the \( \hat{t}_i \)'s are positive.

The following lemma, proved later in this section, will allow us to conclude Lemma 14.

**Lemma 17.** Let \( \Delta = 6 \) and \( B \) be a non-negative integer less than or equal to \( \Delta - 1 = 5 \).
Recall that \( \alpha = 1 - 10^{-4} \). There exists a constant \( \varepsilon_B \leq \alpha \) such that for all non-negative integers \( b_2, b_3, b_4, b_5, b_6 \) with \( b_2 + b_3 + b_4 + b_5 + b_6 = B \), it holds that \( \kappa^{(3)}(\hat{t}) \leq \varepsilon_B \).

In particular, we will show that
\[
\varepsilon_0 = 0, \quad \varepsilon_1 = 6/10, \quad \varepsilon_2 = 7/10, \quad \varepsilon_3 = 83/100, \quad \varepsilon_4 = 91/100, \quad \varepsilon_5 = \alpha = 1 - 10^{-4}. \tag{63}
\]

Assuming Lemma 17 for the moment, we give the proof of Lemma 14, which we restate here for convenience.

**Lemma 14.** Let \( \Delta = 6, \chi = 1/2, \psi = 13/10, \delta = 9789/10000, \alpha = 1 - 10^{-4} \) and \( \Phi \) be defined from (46). There exists a constant \( U > 0 \) such that, for all \( d, b, w \) satisfying (30), it holds that
\[
\kappa^{d,b_2,b_3,w}(r) \leq \begin{cases} 
1, & \text{when } d \leq \Delta - 1, \\
U, & \text{when } d = \Delta,
\end{cases} \tag{47}
\]
for all \( (1/2)^{-1} \leq r \leq 1 \).

**Proof of Lemma 14.** We first derive the bound \( \kappa(r) \leq 1 \) when \( d \leq \Delta - 1 = 5 \). Recall that the quantity \( \kappa(r) \) (as given in (32)) is equal to the quantity \( \tilde{\kappa}(t) \) (as given in (50)). Also, we have shown that
\[
\tilde{\kappa}(t) \leq \frac{1}{\alpha} \kappa^{(1)}(\hat{t}), \tag{56}
\]
where \( \kappa^{(1)}(\hat{t}) \) is as in (57). We have also shown that
\[
\kappa^{(1)}(\hat{t}) \leq \kappa^{(2)}(\hat{t}), \tag{59}
\]
where \( \kappa^{(2)}(\hat{t}) \) is as in (60). Moreover, we showed that
\[
\kappa^{(2)}(\hat{t}) \leq \kappa^{(3)}(\hat{t}) + \frac{(d-B) M}{\psi - 1}, \tag{62}
\]
where \( \kappa^{(3)}(\hat{t}) \) is as in (61), \( B \) is a non-negative integer less than or equal to \( \Delta - 1 = 5 \) satisfying \( B = b_2 + b_3 + b_4 + b_5 + b_6 \) and \( M = 25/1000 \) is as in Lemma 16. Lastly, by Lemma 17, we have
\[
\kappa^{(3)}(\hat{t}) \leq \varepsilon_B \]
where the constants \( \varepsilon_B \) are as in (63). Combining all the above we obtain that
\[
\kappa(r) \leq \frac{1}{\alpha} \left( \varepsilon_B + \frac{(d - B)M}{\psi - 1} \right). \tag{64}
\]
It is a matter of numerical calculations to check that
\[
\frac{1}{\alpha} \left( \varepsilon_B + \frac{(d - B)M}{\psi - 1} \right) \leq 1
\]
for all \( B = 0, 1, \ldots, 5 \) and \( d = \Delta - 1 = 5 \), see Section 10.1 for the explicit calculations. This completes the proof of the lemma for \( d \leq \Delta - 1 = 5 \).

We next consider the case \( d = \Delta \), i.e., we show that there exists a constant \( U > 0 \) such that \( \kappa(r) \leq U \). This will follow by continuity arguments. More precisely, first note that inequalities (56), (59) and (62) still hold in the case where \( d = \Delta \), with the minor modification that in (62), the integer \( B \) (which, recall, corresponds to the number of clauses where the variable \( x \) occurs in \( C \) with arity \( \leq 6 \)) can be as large as (but not bigger than) \( \Delta \). Let us fix \( B \) to be a non-negative integer \( \leq \Delta = 6 \). Observe that there are finitely many possibilities for the non-negative integers \( b_2, b_3, b_4, b_5, b_6 \) such that \( b_2 + b_3 + b_4 + b_5 + b_6 = B \). For each such choice, the quantity \( \kappa^{(3)}(\hat{t}) \) is a continuous function of the (finitely many) variables \( \hat{t}_1, \ldots, \hat{t}_B \) and hence is bounded above by an absolute constant when \( 0 \leq \hat{t} \leq (1/2) \mathbf{1} \). It follows that for every non-negative integer \( B \leq \Delta = 6 \), there exists an absolute constant \( U_B > 0 \) such that \( \kappa^{(3)}(\hat{t}) \leq U_B \). Thus, analogously to (64), we obtain the bound
\[
\kappa(r) \leq \max_{B = 0, \ldots, 6} \left\{ \frac{1}{\alpha} \left( U_B + \frac{(\Delta - B)M}{\psi - 1} \right) \right\} =: U. \tag{66}
\]
Note that \( U \), as defined in (66), is a constant. The desired bound on \( \kappa(r) \) when \( d = \Delta \) follows. This concludes the proof of Lemma 14.

The remainder of this section will focus on the proof of Lemma 17. We begin our considerations by reducing the number of variables. We first need the following transformation. Namely, for \( w = 1, 2, \ldots \) and all \( i \in [B] \) such that \( w_i = w \), we set \( y_i := 1 - \hat{t}_i^w \) (note that \( y_i \in \left[ 1 - (1/2)^w, 1 - 1/(2^{\Delta - 1} + 1)^w \right] \)). For \( w = 1, 2, \ldots \), consider the functions \( g_w(y) \) defined for \( y \in \left[ 1 - (1/2)^w, 1 - 1/(2^{\Delta - 1} + 1)^w \right] \).
\[
g_w(y) := \frac{(1 - y)^b}{y} h((1 - y)^{1/w}). \tag{67}
\]
The quantity \( \kappa^{(3)}(\hat{t}) \) as a function of \( y = \{y_i\}_{i=1}^B \) and the functions \( g_w \) then becomes
\[
\kappa^{(4)}(y) = \delta b_2 \frac{2K^{(2)}_\delta \sum_{i=1}^{b_3} \left( \frac{1}{\delta} \right)^{b_3-1} g_2(y_i) + \sum_{i=b_3+1}^{B} w_i K^{(w)}_\delta g_{w_i}(y_i)}{\psi - \left( \prod_{i=1}^{b} y_i \right)^{\chi}} \tag{68}
\]
Let \( \hat{y}_w \) be the geometric mean of those \( y_i \)'s with \( w_i = w \) (note that the number of such \( i \)'s is equal to \( b_{w+1} \)). More precisely, for \( b_{w+1} > 0 \), let
\[
\hat{y}_w := \left( \prod_{i: y_i = w} y_i \right)^{1/b_{w+1}}.
\]
and when \( b_{w+1} = 0 \), let \( \hat{y}_w = 1 \). Note that

\[
\prod_{i=1}^{B} y_i = \prod_{w=1}^{5} (\hat{y}_w)^{b_{w+1}}.
\] (69)

Let \( \hat{y} = \{ \hat{y}_i \}_{i=1, \ldots, B} \). Our goal will be to bound \( \kappa^{(4)}(y) \) by a function of \( \hat{y} \).

**Lemma 18.** For \( w = 1, 2, \ldots, 5 \), the function \( g_w(e^z) \) is a concave function of \( z \) in the interval \( \left[ \ln \left( 1 - (1/2)^w \right), 0 \right] \).

**Proof.** For \( z \in \left[ \ln \left( 1 - (1/2)^w \right), 0 \right] \), let \( f(z) = g_w(e^z) \). Our goal is to show that for \( w = 1, \ldots, 5 \), it holds that

\[
f''(z) < 0 \text{ for all } z \in \left[ \ln \left( 1 - (1/2)^w \right), 0 \right].
\] (70)

For convenience, we use Mathematica’s RESOLVE function, see Section 10.3 for details. \( \square \)

Lemma 18 and Jensen’s inequality yield that

\[
\sum_{i, \omega_i = w} g_w(y_i) = \sum_{i, \omega_i = w} g_w(e^{\ln y_i}) = b_{w+1} g_w(e^{\frac{1}{w+1} \sum_{i, \omega_i = w} \ln y_i}) \leq b_{w+1} g_w(\hat{y}_w). \] (71)

To bound \( \sum_{i=1}^{b_3} \left( \frac{1}{\delta} \right)^{b_3-i} g_2(y_i) \) by a function of \( \hat{y}_2 \), we will use the following lemma (proved in Section 6.6).

**Lemma 19.** Let \( \Delta = 6 \) and \( b_3 \) be a non-negative integer less than or equal to \( \Delta - 1 = 5 \). There exists a constant \( C_\delta^{(b_3)}(\delta) \geq 0 \) so that for \( y_1, \ldots, y_{b_3} \in [3/4, 1 - 1/(2^d + 1)^2] \) it holds that

\[
\sum_{i=1}^{b_3} \left( \frac{1}{\delta} \right)^{b_3-i} g_2(y_i) \leq b_3 C_\delta^{(b_3)}(\delta) g_2(\hat{y}_1 \cdots \hat{y}_{b_3}). \] (72)

In particular, we will show that inequality (72) holds with \( C_\delta^{(b_3)} = \frac{1}{\delta^{b_3-1}} \left( \frac{1}{\delta^{b_3-1}} \right)^{1/p} \), where \( p = 27/2 \). For \( \delta = 9789/10000 \), we have the following upper bounds on the values of \( C_\delta^{(b_3)} \):

\[
C_\delta^{(0)} = 0, \quad C_\delta^{(1)} = 1, \quad C_\delta^{(2)} = 102/100, \quad C_\delta^{(3)} = 103/100, \quad C_\delta^{(4)} = 104/100, \quad C_\delta^{(5)} = 105/100. \] (73)

Using (69), (71) and (72) we obtain that

\[
\kappa^{(4)}(y) \leq \kappa^{(5)}(\hat{y}),
\]

where

\[
\kappa^{(5)}(\hat{y}) := \delta^{b_3} \cdot \frac{2b_3 R_\delta^{(2)} C_\delta^{(b_3)} g_2(\hat{y}_2) + \sum_{w: w \notin [5], w \neq 2} \sum_{w: \omega_i \neq 2} w b_{w+1} K_\delta^{(w)}(\hat{y}_w)}{\psi - \prod_{w=1}^{6} (\hat{y}_w)^{b_{w+1} \chi}},
\]

with the values of \( K_\delta^{(w)} \) as in Lemma 15 (cf. equation (52)) and the values of \( C_\delta^{(b_3)} \) as in Lemma 19 (cf. equation (73)).
We next define the following constants \( \tau_{b_2, b_3} \) for non-negative integers \( b_2, b_3 \) satisfying \( b_2 + b_3 \leq \Delta - 1 = 5 \) (they are all at most \( \alpha \) and we will show that they bound \( \kappa^{(5)}(\hat{y}) \)):

\[
\begin{align*}
\tau_{0,0} &= 0, & \tau_{0,1} &= \tau_{1,0} = 42/100, \\
\tau_{0,2} &= 54/100, & \tau_{1,1} &= 59/100, & \tau_{2,0} &= 63/100, \\
\tau_{0,3} &= 72/100, & \tau_{1,2} &= 74/100, & \tau_{2,1} &= 76/100, & \tau_{3,0} &= 79/100, \\
\tau_{0,4} &= 864/1000, & \tau_{1,3} &= 868/1000, & \tau_{2,2} &= 876/1000, & \tau_{3,1} &= 886/1000, & \tau_{4,0} &= 901/1000, \\
\tau_{b_2, b_3} &= \alpha \text{ when } b_2 + b_3 = 5.
\end{align*}
\]

(74)

**Lemma 20.** Let \( b_4 = b_5 = b_6 = 0 \). For all non-negative integers \( b_2, b_3 \) such that \( b_2 + b_3 \leq \Delta - 1 = 5 \), it holds that \( \kappa^{(5)}(\hat{y}) \leq \tau_{b_2, b_3} \).

**Proof of Lemma 20.** For \( b_4 = b_5 = b_6 = 0 \), the quantity \( \kappa^{(5)}(\hat{y}) \) simplifies into

\[
\kappa^{(5)}(\hat{y}) := \delta^{b_2} \cdot \frac{2b_3K^{(2)}_\delta C^{(b_3)}_\delta g_2(\hat{y}_2) + b_2 g_1(\hat{y}_1)}{\psi - (\hat{y}_1)^{b_2x}(\hat{y}_2)^{b_3x}},
\]

(75)

where we used that \( K^{(1)}_\delta = 1 \) (note that the values of the variables \( \hat{y}_3, \hat{y}_4, \hat{y}_5 \) do not affect the value of \( \kappa^{(5)} \) when \( b_4 = b_5 = b_6 = 0 \)).

To bound \( \kappa^{(5)}(\hat{y}) \), we need a couple of transformations. The first one is simple: set \( v_1 := 1 - \hat{y}_1 \) and \( v_2 := (1 - \hat{y}_2)^{1/2} \) and note that \( v_1, v_2 \in [1/(2^{\Delta-1} + 1), 1/2] \). From the definition of the functions \( g_1, g_2 \) (cf. equation (67)), we have that

\[
\begin{align*}
g_1(\hat{y}_1) &= \frac{1 - \hat{y}_1}{\hat{y}_1} h(1 - \hat{y}_1) = \frac{v_1}{1 - v_1} h(v_1) = v_1 \left( \psi - \left( \frac{v_1}{1 - v_1} \right)^x \right), \\
g_2(\hat{y}_2) &= \frac{1 - \hat{y}_2}{\hat{y}_2} h((1 - \hat{y}_2)^{1/2}) = \frac{v_2^2}{1 - v_2} h(v_2) = \frac{v_2^2}{1 + v_2} \left( \psi - \left( \frac{v_2}{1 - v_2} \right)^x \right).
\end{align*}
\]

(76)

It follows that the quantity \( \kappa^{(5)}(\hat{y}) \) as a function of \( v_1, v_2 \) becomes

\[
\kappa^{(6)}(v_1, v_2) := \delta^{b_2} \cdot \frac{2b_3K^{(2)}_\delta C^{(b_3)}_\delta \frac{v_2}{1 + v_2} \left( \psi - \left( \frac{v_2}{1 - v_2} \right)^x \right) + b_2 v_1 \left( \psi - \left( \frac{v_1}{1 - v_1} \right)^x \right)}{\psi - (1 - v_1)^{b_2x}(1 - v_2)^{b_3x}}.
\]

We will show that for all \( v_1, v_2 \in [0, 1/2] \), it holds that \( \kappa^{(6)}(v_1, v_2) \leq \tau_{b_2, b_1} \). To do this, we will use Mathematica’s Resolve function. We first need to rationalize the expressions to keep the computations feasible (this can be achieved for \( \chi = 1/2 \)). This brings us to the second (and final) transformation. In particular, set

\[
\begin{align*}
v_1 &= \frac{4z_1^2}{(1 + z_1^2)^2}, & v_2 &= \frac{4z_2^2}{1 + 4z_2^4}
\end{align*}
\]

for \( 0 \leq z_1 \leq \sqrt{2} - 1 \) and \( 0 \leq z_2 \leq \frac{1}{2}(\sqrt{3} - 1) \). Under these transformations, for \( \chi = 1/2 \), we obtain that

\[
\begin{align*}
(\frac{v_1}{1 - v_1})^{1/2} &= \frac{2z_1}{1 - z_1^2}, & (1 - v_1)^{1/2} &= 1 - z_1^2, & (\frac{v_2}{1 - v_2})^{1/2} &= \frac{2z_2}{1 - 2z_2^2}, & (1 - v_2)^{1/2} &= \frac{1 - 4z_2^2}{1 + 4z_2^4}.
\end{align*}
\]

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The quantity $\kappa^{(0)}(v_1, v_2)$ in terms of $z_1, z_2$ thus becomes

$$\kappa^{(7)}(z_1, z_2) := \delta^b 2 \frac{b_2}{(1+z_1^2)} \frac{4z_1^2}{(1+z_1^2)^2} \left( \psi - \frac{2z_1}{1-z_1^2} \right) + 2b_3 K_\delta^{(2)}(b_3) \frac{C_\delta^{(b_3)}}{C_\delta^{(b_3)} + 6z_1^2} \frac{(1-z_1^2)^2}{(1+z_1^2)^2} \left( \psi - \frac{2z_2}{1-2z_2^2} \right).$$

Our goal is to show that, for $b_2, b_3 \geq 0$ satisfying $b_2 + b_3 \leq \Delta - 1 = 5$, there do not exist $z_1, z_2$ in the range $0 \leq z_1 \leq \sqrt{2} - 1$ and $0 \leq z_2 \leq \frac{1}{2}(\sqrt{3} - 1)$ such that $\kappa^{(7)}(z_1, z_2) > \tau_{b_2, b_3}$ where the constants $\tau$ are as in (74). This can be done symbolically using Mathematica. We give the code in Section 10.5.

We use Lemma 20 to show the following.

**Lemma 21.** Let $b_5 = b_6 = 0$ and $B$ be a non-negative integer less than or equal to $\Delta - 1 = 5$. For all non-negative integers $b_2, b_3, b_4$ such that $b_2 + b_3 + b_4 = B$, it holds that $\kappa^{(5)}(\hat{y}) \leq \tau_{B,0}$ where the constants $\tau_{B,0}$ are given by (74).

**Proof of Lemma 21.** We may assume that $b_4 \geq 1$ (when $b_4 = 0$, the bounds in the lemma follow immediately from Lemma 20). For $b_5 = b_6 = 0$, the quantity $\kappa^{(5)}(\hat{y})$ simplifies into

$$\kappa^{(5)}(\hat{y}) := \delta^b \left( \frac{b_2}{(1+z_1^2)} \frac{g_1(\hat{y}) + 2b_3 K_\delta^{(2)} C_\delta^{(b_3)} g_2(\hat{y}) + 3b_4 K_\delta^{(3)} g_3(\hat{y})}{\psi - (\hat{y})^{b_1\lambda}(\hat{y})^{b_2\lambda}(\hat{y})^{b_3\lambda}} \right),$$

where we used that $K_\delta^{(1)} = 1$ (note that the values of the variables $\hat{y}_1, \hat{y}_3$ do not affect the value of $\kappa^{(4)}$ when $b_5 = b_6 = 0$). The proof splits into two cases depending on whether $b_3$ is zero.

**Case I:** $b_3 \geq 1$. Let $A := (\hat{y}_1)^{b_2\lambda}(\hat{y}_2)^{b_3\lambda}$. Since $\hat{y}_1, \hat{y}_2 \in [0, 1]$, we have the crude bound $0 \leq A \leq 1$. By Lemma 20 (see also (75)), we have that

$$\delta^b 2 \frac{b_2}{(1+z_1^2)} \frac{g_1(\hat{y}) + 2b_3 K_\delta^{(2)} C_\delta^{(b_3)} g_2(\hat{y})}{\psi - A} \leq \tau_{b_2, b_3},$$

where the values of the constants $\tau_{b_2, b_3}$ are as in Lemma 20 (cf. equation (74)). It follows that

$$\kappa^{(5)}(\hat{y}) \leq \kappa^{(6)}(A, \hat{y}_3),$$

where $\kappa^{(6)}(A, \hat{y}_3) := \frac{\tau_{b_2, b_3}(\psi - A) + 3b_4 K_\delta^{(3)} g_3(\hat{y}_3)}{\psi - A(\hat{y}_3)^{b_4\lambda}}$.

We next perform a transformation for the variable $\hat{y}_3$ (similar to the one used in the proof of Lemma 20), namely, we set $v_3 := (1 - \hat{y}_3)^{1/3}$ so that $v_3 \in [0, 1/2]$. From the definition of the function $g_3$ (cf. equation (67)), we have that

$$g_3(\hat{y}_3) = \frac{1 - \hat{y}_3}{\hat{y}_3} h((1 - \hat{y}_3)^{1/3}) = \frac{v_3^3}{1 - v_3^3} h(v_3) = \frac{v_3^3}{1 + v_3 + v_3^2} \left( \psi - \left( \frac{v_3^3}{1 - v_3^3} \right)^{1/3} \right).$$

It follows that the quantity $\kappa^{(6)}(A, \hat{y}_3)$ as a function of $A, v_3$ can now be written as

$$\kappa^{(7)}(A, v_3) := \frac{\tau_{b_2, b_3}(\psi - A) + 3b_4 K_\delta^{(3)} \frac{v_3^3}{1 + v_3 + v_3^2} \left( \psi - \left( \frac{v_3^3}{1 - v_3^3} \right)^{1/3} \right)}{\psi - A(\hat{y}_3)^{b_4\lambda}}.$$
We will show that
\[ \kappa^{(7)}(A, v_3) \leq \tau_{B,0} \text{ for all } 0 \leq A \leq 1, \ 0 \leq v_3 \leq 1/2 \text{ (recall, } B = b_2 + b_3 + b_4). \]  \hspace{1cm} (78)

We use Mathematica’s RESOLVE function, see Section 10.6 for details.

**Case II:** \( b_3 = 0 \). For \( b_3 = b_5 = b_6 = 0 \), the quantity \( \kappa^{(4)}(\hat{y}) \) simplifies into

\[ \kappa^{(5)}(\hat{y}) := \delta^{b_2} \cdot \frac{b_2 g_1(\hat{y}_1) + 3 b_4 K^{(3)}_\delta g_3(\hat{y}_3)}{\psi - (\hat{y}_1)^{b_2 x}(\hat{y}_3)^{b_4 x}}. \]

We next perform a transformation on the variables \( \hat{y}_1, \hat{y}_3 \) (similar to the one used in the proof of Lemma 20), namely, we set \( v_1 = 1 - \hat{y}_1 \) and \( v_3 := (1 - \hat{y}_3)^{1/3} \) so that \( v_1, v_3 \in [0, 1/2] \). Using (76) and (77), we obtain the following expression for \( \kappa^{(5)} \) in terms of \( v_1, v_3 \):

\[ \kappa^{(8)}(v_1, v_3) := \delta^{b_2} \frac{b_2 v_1 \left( \psi - \left( \frac{v_1}{1 - v_1} \right)^x \right) + 3 b_4 K^{(3)}_\delta \frac{v_3^3}{1 + v_3 + v_3^2} \left( \psi - \left( \frac{v_1}{1 - v_1} \right)^x \right)}{\psi - (1 - v_1)^{b_2 x}(1 - v_3)^{b_4 x}}. \]  \hspace{1cm} (79)

This quantity is still too complicated for Mathematica to resolve efficiently, so we will need one more transformation. In particular, let \( u_1, u_3 \) be positive reals defined by

\[ v_1 = \frac{u_1^2}{1 + u_1^2}, \quad v_3 = \frac{u_3^2}{1 + u_3^2}, \]

and note that \( 0 \leq u_1, u_3 \leq 1 \). The quantity \( \kappa^{(8)} \) in terms of \( u_1, u_3 \) becomes:

\[ \kappa^{(9)}(u_1, u_3) := \delta^{b_2} \frac{2 u_1^2 \left( \psi - u_1 \right) + 3 b_4 K^{(3)}_\delta \frac{u_3^6}{3 u_1^6 + 6 u_1^4 + 4 u_1^2 + 1} \left( \psi - u_3 \right)}{\psi - (1 + u_1^2)^{b_2 x}(1 - (u_3^3) \cdot b_4 x)}. \]

We will show that

\[ \kappa^{(9)}(u_1, u_3) \leq \tau_{B,0} \text{ for all } 0 \leq u_1, u_3 \leq 1 \text{ (note, } B = b_2 + b_4). \]  \hspace{1cm} (80)

We use Mathematica’s RESOLVE function, see Section 10.6 for details.

This completes the case analysis and therefore the proof of Lemma 21. \( \square \)

**Lemma 22.** Let \( b_0 = 0 \) and \( B \) be a non-negative integer less than or equal to \( \Delta - 1 = 5 \). For all non-negative integers \( b_2, b_3, b_4, b_5 \) such that \( b_2 + b_3 + b_4 + b_5 = B \), it holds that \( \kappa^{(5)}(\hat{y}) \leq \tau_{B,0} \) where the constants \( \tau_{B,0} \) are given by (74).

**Proof of Lemma 22.** We may assume that \( b_5 \geq 1 \) (when \( b_5 = 0 \), the bounds in the lemma follow immediately from Lemma 21). For \( b_6 = 0 \) (note that the value of the variable \( \hat{y}_5 \) does not affect the value of \( \kappa^{(5)} \)), the quantity \( \kappa^{(5)}(\hat{y}) \) becomes

\[ \kappa^{(5)}(\hat{y}) := \delta^{b_2} \cdot \frac{b_2 g_1(\hat{y}_1) + 2 b_3 K^{(2)}_\delta C^{(b_3)}(b_3) g_2(\hat{y}_2) + 3 b_4 K^{(3)}_\delta g_3(\hat{y}_3) + 4 b_5 K^{(4)} g_4(\hat{y}_4)}{\psi - (\hat{y}_1)^{b_2 x}(\hat{y}_2)^{b_3 x}(\hat{y}_3)^{b_4 x}(\hat{y}_4)^{b_5 x}}. \]

Let \( A := (\hat{y}_1)^{b_2 x}(\hat{y}_2)^{b_3 x}(\hat{y}_3)^{b_4 x}. \) Since \( \hat{y}_1, \hat{y}_2, \hat{y}_3 \in [0, 1] \), we have the crude bound \( 0 \leq A \leq 1 \). By Lemma 21, we have that

\[ \delta^{b_2} \cdot \frac{b_2 g_1(\hat{y}_1) + 2 b_3 K^{(2)}_\delta C^{(b_3)}(b_3) g_2(\hat{y}_2) + 3 b_4 K^{(3)}_\delta g_3(\hat{y}_3)}{\psi - A} \leq \tau_{B',0}, \text{ where } B' = b_2 + b_3 + b_4. \]

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where the values of the constants \( \tau_{B',0} \) are given by equation (74). Using that \( \delta b_2 \leq 1 \), it follows that
\[
\kappa^{(5)}(\hat{y}) \leq \kappa^{(6)}(A, \hat{y}_4), \quad \text{where} \quad \kappa^{(6)}(A, \hat{y}_4) := \frac{\tau_{B',0}(\psi - A) + 4b_5 K_5^{(4)} g_4(\hat{y}_4)}{\psi - A(\hat{y}_4)^{b_6}}.
\]
We next perform a transformation on the variable \( \hat{y}_4 \) (similar to the one used in the proof of Lemma 20), namely, we set \( v_4 := (1 - \hat{y}_4)^{1/4} \) so that \( v_4 \in [0, 1/2] \). From the definition of the function \( g_4 \) (cf. equation (67)), we have that
\[
g_4(\hat{y}_4) = \frac{1 - \hat{y}_4}{\hat{y}_4} h((1 - \hat{y}_4)^{1/4}) = \frac{v_4^4}{1 - v_4^4} h(v_4) = \frac{v_4^4}{1 + v_4 + v_4^2 + v_4^3} \left( \psi - \left( \frac{v_4}{1 - v_4} \right)^\psi \right).
\]
It follows that the quantity \( \kappa^{(6)}(A, \hat{y}_4) \) as a function of \( A, v_4 \) can now be written as
\[
\kappa^{(7)}(A, v_4) := \frac{\tau_{B',0}(\psi - A) + 4b_5 K_5^{(4)} \frac{v_4^4}{1 + v_4 + v_4^2 + v_4^3} \left( \psi - \left( \frac{v_4}{1 - v_4} \right)^\psi \right)}{\psi - A(1 - v_4^{b_6})}\]
We will show that
\[
\kappa^{(7)}(A, v_4) \leq \tau_{B,0} \quad \text{for all} \quad 0 \leq A \leq 1, \quad 0 \leq v_4 \leq 1/2 \quad \text{(recall,} \quad B = b_2 + b_3 + b_4 + b_5 = B' + b_5)\].

We use Mathematica’s RESOLVE function, see Section 10.7 for details.

**Lemma 23.** Let \( B \) be a non-negative integer less than or equal to \( \Delta - 1 = 5 \). For all non-negative integers \( b_2, b_3, b_4, b_5, b_6 \) such that \( b_2 + b_3 + b_4 + b_5 + b_6 = B \), it holds that \( \kappa^{(5)}(\hat{y}) \leq \tau_{B,0} \) where the constants \( \tau_{B,0} \) are given by (74).

**Proof of Lemma 23.** We may assume that \( b_6 \geq 1 \) (when \( b_6 = 0 \), the bounds in the lemma follow immediately from Lemma 22). Recall that the quantity \( \kappa^{(5)}(\hat{y}) \) is given by
\[
\kappa^{(5)}(\hat{y}) = \delta b_2 \frac{g_1(\hat{y}_1) + 2b_3 K_3^{(2)} C_\delta^{(b_2)} g_2(\hat{y}_2) + 3b_4 K_4^{(3)} C_\delta^{(b_2)} g_3(\hat{y}_3) + 4b_5 K_5^{(4)} C_\delta^{(b_2)} g_4(\hat{y}_4) + 5b_6 K_6^{(5)} g_5(\hat{y}_5)}{\psi - (\hat{y}_1)^{b_2} + (\hat{y}_2)^{b_3} + (\hat{y}_3)^{b_4} + (\hat{y}_4)^{b_5} + (\hat{y}_5)^{b_6}}.
\]

Let \( A := (\hat{y}_1)^{b_2} + (\hat{y}_2)^{b_3} + (\hat{y}_3)^{b_4} + (\hat{y}_4)^{b_5} + (\hat{y}_5)^{b_6} \). Since \( \hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{y}_4, \hat{y}_5 \in [0, 1] \), we have the crude bound \( 0 \leq A \leq 1 \). By Lemma 20, we have that
\[
\delta b_2 \frac{g_1(\hat{y}_1) + 2b_3 K_3^{(2)} C_\delta^{(b_2)} g_2(\hat{y}_2) + 3b_4 K_4^{(3)} C_\delta^{(b_2)} g_3(\hat{y}_3) + 4b_5 K_5^{(4)} C_\delta^{(b_2)} g_4(\hat{y}_4) + 5b_6 K_6^{(5)} g_5(\hat{y}_5)}{\psi - A} \leq \tau_{B',0}, \quad \text{where} \quad B' = b_2 + b_3 + b_4 + b_5.
\]
where the values of the constants \( \tau_{B',0} \) are given by equation (74). Using that \( \delta b_2 \leq 1 \), it follows that
\[
\kappa^{(5)}(\hat{y}) \leq \kappa^{(6)}(A, \hat{y}_{5}), \quad \text{where} \quad \kappa^{(6)}(A, \hat{y}_{5}) := \frac{\tau_{b_2 b_3}(\psi - A) + 5b_6 K_6^{(5)} g_5(\hat{y}_5)}{\psi - A(\hat{y}_5)^{b_6}}.
\]
We next perform a transformation on the variable \( \hat{y}_5 \) (similar to the one used in the proof of Lemma 20), namely, we set \( v_5 := (1 - \hat{y}_5)^{1/5} \) so that \( v_5 \in [0, 1/2] \). From the definition of the function \( g_5 \) (cf. equation (67)), we have that
\[
g_5(\hat{y}_5) = \frac{1 - \hat{y}_5}{\hat{y}_5} h((1 - \hat{y}_5)^{1/5}) = \frac{v_5^5}{1 - v_5^5} h(v_5) = \frac{v_5^5}{1 + v_5 + v_5^2 + v_5^3 + v_5^4} \left( \psi - \left( \frac{v_5}{1 - v_5} \right)^\psi \right).
\]
It follows that the quantity $\kappa^{(6)}(A, \hat{y}_5)$ as a function of $A, v_5$ can now be written as

$$\kappa^{(7)}(A, v_5) := \frac{\tau_{B',0}(\psi - A) + 5b_6 K^{(5)}_\delta \frac{v_5^2}{1+v_5^2+v_5^2+v_5^2} (\psi - (\frac{v_5}{1-v_5})^x)}{\psi - A(1-v_5)^{b_6x}}.$$ 

We will show that

$$\kappa^{(7)}(A, v_5) \leq \tau_{B,0} \text{ for all } 0 \leq A \leq 1, \ 0 \leq v_5 \leq 1/2 \ (\text{recall, } B = B' + b_6). \quad (82)$$

We use Mathematica’s RESOLVE function, see Section 10.8 for details. □

The proof of Lemma 17, which was important in proving Lemma 14, is now immediate.

**Lemma 17.** Let $\Delta = 6$ and $B$ be a non-negative integer less than or equal to $\Delta - 1 = 5$. Recall that $\alpha = 1 - 10^{-4}$. There exists a constant $\varepsilon_B \leq \alpha$ such that for all non-negative integers $b_2, b_3, b_4, b_5, b_6$ with $b_2 + b_3 + b_4 + b_5 + b_6 = B$, it holds that $\kappa^{(3)}(\hat{t}) \leq \varepsilon_B$.

In particular, we will show that

$$\varepsilon_0 = 0, \ \varepsilon_1 = 6/10, \ \varepsilon_2 = 7/10, \ \varepsilon_3 = 83/100, \ \varepsilon_4 = 91/100, \ \varepsilon_5 = \alpha = 1 - 10^{-4}. \quad (63)$$

**Proof of Lemma 17.** By the definition of $\hat{y}$, we have that $\kappa^{(3)}(\hat{t}) = \kappa^{(5)}(\hat{y})$. Now, just use Lemma 23 and observe that the $\varepsilon_B$ in (63) and the constants $\tau_{B,0}$ in (74) satisfy $\tau_{B,0} \leq \varepsilon_B$. □

### 6.4 Simplifying the optimisation using geometric means

In this section, we prove Lemma 15, which we restate here for convenience. Roughly, the lemma bounds the contribution of a clause with arity $w+1$ to the decay rate. The main accomplishment here is the significant reduction of the number of variables; initially the contribution of such a clause is a function of $w$ variables $t_1, \ldots, t_w$ which roughly correspond to the probability that a variable in the clause takes the value 0 (in an appropriate formula). The lemma shows that we can reduce the number of variables to 1 by considering the geometric mean of the $t_j$’s. The challenge here is to deal with the asymmetry caused by the $\delta$ terms (which account for the potential creation of arity 2 clauses) without introducing too much slack in the argument, especially for small arity clauses ($w \leq 4$).

**Lemma 15.** Let $\Delta = 6$ and $\delta \in (0, 1]$. For $w = 1, 2, \ldots$, there exists a constant $K^{(w)}_\delta \geq 1$ so that the following inequality holds for all $t_1, \ldots, t_w \in [0, 1/2]$:

$$\sum_{j=1}^{w} \frac{1}{\delta^{(j-1)(\Delta-1)}} h(t_j) \leq w K^{(w)}_\delta \ h(t), \quad (52)$$

where $t$ is the geometric mean of the $t_j$’s, i.e., $t = (t_1 \cdots t_w)^{1/w}$.

In particular, for $\delta = 9789/10000, \chi = 1/2, \psi = 13/10, \text{ and } \Delta = 6$, the inequalities (52) hold with the following values of the constants $K^{(w)}_\delta$:

$$K^{(1)}_\delta = 1, \ K^{(2)}_\delta = 1069/1000, \ K^{(3)}_\delta = 1160/1000, \ K^{(4)}_\delta = 1225/1000, \ K^{(w)}_\delta = \left(\frac{1}{\delta}\right)^{(w-1)(\Delta-1)} \text{ for } w \geq 5. \quad (53)$$

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Recall that the function \( h(t) \) is given by
\[
h(t) = (1 - t) \left[ \psi - \left( \frac{t}{1-t} \right]^\chi \right]
\] for \( t \in [0, 1/2] \),
where \( \chi = 1/2, \psi = 13/10 \). We begin with the following lemma.

**Lemma 24.** The function \( h(e^y) \) is a concave function of \( y \) in the interval \((-\infty, \ln(1/2)]\).

**Proof of Lemma 24.** Let \( f(y) := h(e^y) \) for \( y \in (-\infty, \ln(1/2)] \). We will show that \( f''(y) < 0 \) for all \( y \in (-\infty, \ln(1/2)] \).

For convenience, we use Mathematica’s `Resolve` function, see Section 10.9 for the code. \( \square \)

As an immediate consequence of Lemma 24 and Jensen’s inequality, we obtain the following inequality for \( w = 2, 3, \ldots \), for all \( t_1, \ldots, t_w \in [0, 1/2] \):
\[
\sum_{j=1}^{w} h(t_j) = \sum_{j=1}^{w} h(e^{\ln t_j}) \leq w h(e^{\frac{1}{w} \sum_{j=1}^{w} \ln t_j}) = w h(t),
\]
where \( t \) is the geometric mean of the \( t_i \)'s, i.e., \( t = (t_1 \cdots t_w)^{1/w} \). Using that \( \delta \in (0, 1] \) and inequality (84), it follows that
\[
\frac{1}{\delta (j-1)(\Delta-1)} h(t_j) \leq \frac{1}{\delta (w-1)(\Delta-1)} \sum_{j=1}^{w} h(t_j) \leq w \frac{1}{\delta (w-1)(\Delta-1)} h(t).
\]
This proves the bounds on \( K_\delta^{(w)} \) stated in the lemma for all integer \( w \geq 5 \).

For the bounds on \( K_\delta^{(w)} \) stated in the lemma for \( w = 2, 3, 4 \) we will have to work harder. Our goal is to prove the following inequalities for \( t_i \in [0, 1/2] \) (\( i = 1, 2, \ldots \)):
\[
h(t_1) + \frac{1}{\delta^5} h(t_2) \leq 2K_\delta^{(2)} h(\sqrt{t_1 t_2})
\]
\[
h(t_1) + \frac{1}{\delta^5} h(t_2) + \frac{1}{\delta^{10}} h(t_3) \leq 3K_\delta^{(3)} h(\sqrt[3]{t_1 t_2 t_3})
\]
\[
h(t_1) + \frac{1}{\delta^5} h(t_2) + \frac{1}{\delta^{10}} h(t_3) + \frac{1}{\delta^{15}} h(t_4) \leq 4K_\delta^{(4)} h(\sqrt[4]{t_1 t_2 t_3 t_4})
\]
To prove these, we will need the following inequalities.

**Lemma 25.** Let \( A_1, A_2 > 0 \) be real numbers. There exists \( A > 0 \) such that for \( t_1, t_2 \in [0, 1/2] \), it holds that
\[
A_1 h(t_1) + A_2 h(t_2) \leq A h(\sqrt{t_1 t_2}),
\]
In particular, inequality (88) holds for the following values of \( A_1, A_2, A \):
\[
\begin{align*}
A_1 = 1, & \quad A_2 = \frac{1}{\delta^5}, & \quad A = 2K_\delta^{(2)}, \\
A_1 = 2, & \quad A_2 = \frac{2}{\delta^5} K_\delta^{(2)}, & \quad A = 4 \cdot \frac{1120}{1000}, \\
A_1 = 1, & \quad A_2 = \frac{1}{\delta^{15}}, & \quad A = \frac{5}{2}, \\
A_1 = \frac{2}{\delta^5} K_\delta^{(2)}, & \quad A_2 = \frac{5}{2}, & \quad A = 4 \cdot K_\delta^{(4)},
\end{align*}
\]
where \( K_\delta^{(2)} = 1069/1000 \) and \( K_\delta^{(4)} = 1225/1000 \) are as in Lemma 15.
Proof. The existence of such an $A$ follows by standard continuity and compactness arguments. The positivity of $A$ is also easy to prove. We thus focus on the more intricate task of verifying (88) for the values of $A_1, A_2, A$ given in the statement of the lemma.

We will use Mathematica’s Resolve function. To do this, we first need to rationalize the expressions which can be achieved for $\chi = 1/2$. In particular, we will use the transformations

$$t_1 = \frac{4x_1^2}{(1 + x_1^2)^2}, \quad t_2 = \frac{4x_2^2}{(1 + x_2^2)^2}$$

for $0 \leq x_1, x_2 \leq \sqrt{2} - 1$. Under these transformations, for $\chi = 1/2$, we obtain that

$$\left( \frac{t_1}{1 - t_1} \right)^\chi = \frac{2x_1}{1 - x_1^2}, \quad \left( \frac{t_2}{1 - t_2} \right)^\chi = \frac{2x_2}{1 - x_2^2}, \quad \sqrt{t_1t_2} = \frac{4x_1x_2}{(1 + x_1^2)(1 + x_2^2)}.$$  

(94)

We are quite close to rationalizing the desired inequality, we only have to address the rationalization of $\left(\frac{\sqrt{t_1t_2}}{1 - \sqrt{t_1t_2}}\right)^x$. Unfortunately, we will have to explicitly eradicate the radical for this expression.

In particular, inequality (88) is equivalent to

$$\frac{A_1(1 - t_1) \left[ \psi - \left( \frac{t_1}{1 - t_1} \right)^\chi \right] + A_2(1 - t_2) \left[ \psi - \left( \frac{t_2}{1 - t_2} \right)^\chi \right]}{A(1 - \sqrt{t_1t_2})} \leq \psi - \left( \frac{\sqrt{t_1t_2}}{1 - \sqrt{t_1t_2}} \right)^\chi.$$  

(95)

Inequality (95) will follow from the following inequalities:

$$\frac{A_1(1 - t_1) \left[ \psi - \left( \frac{t_1}{1 - t_1} \right)^\chi \right] + A_2(1 - t_2) \left[ \psi - \left( \frac{t_2}{1 - t_2} \right)^\chi \right]}{A(1 - \sqrt{t_1t_2})} \leq \psi,$$  

(96)

and

$$\frac{\sqrt{t_1t_2}}{1 - \sqrt{t_1t_2}} \leq \left( \psi - \frac{A_1(1 - t_1) \left[ \psi - \left( \frac{t_1}{1 - t_1} \right)^\chi \right] + A_2(1 - t_2) \left[ \psi - \left( \frac{t_2}{1 - t_2} \right)^\chi \right]}{A(1 - \sqrt{t_1t_2})} \right)^2.$$  

(97)

Note that (96) allows us to take square roots in (97), and thus (95) follows.

It remains to prove (96) and (97). Using the substitutions (93) and (94), inequalities (96) and (97) are equivalent to

$$0 \leq \psi - \frac{A_1 \left( 1 - x_1^2 \right)^2 \left( \psi - \frac{2x_1}{1 - x_1^2} \right) + A_2 \left( 1 - x_2^2 \right)^2 \left( \psi - \frac{2x_2}{1 - x_2^2} \right)}{A \left( 1 - \frac{4x_1x_2}{(1 + x_1^2)(1 + x_2^2)} \right)},$$  

(98)

and

$$\frac{4x_1x_2}{(1 + x_1^2)(1 + x_2^2) - 4x_1x_2} \leq \left( \psi - \frac{A_1 \left( 1 - x_1^2 \right)^2 \left( \psi - \frac{2x_1}{1 - x_1^2} \right) + A_2 \left( 1 - x_2^2 \right)^2 \left( \psi - \frac{2x_2}{1 - x_2^2} \right)}{A \left( 1 - \frac{4x_1x_2}{(1 + x_1^2)(1 + x_2^2)} \right)} \right)^2,$$  

(99)

respectively. The last inequalities involve rational expressions and can be resolved using Mathematica for the values of $A_1, A_2, A$ given in the statement of the lemma, the code can be found in Section 10.10.  

□

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We now return to the task of proving the inequalities (85), (86) and (87). Inequality (85) is an immediate consequence of Lemma 25 (cf. the values in (89)).

To prove (86), we will use the following inequality
\[ h(t_1) + \frac{1}{\delta^5} h(t_2) + \frac{1}{\delta^{10}} h(t_3) + h(t_4) \leq 4 \cdot \frac{1120}{1000} h(\sqrt[t_1 t_2 t_3 t_4]{).} \] (100)
Applying (100) with \( t_4 = \sqrt[t_1 t_2 t_3]{\) (note that with this value of \( t_4 \) it holds that \( \sqrt[t_1 t_2 t_3]{h} = \sqrt[t_1 t_2 t_3]{\) yields
\[ h(t_1) + \frac{1}{\delta^5} h(t_2) + \frac{1}{\delta^{10}} h(t_3) \leq 3 \left( \frac{4 \cdot \frac{1120}{1000} - 1}{3} \right) h(\sqrt[t_1 t_2 t_3]{) = 3K^{(3)}_{\delta}(h(\sqrt[t_1 t_2 t_3]{), \]
which proves (86). It remains to prove (100), which follows by adding the following inequalities:
\[ h(t_1) + h(t_4) \leq 2h(\sqrt[t_1 t_4]{) \] (101)
\[ \frac{1}{\delta^5} h(t_2) + \frac{1}{\delta^{10}} h(t_3) \leq \frac{2}{\delta^5} K^{(2)}_{\delta}(h(\sqrt[t_2 t_3]{) \] (102)
\[ 2h(\sqrt[t_1 t_4]{) + \frac{2}{\delta^5} K^{(2)}_{\delta}(h(\sqrt[t_2 t_3]{) \leq 4 \cdot \frac{1120}{1000} h(\sqrt[t_1 t_2 t_3 t_4]{) \] (103)
Inequality (101) is an immediate consequence of Lemma 24. Inequality (102) is an immediate consequence of inequality (85) (multiplied by \( 1/\delta^5 \)). For inequality (103), we use the transformations \( u_1 = \sqrt[t_1 t_4]{\) and \( u_2 = \sqrt[t_2 t_3]{\), so that we need to show
\[ 2h(u_1) + \frac{2}{\delta^5} K^{(2)}_{\delta}(h(u_2) \leq 4 \cdot \frac{1120}{1000} h(\sqrt[u_1 u_2]{) \] (104)
for \( u_1, u_2 \in [0, 1/2] \), which follows from Lemma 25 (cf. the values (90)).

Finally, we conclude with the proof of inequality (87). This is obtained by adding the following three inequalities:
\[ \frac{1}{\delta^5} h(t_2) + \frac{1}{\delta^{10}} h(t_3) \leq \frac{2}{\delta^5} K^{(2)}_{\delta}(h(\sqrt[t_2 t_3]{), \] (105)
\[ h(t_1) + \frac{1}{\delta^{15}} h(t_4) \leq \frac{5}{2} h(\sqrt[t_1 t_4]{, \] (106)
\[ \frac{2}{\delta^5} K^{(2)}_{\delta}(h(\sqrt[t_2 t_3]{) + \frac{5}{2} h(\sqrt[t_1 t_4]{) \leq 4K^{(4)}_{\delta}(h(\sqrt[t_1 t_2 t_3 t_4]{). \] (107)
Inequality (105) is an immediate consequence of (85) (again, multiplied by \( 1/\delta^5 \)). Inequality (106) follows from Lemma 25 (cf. the values (91)). Finally, inequality (107) can be proved using an analogous transformation as the one used to prove (103); the required analogue of inequality (104) has been proved in Lemma 25 (cf. the values (92)).

This concludes the proof of Lemma 15.

6.5 Eliminating large arity clauses from consideration
In this section, we prove Lemma 16, which we restate here for convenience. Recall that \( w_i \) is the arity of the \( i \)-th clause containing \( x \) minus one. Intuitively, clauses with large arity should not affect significantly the correlation decay. The lemma captures this in a quantitative way which is sufficient for our needs (for clauses with \( w_i \geq 6 \).
Lemma 16. Let \( i \) be such that \( w_i \geq 6 \). Then, for all \( \hat{i} \) such that \( 0 \leq \hat{i} \leq (1/2)1 \), it holds that

\[
\rho_1^{(1)}(\hat{i}) \leq \frac{1}{a} \hat{F}(\hat{i}) M, \text{ where } M = 25/1000,
\]

and \( \hat{F}(\hat{i}) \) is given by (51).

Proof. We will show that for all integers \( w \geq 6 \) and all \( t \in [0,1/2] \), it holds that

\[
\frac{t^w}{1-t^w} \leq \frac{63}{2^w - 1} \frac{t^6}{1-t^6}.
\]

We will also show that for \( \chi = 1/2, \psi = 13/10 \), it holds that

\[
\max_{t \in [0,1/2]} \frac{t^6}{1-t^6} h(t) \leq M_1, \text{ where } M_1 = \frac{1}{410}.
\]

Finally, we will show that for integer \( w \geq 6 \), \( l_w = \lceil \log_6(w + 1) \rceil \), \( \delta = 9789/10000 \), it holds that

\[
\frac{63 M_1}{2^w - 1} w^\delta \kappa_1^\delta(w) \alpha^{1-w} \leq \frac{M}{\alpha},
\]

where \( \kappa_1^\delta(w) \) for \( w \geq 6 \) is given by Lemma 15 and \( M = 25/1000 \) is as in the statement of the lemma. The lemma then follows by multiplying (108), (109) and (110).

We start with the verification of (108). Note that (108) holds at equality for \( \hat{t}_i = 1/2 \), so it suffices to show that for all integer \( w \geq 6 \), the function

\[
f(t) := \frac{t^w (1-t^6)}{p^6(1-t^w)} = \frac{t^{w-6} (1-t^6)}{1-t^w}
\]

is increasing for \( t \in [0,1/2] \). For \( w = 6 \), there is nothing to show, so we may assume that \( w \geq 7 \). We then calculate (see Section 10.2 for the calculation)

\[
f'(t) = \frac{t^{w-7} p(t)}{(t^w - 1)^2}, \text{ where } p(t) := 6t^w - t^6 w + w - 6.
\]

so we only need to show that \( p(t) \geq 0 \) for \( t \in [0,1/2] \). Note that \( p'(t) = 6w(t^{w-1} - t^6) \leq 0 \) for all \( t \in [0,1/2] \) since \( w \geq 7 \). It follows that

\[
p(t) \geq p(1/2) = 6(1/2)^w + (63/64)w - 6 \geq (63/64)w - 6 \geq 1/2,
\]

where in the last inequality we again used that \( w \geq 7 \). This completes the verification of (108).

We next verify (109). For convenience, we use Mathematica’s Resolve function for that, see Section 10.2.

Finally, we verify (110). Since \( l_w \leq \log_6(w + 1) + 1 \) and \( \alpha = 1 - 10^{-4} < 1 \), it suffices to show that for all \( w \geq 6 \) it holds that

\[
\frac{63 M_1}{2^w - 1} w^\delta \left( \frac{1}{\delta} \right)^{(w-1)} (\Delta-1)^{-\log_6(w+1)} \alpha^{-\log_6(w+1)} \leq M.
\]
It is a matter of numerical calculations to show that $\alpha^{-1} \leq \exp(11 \cdot 10^{-5})$. Thus, to show (112), it suffices to show that

$$\frac{63M_1}{2^w - 1} w \left( \frac{1}{\delta} \right)^{(w-1)(\Delta-1)} (w + 1)^{11 \cdot 10^{-5}/\ln 6} \leq M. \quad (113)$$

We view the lhs in (113) as a function of $w$, say $f(w)$. We will prove that

$$f(6) \leq M \quad \text{and} \quad f'(w) \leq 0 \quad \text{for all } w \geq 6, \quad (114)$$

from which inequality (113) follows. To show (114), we use Mathematica’s RESOLVE function, see Section 10.2 for details. This concludes the proof of Lemma 16.

\[ \square \]

6.6 The contribution of arity 3 clauses

In this section, we give the proof of Lemma 19. Roughly, the lemma bounds the aggregate contribution of arity 3 clauses along with the effect of the creation of arity 2 clauses (due to the pinnings when processing arity 3 clauses). This was used to further reduce the number of variables.

To prove Lemma 19, we will use Hölder’s inequality. Let $p > 1$. For $q = p/(p-1)$ and positive real numbers $\alpha_i, \beta_i$ for $i \in [b_3]$, it holds that

$$\sum_{i=1}^{b_3} \alpha_i \beta_i \leq \left( \sum_{i=1}^{b_3} \alpha_i^p \right)^{1/p} \left( \sum_{i=1}^{b_3} \beta_i^q \right)^{1/q}.$$ 

This yields

$$\sum_{i=1}^{b_3} \left( \frac{1}{\delta} \right)^{b_3-i} g_2(y_i) \leq \left( \sum_{i=1}^{b_3} \left( \frac{1}{\delta} \right)^{(b_3-i)p} \right)^{1/p} \left( \sum_{i=1}^{b_3} (g_2(y_i))^q \right)^{1/q}. \quad (115)$$

For $p = 27/2$, we have $q = 27/25$. Recall from equation (67) that the function $g_2(y)$ is given by

$$g_2(y) := \frac{(1-y)h((1-y)^{1/2})}{y} = \frac{(1-y)(1-(1-y)^{1/2})}{y} \left( y - \frac{(1-y) + (1-y)^{1/2}}{y} \right)^x. \quad (67)$$

**Lemma 26.** For $q = 27/25$ and $\Delta = 6$, the function $(g_2(e^t))^q$ is a concave function of $t$ in the interval $[\ln(3/4), \ln(1 - \frac{1}{(2^4-1)\cdot 7^q})]$.

**Proof of Lemma 26.** Let $\tilde{g}_2(t) := (g_2(e^t))^q$. Our goal is to show that $\tilde{g}_2(t) \leq 0$ for all $t \in [\ln(3/4), \ln(1 - \frac{1}{(2^4-1)\cdot 7^q})]$. We have

$$\tilde{g}_2'(t) = q(g_2'(e^t))^{q-1}g_2'(e^t)e^t,$$

$$\tilde{g}_2''(t) = q(q-1)(g_2'(e^t))^{q-2}(g_2''(e^t))^2e^{2t} + q(g_2'(e^t))^{q-1}g_2''(e^t)e^{2t} + q(g_2'(e^t))^{q-1}g_2'(e^t).$$

Observe that $g_2(e^t)e^t > 0$ for all $t \in [\ln(3/4), \ln(1 - \frac{1}{(2^4-1)\cdot 7^q})]$. Thus, using the transformation $y = e^t$, it suffices to show that

$$\frac{(q-1)}{y} (g_2'(y))^2 + g_2(y)(g_2'(y) + yg_2''(y)) \leq 0, \quad (116)$$

for all $y \in [3/4, 1 - \frac{1}{(2^4-1)\cdot 7^q}]$. For convenience, we verify (116) using Mathematica’s RESOLVE function, see Section 10.11 for the code.

\[ \square \]
We are now ready to prove Lemma 19, which we restate here for convenience.

**Lemma 19.** Let $\Delta = 6$ and $b_3$ be a non-negative integer less than or equal to $\Delta - 1 = 5$. There exists a constant $C^{(b_3)}_\delta \geq 0$ so that for $y_1, \dotsc, y_b \in [3/4, 1 - 1/(2^d + 1)^2]$ it holds that

$$\sum_{i=1}^{b_3} \left( \frac{1}{\delta} \right)^{b_3-i} g_2(y_i) \leq b_3 C^{(b_3)}_\delta g_2(\sqrt[3]{y_1 \cdots y_b}).$$  \hspace{1cm} (72)

In particular, we will show that inequality (72) holds with $C^{(b_3)}_\delta = \frac{1}{\delta^{1 - 1/p}} \left( \frac{1 - \delta^{b_3} p}{\delta^{1 - 1/p}} \right)^{1/p}$, where $p = 27/2$. For $\delta = 9789/10000$, we have the following upper bounds on the values of $C^{(b_3)}_\delta$:

$C^{(0)}_\delta = 0$, $C^{(1)}_\delta = 1$, $C^{(2)}_\delta = 102/100$, $C^{(3)}_\delta = 103/100$, $C^{(4)}_\delta = 104/100$, $C^{(5)}_\delta = 105/100$. \hspace{1cm} (73)

**Proof of Lemma 19.** For $p = 27/2$ and $q = 27/25$, inequality (115) and Lemma 26 yield that

$$\sum_{i=1}^{b_3} \left( \frac{1}{\delta} \right)^{b_3-i} g_2(y_i) \leq b_3^{1/q} \left( \sum_{i=1}^{b_3} \left( \frac{1}{\delta} \right)^{(b_3-i)p} \right)^{1/p} g_2(\sqrt[3]{y_1 \cdots y_b}).$$

Note that for $b_3 = 1, \dotsc, 5$ it holds that

$$\sum_{i=1}^{b_3} \left( \frac{1}{\delta} \right)^{(b_3-i)p} = \frac{1}{\delta^{b_3} - 1} = \frac{1 - \delta^{b_3} p}{1 - \delta^p}.$$  \hspace{1cm} (74)

Using this and $q = p/(p-1)$, we obtain that

$$\sum_{i=1}^{b_3} \left( \frac{1}{\delta} \right)^{b_3-i} g_2(y_i) \leq b_3 C^{(b_3)}_\delta g_2(\sqrt[3]{y_1 \cdots y_b}).$$

where $C^{(b_3)}_\delta = \frac{1}{\delta^{1 - 1/p}} \left( \frac{1 - \delta^{b_3} p}{\delta^{1 - 1/p}} \right)^{1/p}$, as desired. The numerical bounds in (73) on the values of $C^{(b_3)}_\delta$ for $b_3 = 1, \dotsc, 5$ can be verified by a direct calculation using Mathematica, see Section 10.4. This concludes the proof of Lemma 19. \hspace{1cm} \blacksquare

## 7 Proof of the main theorems

In this section, we give the proofs of Theorems 3 and 2. First, we give the proof of Lemma 5 which we restate here for convenience.

**Lemma 5.** Let $k$ and $\Delta$ be positive integers. Suppose that there is a polynomial-time algorithm (in $n$ and $1/\varepsilon$) that takes an $n$-variable formula $C \in C_{k,\Delta}$, a variable $x$ of $C$, and an $\varepsilon > 0$ and computes a quantity $\hat{R}(C, x)$ satisfying $|\hat{R}(C, x) - R(C, x)| \leq \varepsilon$. Then, there exists an FPTAS which approximates $Z(C)$ for every $C \in C_{k,\Delta}$.

**Proof.** The proof is actually identical to the argument in [10, Appendix A], it is just for the sake of explicitness that we need to formally check that we invoke the algorithm that
computes $\hat{R}(C,x)$ only on formulas $C$ whose clauses have arity at least $k$ (and max degree $\Delta$).

Let $\varepsilon > 0$ and $C$ be a monotone CNF formula $C$ with max degree $\Delta$ whose clauses have arity at least $k$. Let $x_1, \ldots, x_n$ be the variables in $C$. Let $C_i$ be the formula obtained by $C$ by setting $x_1 = \cdots = x_i = 1$ and removing all the clauses that are satisfied (i.e., all clauses that contain a variable from $x_1, \ldots, x_i$). We have

$$\frac{1}{Z(C)} = \Pr_C(x_1 = \ldots = x_n = 1) = \prod_{i=1}^{n-1} \Pr_{C_i}(x_{i+1} = 1 | x_1 = \cdots = x_i = 1)$$

(117)

$$= \prod_{i=1}^{n-1} \Pr_{C_i}(x_{i+1} = 1) = \prod_{i=1}^{n-1} \frac{1}{1 + R(C_i, x_{i+1})}.$$

Note that every $C_i$ is a monotone CNF formula with max degree $\Delta$ whose clauses have arity at least $k$. By the assumption in the lemma, we can compute (in $\text{poly}(n, 1/\varepsilon)$ time) quantities $\hat{R}(C_i, x_{i+1})$ such that

$$\hat{R}(C_i, x_{i+1}) - R(C_i, x_{i+1}) \leq \varepsilon/(100n) \text{ for all } i = 1, \ldots, n - 1.$$

(118)

Let

$$\hat{Z}(C) = \prod_{i=1}^{n-1} (1 + \hat{R}(C_i, x_{i+1})).$$

(119)

It is not hard to conclude from (117), (118) and (119) that $(1 - \varepsilon)Z(C) \leq \hat{Z}(C) \leq (1 + \varepsilon)Z(C)$. This completes the proof.

We are now ready to give the proof of Theorems 2 and 3, which we restate here.

**Theorem 2.** There is an FPTAS for $\#\text{HyperIndSet}(3,6)$.

**Theorem 3.** There exists a constant $k_0$ such that for all positive integers $k \geq k_0$ and $\Delta$ satisfying $k \geq 1.66\Delta$ there is an FPTAS for the problem $\#\text{HyperIndSet}(k, \Delta)$.

**Proof of Theorems 2 and 3.** We first prove Theorem 2. First, reformulate $\#\text{HyperIndSet}(3,6)$ in terms of the monotone CNF problem, as was explicitly done in Section 2.2. Then, just invoke Lemmas 5 and 10 to obtain Theorem 2.

The proof of Theorem 3 is completely analogous, we just need to specify the value of $k_0$ in the statement of Theorem 3. To do this, let $k'_0$ be the lower bound such that Lemma 11 holds for all $k \geq k'_0$. Let $\Delta_0 = 3/(1.66 - \beta)$. Then let $k_0$ be the value of $k_0$ in the statement of Theorem 3 be $\max(k'_0, 1.66\Delta_0 + 3)$. Now suppose that $k \geq k_0$ and $\Delta$ are positive integers satisfying $k \geq 1.66\Delta$. Whether or not $\Delta \leq \Delta_0$, it is easy to see that $k \geq \beta\Delta + 3$, so Lemma 11 applies. Invoking Lemmas 5 and 11, just as before, yields Theorem 3.

### 8 Hardness for Counting

In this section, we prove the hardness results stated in the Introduction. For this section, it is convenient to return to the original hypergraph independent set formulation of the problem (instead of the monotone CNF formulation). The proof is via a reduction to the independent set model on graphs which was used by Bordewich et al. [2]. The precise inapproximability
results for the hard-core model had not yet been proved at the time [2] was written, so we carry out the details explicitly to obtain the bound that their reduction gives.

Namely, we will use the inapproximability result of Sly and Sun [16] for the hard-core model. We first remind the reader the relevant definitions. Let \( \lambda > 0 \). For a graph \( G = (V, E) \), the hard-core model with parameter \( \lambda \) is a probability distribution over the set of independent sets of \( G \); each independent set \( I \) of \( G \) has weight proportional to \( \lambda^{|I|} \). The normalizing factor of this distribution is the partition function \( Z_G(\lambda) \), formally defined as \( Z_G(\lambda) := \sum_I \lambda^{|I|} \) where the sum ranges over all independent sets \( I \) of \( G \).

**Theorem 27** ([16]). For \( \Delta \geq 3 \), let \( \lambda_c(\Delta) := (\Delta - 1)^{\Delta - 1}/(\Delta - 2)^{\Delta} \). For all \( \lambda > \lambda_c(\Delta) \), it is NP-hard to approximate \( Z_G(\lambda) \) on \( \Delta \)-regular graphs \( G \), even within an exponential factor.

**Theorem 28.** Let \( k \geq 2 \), \( \Delta \geq 3 \) be integers. Suppose that \( 2^{k/2} - 1 < (\Delta - 2)^\Delta/(\Delta - 1)^\Delta \). Then, it is NP-hard to approximate \( \#\text{HyperIndSet}(k, \Delta) \), even within an exponential factor.

**Proof.** Let \( k \geq 2 \), \( \Delta \geq 3 \) be integers satisfying \( 2^{k/2} - 1 < (\Delta - 2)^\Delta/(\Delta - 1)^\Delta \) and let
\[
\lambda := 1/(2^{k/2} - 1).
\]
Note that \( \lambda > \lambda_c(\Delta) \) where \( \lambda_c(\Delta) \) is as in Theorem 27. For convenience, let \( k' := [k/2] \) in what follows.

Let \( G = (V, E) \) be a \( \Delta \)-regular graph and set \( n := |V| \). We will construct a \((2k')\)-uniform hypergraph \( H = (U, \mathcal{F}) \) with maximum degree \( \Delta \) such that \( |U| = k'|V| \), \( |\mathcal{F}| = |E| \) and
\[
Z_H = (2^{k'} - 1)^n Z_G(\lambda). \tag{120}
\]
Note that the size of \( H \) is larger than the size of \( G \) only by a constant factor. It thus follows that if we could approximate \( \#\text{HyperIndSet}(k, \Delta) \) within an arbitrarily small exponential factor, we could also approximate \( Z_G(\lambda) \) within an (arbitrarily small) exponential factor for all \( \Delta \)-regular graphs \( G \), contradicting Theorem 27.

It remains to construct the hypergraph \( H = (U, \mathcal{F}) \). Let
\[
U = \bigcup_{v \in V} \{u_v, 1, \ldots, u_v, k'\}, \quad \mathcal{F} = \bigcup_{(v, w) \in E} \{\{u_v, 1, \ldots, u_v, k', u_w, 1, \ldots, u_w, k'\}\}.
\]
In words, every vertex \( v \) of \( G \) maps to a (distinct) set of \( k' \) vertices in \( H \), the set \( \{u_v, 1, \ldots, u_v, k'\} \), which we will henceforth denote as \( S_v \). Further, each edge \((v, w)\) in \( G \) maps to a hyperedge in \( H \) which is given by \( S_v \cup S_w \). It is clear from the construction that every vertex of \( H \) has degree \( \Delta \) (since \( G \) is a \( \Delta \)-regular graph) and, further, that every hyperedge of \( H \) has arity \( 2k' \geq k \). Also, note that \(|U| = k'|V|\) and \(|\mathcal{F}| = |E|\).

We complete the proof by showing (120). To do this, we will map independent sets of the hypergraph \( H \) to independent graphs of \( G \) as follows. Let \( I_H \) be an independent set of \( H \). Define \( I_G \) to be the subset of vertices of \( G \) such that \( v \in I_G \) iff \( S_v \cap I_H = S_v \). It is immediate that \( I_G \) is an independent set of \( G \). In fact, it is not hard to see that for every independent set \( I_G \) of \( G \) there are exactly \((2^{k'} - 1)^{n-|I_G|}\) independent sets of \( H \) that map to \( I_G \). From this, (120) follows, thus completing the proof.

**Corollary 29.** Let \( k = 6 \), \( \Delta = 22 \). It is NP-hard to approximate \( \#\text{HyperIndSet}(k, \Delta) \), even within an exponential factor.
Proof. Just plug the values of $k, \Delta$ to check that the inequality $2^{\lfloor k/2 \rfloor} - 1 < \frac{(\Delta - 2)^{\Delta}}{(\Delta - 1)^{\Delta - 1}}$ holds. Then, apply Theorem 28.

The following corollary is a crude estimate of the range of $\Delta$ in which $\#\text{HyperIndSet}(k, \Delta)$ is hard to approximate (by applying Theorem 28).

**Corollary 30.** Let $k \geq 2$. For all integer $\Delta \geq 20 \cdot 2^{k/2}$, it is NP-hard to approximate $\#\text{HyperIndSet}(k, \Delta)$, even within an exponential factor.

**Proof.** For $\Delta \geq 20 \cdot 2^{k/2}$, we have that

\[
\frac{(\Delta - 2)^{\Delta}}{(\Delta - 1)^{\Delta - 1}} = (\Delta - 2) \left(1 - \frac{1}{\Delta - 1}\right)^{\Delta - 1} > \frac{\Delta - 2}{e^2} > 2^{\lfloor k/2 \rfloor} - 1.
\]

Now apply Theorem 28.

\qed

## 9 The Uniqueness Threshold on the Infinite Hypertree

We denote by $T_{k, \Delta}$ the infinite $(\Delta - 1)$-ary $k$-uniform hypertree with root vertex $\rho$. Also, for $n = 0, 1, 2, \ldots$, denote by $T_{k, \Delta}(n)$ the subtree of $T_{k, \Delta}$ obtained by the first $n$ levels, i.e., $T_{k, \Delta}(n)$ is the tree induced by the set of vertices at distance $\leq n$ from $\rho$ in $T_{k, \Delta}$. We denote by $V_n$ the vertex set of $T_{k, \Delta}(n)$ and by $L_n$ the leaves of the tree, i.e., vertices with degree 1 in $T_{k, \Delta}(n)$.

Denote by $\mu_n$ the Gibbs distribution of the independent set model on $T_{k, \Delta}(n)$ (see Section 2.1). For a configuration $\sigma : V_n \to \{0, 1\}$, we denote by $\sigma_{L_n}$ the restriction of $\sigma$ to the set $L_n$ and by $\sigma_\rho$ the spin of the root $\rho$.

**Definition 31.** Let $k \geq 2, \Delta \geq 2$ be integers. The independent set model has uniqueness on $T_{k, \Delta}$ iff

\[
\limsup_{n \to \infty} \max_{\eta, \eta' \in L_n \to \{0, 1\}} |\mu_n(\sigma_\rho = 1 | \sigma_{L_n} = \eta) - \mu_n(\sigma_\rho = 1 | \sigma_{L_n} = \eta')| = 0. \tag{121}
\]

We will use $\sigma_{L_n} = 1$ to denote that, in the configuration $\sigma$, all vertices in $L_n$ are assigned the spin 1. For $n = 0, 1, 2, \ldots$, define

\[
p_n = \mu_n(\sigma_\rho = 1 | \sigma_{L_n} = 1) \tag{122}
\]

When $\sigma_\rho = 1$, in each of the $\Delta - 1$ hyperedges that include $\rho$, at least one of the $k - 1$ vertices (other than $\rho$) must have spin 0. When $\sigma_\rho = 0$, any configuration on the neighbours of $\rho$ is allowed. By considering the (normalised) weight of such configurations on $T_{k, \Delta}(n + 1) \backslash \rho$, it is not hard to see that the sequence $p_n$ satisfies the following recursion for every integer $n \geq 0$:

\[
p_{n+1} = f(p_n), \quad \text{where} \quad f(z) := \frac{(1 - z^{k-1})^{\Delta - 1}}{1 + (1 - z^{k-1})^{\Delta - 1}}. \tag{123}
\]

For any configuration $\eta : L_n \to \{0, 1\}$, we will see that $\mu_n(\sigma_\rho = 1 | \sigma_{L_n} = \eta)$ is sandwiched between $p_n$ and $p_{n+1}$. This yields the following.

**Lemma 32.** Let $k \geq 2, \Delta \geq 2$ be integers. The independent set model has uniqueness on $T_{k, \Delta}$ iff

\[
\limsup_{n \to \infty} |p_{n+1} - p_n| = 0. \tag{124}
\]
Proof. Let $n$ be a non-negative integer and let $\eta$ be an arbitrary configuration on $L_n$, i.e., $\eta : L_n \to \{0, 1\}$. We will show that

$$
\begin{align*}
  p_{n+1} &\leq \mu_n(\sigma_\rho = 1 \mid \sigma_{L_n} = \eta) \leq p_n \text{ for even integers } n, \\
  p_n &\leq \mu_n(\sigma_\rho = 1 \mid \sigma_{L_n} = \eta) \leq p_{n+1} \text{ for odd integers } n.
\end{align*}
$$

(125)

Let us first conclude the lemma assuming (125). From (125), we obtain that for all $n$, it holds that

$$
\max_{\eta, \eta' : L_n \to \{0, 1\}} |\mu_n(\sigma_\rho = 1 \mid \sigma_{L_n} = \eta) - \mu_n(\sigma_\rho = 1 \mid \sigma_{L_n} = \eta')| = |p_{n+1} - p_n|.
$$

It follows that the conditions in (121) and (124) are equivalent, which yields the statement in the lemma.

We next show (125). The proof is by induction on $n$. The claim is trivial for $n = 0$ since $p_0 = 1$ and $p_1 = 0$. So assume that the claim holds for all non-negative integers less than $n$, we will show it for $n$.

Set $d := \Delta - 1$. Let $e_1, \ldots, e_d$ be the $d$ hyperedges containing $\rho$ and for $i \in [d]$ denote by $v_{i,1}, \ldots, v_{i,k-1}$ the vertices in $e_i$ other than $\rho$, i.e.,

$$
e_1 = \{\rho, v_{1,1}, \ldots, v_{1,k-1}\}, \ldots, e_d = \{\rho, v_{d,1}, \ldots, v_{d,k}\}.
$$

For $i \in [d]$ and $j \in [k - 1]$, let $T_{i,j}$ be the subtree of $T_{k,\Delta}(n)$ rooted at $v_{i,j}$. Denote by $S_{i,j}$ the leaves of $T_{i,j}$ and by $\eta_{i,j}$ the restriction of $\eta$ on $S_{i,j}$. Let $\mu_{T_{i,j}}$ be the Gibbs distribution of $T_{i,j}$ in the independent set model. Note that $\bigcup_{i\in[d],j\in[k-1]} S_{i,j} = L_n$. Finally, let

$$
\begin{align*}
  q_{i,j} &:= \mu_{T_{i,j}}(\sigma_{v_{i,j}} = 1 \mid \sigma_{S_{i,j}} = \eta_{i,j}), \\
  q &:= \mu_n(\sigma_\rho = 1 \mid \sigma_{L_n} = \eta).
\end{align*}
$$

It is simple to see that

$$
q = \frac{\prod_{i=1}^{d}(1 - \prod_{j=1}^{k-1} q_{i,j})}{1 + \prod_{i=1}^{d}(1 - \prod_{j=1}^{k-1} q_{i,j})}, \text{ or equivalently that } \frac{q}{1-q} = \prod_{i=1}^{d}(1 - \prod_{j=1}^{k-1} q_{i,j}).
$$

(126)

For $i \in [d]$ and $j \in [k - 1]$, note that $T_{i,j}$ is isomorphic to $T_{k,\Delta}(n - 1)$ and hence we can use the induction hypothesis to bound $q_{i,j}$. Let us consider first the case where $n$ is odd. Then, we have that

$$
p_n \leq q_{i,j} \leq p_{n-1}.
$$

(127)

It follows from (126) and (127) that

$$
(1 - (p_{n-1})^{k-1})^d \leq \frac{q}{1-q} \leq (1 - (p_n)^{k-1})^d, \text{ so that } p_n \leq q \leq p_{n+1},
$$

(128)

where to derive the last inequality we used that the sequence $p_n$ satisfies the recursion in (123) and that the function $\frac{x}{1-x}$ is increasing in $x$. The proof for odd $n$ is completely analogous, modulo that the inequalities in (127) and (128) hold in the opposite direction.

This concludes the induction step and hence the proof of (125). The proof of the lemma is thus complete. \qed
Lemma 33. Let \( k \geq 2, \Delta \geq 2 \) be integers. Let \( f(z) = \frac{(1 - x^{k-1})^{\Delta-1}}{1 + (1 - x^{k-1})^{\Delta-1}} \) be as in (123). The function \( f \) is increasing in the interval \([0, 1]\). Also, there is unique \( x \in [0, 1] \) such that

\[
f(x) = x,
\]

which further satisfies \( |f'(x)| = \frac{(\Delta-1)(k-1)(1 - z^{k-1})^{\Delta-2} z^{k-2}}{(1 + (1 - z^{k-1})^{\Delta-1})^2} \).

Finally, if \( |f'(x)| < 1 \), the equation

\[
f(f(z)) = z \text{ for } z \in [0, 1]
\]
is uniquely satisfied by \( z = x \).

**Proof.** To see that the function \( f \) is decreasing, we calculate

\[
f'(z) = -\frac{(\Delta-1)(k-1)(1 - z^{k-1})^{\Delta-2} z^{k-2}}{(1 + (1 - z^{k-1})^{\Delta-1})^2},
\]

which clearly shows that \( f \) is decreasing for \( z \in [0, 1] \).

We next show the second part of the lemma. We can rewrite \( z = f(z) \) as

\[
g(z) = 0 \text{ where } g(z) := z - (1 - z)(1 - z^{k-1})^{\Delta-1}.
\]

For the function \( g \), we have that \( g(0) = -1, g(1) = 1 \) and \( g \) is continuous on \([0, 1]\). It thus follows that there exists \( x \) such that \( g(x) = 0 \) which implies that \( f(x) = x \). We next prove that \( x \) is unique, i.e., for all \( z \neq x \) it holds that \( g(z) \neq 0 \). For this, it suffices to show that \( g \) is increasing on \([0, 1]\) or that \( g'(z) > 0 \) for all \( z \in [0, 1] \). We calculate

\[
g'(z) = 1 + (1 - z^{k-1})^{\Delta-1} + (k - 1)(\Delta - 1)(1 - z)(1 - z^{k-1})^{\Delta-2} z^{k-2},
\]

which clearly shows that \( g'(z) \geq 1 \) for all \( z \in [0, 1] \). Finally, to see the expression for \( |f'(x)| \), just use (131) and use that

\[
x = \frac{(1 - x^{k-1})^{\Delta-1}}{1 + (1 - x^{k-1})^{\Delta-1}}, \quad 1 - x = \frac{1}{1 + (1 - x^{k-1})^{\Delta-1}}
\]
to simplify. This proves the second assertion in the lemma.

Finally, we show the last part of the lemma. Namely, suppose that \( |f'(x)| < 1 \). Consider the function

\[
h(z) := f(f(z)) - z, \text{ for } z \in [0, 1].
\]

Clearly, we have that \( h(x) = 0 \). By the assumption \( |f'(x)| < 1 \), we have that \( h'(x) < 0 \) and hence for small \( \varepsilon > 0 \) it holds that \( h(x - \varepsilon) > 0 \) and \( h(x + \varepsilon) < 0 \). Note also that \( h(0) > 0 \) and \( h(1) < 0 \).

For the sake of contradiction, assume that \( h \) has a zero other than \( x \), say at \( z = \rho_1 \) where \( \rho_1 \neq x \). Let \( \rho_2 := f(\rho_1) \) and note that \( f(\rho_2) = \rho_1 \). Observe also that \( h \) has another zero at \( z = \rho_2 \). Also, using that \( f \) is decreasing and \( f(x) = x \), we have \( \rho_2 \neq \rho_1, x \). In fact, we have that \( x \) is between \( \rho_1 \) and \( \rho_2 \). Wlog we may thus assume that \( \rho_1 > x > \rho_2 \) (otherwise we may swap \( \rho_1 \) and \( \rho_2 \)).

We may also assume that \( h'(\rho_1) \geq 0 \). Otherwise, we claim that there exists \( \rho \in (x, \rho_1) \) such that \( h(\rho) = 0 \) and \( h'(\rho) \geq 0 \), so that if \( h'(\rho_1) < 0 \) we could swap our focus from \( \rho_1 \) to \( \rho \).
(the role of $\rho_2 = f(\rho_1)$ would be played by $f(\rho)$). To see the claim, assume that $h'(\rho_1) < 0$ and let $\varepsilon > 0$ be sufficiently small. Then, it holds that $h(\rho_1 - \varepsilon) > 0$. Recall also that $h(x + \varepsilon) < 0$. It follows that there must exist a crossing of $h$ in the interval $(x, \rho_1)$, i.e., a real number $\rho \in (x, \rho_1)$ such that $h(\rho) = 0$ and, for all sufficiently small $\varepsilon'$, it holds that $h(\rho + \varepsilon') < 0$ and $h(\rho - \varepsilon') > 0$. It must thus be the case that $h'(\rho) \geq 0$, as claimed.

Thus, if $|f'(x)| < 1$, we have concluded the existence of $\rho_1, \rho_2$ such that $1 > \rho_1 > \rho_2 > 0$, $\rho_1 = f(\rho_2)$, $\rho_2 = f(\rho_1)$ and $h'(\rho_1) \geq 0$. We obtain a contradiction by showing that $h'(\rho_1) < 0$. We have

$$h'(\rho_1) = f'(\rho_1)f'(\rho_2) - 1 = \frac{(\Delta - 1)^2(k - 1)^2\rho_1^{k-2}\rho_2^{k-2}(1 - \rho_1^{k-1})\Delta^{-2}(1 - \rho_2^{k-1})\Delta^{-2}}{(1 + (1 - \rho_1^{k-1})\Delta^{-1})(1 + (1 - \rho_2^{k-1})\Delta^{-1})} - 1.$$  

(132)

For $k = 2, \Delta = 2$, (132) is trivial, so henceforth we may assume that at least one of $k, \Delta$ is greater or equal than 3.

We can rewrite $\rho_1 = f(\rho_2)$ and $\rho_2 = f(\rho_1)$ as

$$1 - \rho_1 = \frac{1}{1 + (1 - \rho_2^{k-1})\Delta^{-1}}, \quad 1 - \rho_2 = \frac{1}{1 + (1 - \rho_1^{k-1})\Delta^{-1}}.$$  

(133)

Subtracting the equations in (133) gives

$$\rho_1 - \rho_2 = \frac{(1 - \rho_2^{k-1})\Delta^{-1} - (1 - \rho_1^{k-1})\Delta^{-1}}{(1 + (1 - \rho_1^{k-1})\Delta^{-1})(1 + (1 - \rho_2^{k-1})\Delta^{-1})}.$$  

(134)

Let $E := (1 - \rho_2^{k-1})\Delta^{-1} - (1 - \rho_1^{k-1})\Delta^{-1}$ and assume for now that $k \geq 3, \Delta \geq 3$. We use the inequality $x^d - y^d > d(x - y)(xy)^{(d-1)/2}$, which holds for all $d \geq 2$ and real numbers $x > y > 0$ (see, for example, [6, Claim 35]), to obtain the following bound for $E$:

$$E > (\Delta - 1)(\rho_1^{k-1} - \rho_2^{k-1})(1 - \rho_1^{k-1})(\Delta - 2)^2(1 - \rho_2^{k-1})(\Delta - 2)^2$$

$$> (\Delta - 1)(k - 1)(\rho_1 - \rho_2)(\rho_1^{(k-2)/2} - \rho_2^{(k-2)/2})(\Delta - 2)^2(1 - \rho_1^{k-1})(\Delta - 2)^2(1 - \rho_2^{k-1})(\Delta - 2)^2.$$  

(135)

Note that the strict inequality in (135) holds even in the cases where $k = 2, \Delta \geq 3$ or $\Delta = 2, k \geq 3$.

Combining (134) and (135), we obtain that

$$1 > \frac{(\Delta - 1)(k - 1)(\rho_1^{k-2/2} - \rho_2^{k-2/2})(1 - \rho_1^{k-1})(\Delta - 2)^2(1 - \rho_2^{k-1})(\Delta - 2)^2}{(1 + (1 - \rho_1^{k-1})\Delta^{-1})(1 + (1 - \rho_2^{k-1})\Delta^{-1})}.$$  

Squaring the last inequality and using (132), we obtain $h'(\rho_1) < 0$, as desired.

This concludes the proof of Lemma 33. □

**Lemma 34.** Let $k \geq 2, \Delta \geq 2$ be integers. Let $f(z) = \frac{(1 - z^{k-1})\Delta^{-1}}{1 + (1 - z^{k-1})\Delta^{-1}}$ be as in (123) and $x$ be as in Lemma 33.

If $|f'(x)| < 1$, the independent set model has uniqueness on $\mathbb{T}_{k,\Delta}$.

If $|f'(x)| > 1$, the independent set model has non-uniqueness on $\mathbb{T}_{k,\Delta}$.
Proof. Recall the sequence $p_n$ defined in (123). Let $p_n^+ = p_{2n}$ and $p_n^- = p_{2n+1}$. As a consequence of the fact that $f$ is decreasing (cf. Lemma 33) and $p_0^+ = 1$, $p_0^- = 0$, we have that

$$p_n^+ \downarrow p^+, \quad p_n^- \uparrow p^- \quad \text{(136)}$$

where $p^+, p^-$ are real numbers in $[0,1]$. To see the existence of these limits, note that $p_0^+ = 1 \geq p_1^+$ and $p_0^- = 0 \leq p_1^-$. Since $p_{n+1}^+ = f(f(p_n^+))$, a simple induction shows that $p_n^+$ is a decreasing sequence and $p_n^-$ increasing. Since both sequences are bounded, we obtain the existence of the limits in (136). For later use, we remark here that the continuity of $f$ and the recursions $p_{n+1}^+ = f(p_n^+)$ and $p_{n+1}^+ = f(f(p_n^+))$ imply that $p^+, p^-$ satisfy the equalities

$$p^+ = f(p^-) \quad \text{and} \quad p^- = f(p^+), \quad \text{(137)}$$

$$p^+ = f(f(p^+)) \quad \text{and} \quad p^- = f(f(p^-)). \quad \text{(138)}$$

As a consequence of the existence of the limits in (136), we can conclude that the condition $\limsup_{n \to \infty} |p_{n+1} - p_n| = 0$ is equivalent to

$$p^+ = p^- \quad \text{(139)}$$

We are now ready to show the equivalence in the lemma. For the first part, assume that $|f'(x)| < 1$. To show that uniqueness holds on $T_{k,\Delta}$, it suffices to show that (139) holds, i.e., $p^+ = p^-$. We have that $p^+, p^-$ satisfy (138). From the second part of Lemma 33 and the assumption $|f'(x)| < 1$, we thus obtain that $p^+ = x = p^-$, as wanted.

For the second part, it suffices to show the contrapositive. So, assume that uniqueness holds on $T_{k,\Delta}$, we will show that $|f'(x)| \leq 1$ where recall that $x$ is specified by the relation $x = f(x)$ (cf. the second part of Lemma 33). From Lemma 32 we have that $\limsup_{n \to \infty} |p_{n+1} - p_n| = 0$ and hence by our previous arguments, we obtain that (139) also holds. From (137) and using the uniqueness of $x$ (cf. Lemma 33), we obtain that the common value of $p^+$ and $p^-$ is $x$ and thus $p_n \to x$.

By the Mean Value theorem we have that there exists $\xi_n$ between $p_{n+1}$ and $p_n$ such that $|f'(\xi_n)| = |p_{n+1} - p_n|/|p_n - p_{n-1}|$. From $p_n \to x$, we also have that $\xi_n \to x$. Since $p_n \to x$, we have that for infinitely many $n$ it holds that $|p_{n+1} - p_n| \leq |p_n - p_{n-1}|$. For all such $n$, it holds that $|f'(\xi_n)| \leq 1$. Using that $f'$ is continuous and $\xi_n \to x$, we obtain that $|f'(x)| \leq 1$ as desired.

This concludes the proof of Lemma 34. $\square$

The following lemma establishes the intuitive fact that, for the independent set model, uniqueness on $T_{k,\Delta}$ is a monotone property with respect to $\Delta$.

Lemma 35. Let $k \geq 2$ be an integer. There exists $\Delta_c(k) \geq 3$ such that the following holds for all integer $\Delta \geq 2$. The independent set model has uniqueness on $T_{k,\Delta}$ whenever $\Delta < \Delta_c(k)$ and non-uniqueness whenever $\Delta > \Delta_c(k)$.

Proof. Fix an integer $k \geq 2$. For integer $\Delta \geq 2$, parameterise the function $f(z)$ in (123) by $\Delta$, i.e., set

$$f_{\Delta}(z) = \frac{(1 - z^{k-1})\Delta^{-1}}{1 + (1 - z^{k-1})\Delta^{-1}}.$$
Let \(x_\Delta\) be the unique solution of \(z = f_\Delta(z)\) (cf. the second part of Lemma 33). Recall also from Lemma 33 that

\[
|f'_\Delta(x_\Delta)| = (\Delta - 1)(k - 1)h(x_\Delta), \quad \text{where} \quad h(z) := \frac{z^{k-1}(1-z)}{1-z^{k-1}}.
\]  

(140)

We will show that \(|f'_\Delta(x_\Delta)|\) is a (strictly) increasing function of \(\Delta\), i.e., \(|f'_{\Delta+1}(x_{\Delta+1})| > |f'_\Delta(x_\Delta)|\). This follows immediately by multiplying the following inequalities:

\[
\Delta(x_{\Delta+1})^{k-1} > (\Delta - 1)(x_\Delta)^{k-1},
\]

(141)

\[
\left(\frac{x_\Delta}{x_\Delta - 1}\right)^{k-1} \geq \left(\frac{x_{\Delta+1}}{x_{\Delta+1} - 1}\right)^{k-1}.
\]

(142)

Thus, to show the desired monotonicity, we only need to argue for the validity of (141) and (142). We will need the following simple fact:

for all \(z \in (0, 1)\) and all \(\Delta \geq 2\), it holds that \(f_{\Delta+1}(z) < f_\Delta(z)\).

(143)

To see (143), fix \(z \in (0, 1)\) and let \(\Delta \geq 2\). Then we have

\[
\frac{f_{\Delta+1}(z)}{1 - f_{\Delta+1}(z)} = (1 - z^{k-1})^{\Delta} < (1 - z^{k-1})^{\Delta-1} = \frac{f_\Delta(z)}{1 - f_\Delta(z)},
\]

and thus (143) follows.

We next proceed to showing (141) and (142). The crucial step will be to show that \(x_\Delta\) is a decreasing function of \(\Delta\), i.e., \(x_{\Delta+1} < x_\Delta\). Suppose for the sake of contradiction that \(x_\Delta \leq x_{\Delta+1}\) for some \(\Delta\). Using (143) for \(z = x_\Delta\) and the fact that \(f_{\Delta+1}(z)\) is decreasing (by Lemma 33), we have

\[
x_\Delta = f_\Delta(x_\Delta) > f_{\Delta+1}(x_\Delta) \geq f_{\Delta+1}(x_{\Delta+1}) = x_{\Delta+1},
\]

contradiction. It thus follows that \(x_\Delta > x_{\Delta+1}\). From this, it is simple to conclude (142):

\[
\left(\frac{x_\Delta}{x_\Delta - 1}\right)^{k-1} = \sum_{j=0}^{k-2} (x_\Delta)^j \geq \sum_{j=0}^{k-2} (x_{\Delta+1})^j = \frac{(x_{\Delta+1})^{k-1} - 1}{x_{\Delta+1} - 1}.
\]

We next show the slightly harder (141). From \(x_\Delta > x_{\Delta+1}\) and the facts \(x_\Delta = f_\Delta(x_\Delta)\) and \(x_{\Delta+1} = f_{\Delta+1}(x_{\Delta+1})\), it follows that \(f_\Delta(x_\Delta) > f_{\Delta+1}(x_{\Delta+1})\). This in turn yields

\[
(1 - x_{\Delta+1})^{\Delta-1} = \frac{f_\Delta(x_\Delta)}{1 - f_\Delta(x_\Delta)} > \frac{f_\Delta(x_{\Delta+1})}{1 - f_{\Delta+1}(x_{\Delta+1})} = (1 - x_{\Delta+1})^{\Delta}.
\]

Using Bernoulli’s inequality, we thus obtain

\[
1 - x_{\Delta+1}^k > (1 - x_{\Delta+1})^{\Delta-1} = 1 - \frac{\Delta}{\Delta - 1} x_{\Delta+1}^{k-1},
\]

and, by rearranging, we obtain (141).

Since \(|f'_\Delta(x_\Delta)|\) is increasing for \(\Delta \geq 2\), to complete the proof of the lemma, it suffices to show that for \(\Delta = 2\) it holds that \(|f'_\Delta(x_\Delta)| < 1\) and that for some \(\Delta\) it holds that \(|f'_\Delta(x_\Delta)| > 1\).
Note that for all \( z \in (0, 1) \), we have \((k-1)z^k + 1 > k(z^{k-1})\) (the function \((k-1)z^k + 1 - k(z^{k-1})\) is decreasing for \( z \in [0, 1] \) and it has a zero at \( z = 1 \). Rearranging, we obtain that \( h(z) < 1/(k-1) \), so that for \( \Delta = 2 \) we have from (140) that \( |f'_\Delta(x)\Delta| < 1 \) as needed.

For large \( \Delta \), since \( x_\Delta \) is decreasing and bounded, we have that \( x_\Delta \) converges. From \( x_\Delta = f_\Delta(x_\Delta) \), we thus obtain that \( x_\Delta \downarrow 0 \) as \( \Delta \to \infty \). We claim also that \( \Delta(x_\Delta)^{k-1} \to \infty \) as \( \Delta \to \infty \), from which it clearly follows that \( |f'_\Delta(x_\Delta)| \to \infty \) (cf. (140)) and hence \( |f'_\Delta(x_\Delta)| > 1 \) for large \( \Delta \). To see the claimed limit, assume for the sake of contradiction that there was \( M > 0 \) such that \( \Delta(x_\Delta)^{k-1} < M \) for infinitely many \( \Delta \). For all such \( \Delta \), it holds that

\[
x_\Delta = f_\Delta(x_\Delta) = \frac{(1 - x_\Delta^{k-1})^{\Delta-1}}{1 + (1 - x_\Delta^{k-1})^{\Delta-1}} \geq \frac{1}{2}(1 - x_\Delta^{k-1})^{\Delta-1} \geq \frac{1}{2} \left(1 - \frac{M}{\Delta}\right)^{\Delta-1}.
\]

It follows that \( \limsup_{\Delta \to \infty} x_\Delta \geq \frac{1}{2} \exp(-M) > 0 \), contradicting that \( x_\Delta \downarrow 0 \) as \( \Delta \to \infty \).

This completes the proof of Lemma 35. \( \square \)

Corollary 36. Let \( k = 6 \). The independent set model has uniqueness on \( \mathbb{T}_{k, \Delta} \) iff \( \Delta \leq 28 \).

Proof. We will show that for \( \Delta = 28 \), the independent set model has uniqueness on \( \mathbb{T}_{k, \Delta} \) and non-uniqueness for \( \Delta = 29 \). The result then follows from Lemma 35.

For \( \Delta = 28 \), we have that \( |f'(x)| < 0.996 \) where \( x, f(x) \) are as in Lemma 33. For \( \Delta = 29 \), we have that \( |f'(x)| > 1.01 \). Lemma 34 thus shows that the independent set model has uniqueness for \( \Delta = 28 \) and does not have uniqueness for \( \Delta = 29 \), as needed. \( \square \)

The following lemma asserts that, asymptotically in \( k \), the critical value \( \Delta_c(k) \) in Lemma 35 satisfies \( \Delta_c(k) = (1 + o_k(1))2^k/k \). The precise statement is as follows.

Lemma 37. Let \( \Delta_c(k) \) be as in Lemma 35. For all \( \varepsilon > 0 \), there exists an integer \( k_0(\varepsilon) > 0 \) such that the following holds. For all integer \( k \geq k_0(\varepsilon) \),

\[
(1 - \varepsilon)2^k/k \leq \Delta_c(k) \leq (1 + \varepsilon)2^k/k.
\]

Proof. Let \( \varepsilon > 0 \) and \( k_0(\varepsilon) \geq 2 \) be a large constant (depending only on \( \varepsilon \)). We will henceforth assume that \( k \) is an integer satisfying \( k \geq k_0(\varepsilon) \).

Let \( \Delta^+_k := \lceil (1 + \varepsilon)2^k/k \rceil \) and \( \Delta^-_k := \lfloor (1 - \varepsilon)2^k/k \rfloor \). Note that for all sufficiently large \( k \) it holds that \( \Delta^+_k \geq 3 \). Let \( f_k,\pm(z) \) be the function \( f(z) \) in (123) when \( \Delta = \Delta^\pm_k \), i.e.,

\[
f_k,+(z) = \frac{(1 - z^{k-1})\Delta^+_k - 1}{1 + (1 - z^{k-1})\Delta^+_k - 1} \quad \text{and} \quad f_k,-(z) = \frac{(1 - z^{k-1})\Delta^-_k - 1}{1 + (1 - z^{k-1})\Delta^-_k - 1}.
\]

Further, let \( x^\pm_k \) be the unique solution of \( x = f_k,\pm(x) \). We will show that for all sufficiently large \( k \), it holds that

\[
|f'_{k,-}(x^-_k)| < 1 \quad \text{and} \quad |f'_{k,+}(x^+_k)| > 1. \tag{144}
\]

These, together with Lemmas 34 and 35, (144) yields that \( \Delta^-_k \leq \Delta_c(k) \leq \Delta^+_k \), as desired.

The key to showing (144) is that \( x^\pm_k \to 1/2 \) as \( k \to \infty \). Assuming this for the moment, let us conclude (144). Recall from Lemma 33 that

\[
|f'_{k,-}(x^-_k)| = \frac{(\Delta^-_k - 1)(k-1)(x^-_k)^{k-1}(1-x^-_k)}{1-(x^-_k)^{k-1}}, \quad |f'_{k,+}(x^+_k)| = \frac{(\Delta^+_k - 1)(k-1)(x^+_k)^{k-1}(1-x^+_k)}{1-(x^+_k)^{k-1}}.
\]

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Now just using that \( x_k^\pm \to 1/2 \), we obtain that for \( k \to \infty \) it holds that
\[
|f_{k,-}(x_k^-)| \to 1 - \varepsilon, \quad |f_{k,+}(x_k^+)| \to 1 + \varepsilon.
\]
This shows that (144) holds for all sufficiently large \( k \).

We next show that \( x_k^\pm \to 1/2 \) as \( k \to \infty \). From \( x_k^\pm = f_{k,\pm}(x_k^\pm) \), we obtain that
\[
\frac{x_k^\pm}{1 - x_k^\pm} = \left(1 - (x_k^\pm)^{-k-1}\right)^{-1},
\]
from which it clearly follows that \( x_k^\pm \leq 1/2 \). Thus, it suffices to show that for every \( \varepsilon' > 0 \), for all sufficiently large \( k \), it holds that \( x_k^\pm > (1/2) - \varepsilon' \).

From (145), by taking logarithms and then the \( k \)-th root, we obtain
\[
(x_k^\pm - 1)^{1/k} = g(x_k^\pm)^{1/k}, \quad \text{with } g(z) := \frac{\ln \left(\frac{z}{1-z}\right)}{\ln \left(1-z^{k-1}\right)}.
\]
We will use that the function \( g(z) \) is decreasing for \( z \in (0, 1/2) \), since
\[
g'(z) = \frac{(k-1)z^{k-2}\ln \left(\frac{z}{1-z}\right)}{(1-z^{k-1})\ln^2(1-z^{k-1})} + \frac{\left(\frac{z}{1-z} + \frac{1}{1-z}\right)(1-z)}{z \ln(1-z^{k-1})} \leq 0.
\]
We will also use that for any \( z \in (0, 1/2) \) it holds that
\[
\lim_{k \to \infty} (g(z))^{1/k} = 1/z, \quad \text{since } \lim_{k \to \infty} \left(-\ln \left(\frac{z}{1-z}\right)^{1/k}\right) \to 1 \quad \text{and } \quad \lim_{k \to \infty} \left(-\ln(1-z^{k-1})\right)^{1/k} \to z.
\]

Now, for the sake of contradiction, assume that there was \( \varepsilon' > 0 \) such that \( x_k \leq (1/2) - \varepsilon' \) for infinitely many \( k \). For all such \( k \), we would have that \( g(x_k^\pm) \geq g((1/2) - \varepsilon') \) and thus, by taking the \( \lim \sup \) in (146), we obtain
\[
2 = \limsup_{k \to \infty} (x_k^\pm - 1)^{1/k} = \limsup_{k \to \infty} (g(x_k^\pm))^{1/k} \geq \limsup_{k \to \infty} (g((1/2) - \varepsilon'))^{1/k} = \frac{1}{2 - \varepsilon'} > 2,
\]
contradiction. This proves that \( x_k^\pm \to 1/2 \) as \( k \to \infty \), thus concluding the proof of Lemma 37.

\[\square\]

References


10 Computer Assisted Proofs

The code in this section can be executed by copying it in a Mathematica cell.

10.1 Mathematica Code for Proof of Lemma 14

(** Verification of (65) **)  
\[
\alpha = 1 - 10^{-4}; \Delta = 6; d_0 = \Delta - 1; 
\varepsilon[0]=0; \varepsilon[1]=6/10; \varepsilon[2]=7/10; \varepsilon[3]=83/100; \varepsilon[4]=91/100; \varepsilon[5]=\alpha; 
M = 25/1000; 
\]

For[BB = 0, BB<=d0, BB++,  
  Print["Running for BB=", BB]; 
  Print[Resolve[1/\alpha (\varepsilon[BB] + (d_0 - BB) M) > 1]]; 
];

10.2 Mathematica Code for Lemma 16

(** Verification of (109) **)  
\[
M_1 = 1/410; 
h = (1-t) (\psi - (t/(1-t))^\chi)/.{\chi \rightarrow 1/2, \psi \rightarrow 13/10}; 
f = t^6/(1 - t^6) h; 
\]

Resolve[Exists[t, f > M_1 && 0 <= t <= 1/2]]

(** Verification of (111) **)  
\[
f = (t^{(w - 6)} (1 - t^6))/(1 - t^w); 
p = 6 t^w - t^6 w + w - 6; 
\]

Simplify[D[f, t] - (t^{(w - 7)} p)/(1 - t^w)^2]

(** Verification of (114) **)  
\[
M = 25/1000; \ M_1 = 1/410; \Delta = 6; \delta = 9789/10000; 
K_w = (1/\delta)^((w - 1) (\Delta - 1)); 
f = (63 M_1/(2^w - 1)) \times K_w \times w \times (w + 1)^{-1}(11*10^{-5}/\log[6]); 
f_1 = D[f, w]; 
\]

Resolve[(f /. \{w \rightarrow 6\}) > M]  
Resolve[Exists[w, f_1 > 0 && w >= 6]]
10.3 Mathematica Code for Proof of Lemma 18

(** Verification of (70) **) 

\[ h = (1-t) \left( \psi - \left( \frac{t}{(1-t)} \right)^{\chi} \right) / \{ \chi \rightarrow 1/2, \psi \rightarrow 13/10 \}; \]

For[w = 1, w <= 5, w++,
   Print["Running for w=", w];
   g[w] = (1 - y)/y * (h /. {t -> (1 - y)^(1/w)});
   f[w] = g[w] /. {y -> Exp[z]};
   f2[w] = D[f[w], {z, 2}];
   Print[Resolve[Exists[z, f2[w] > 0 && Log[1 - (1/2)^w] <= z <= 0]]];
];

10.4 Mathematica Code for Proof of Lemma 19

(** Verification of (73) **) 

Delta = 6; delta = 9789/10000; p = 27/2;
Cproof = (1/delta)^(b3-1) * \left( \frac{1 - \delta^{b3 \cdot p}}{b3 \cdot (1 - \delta^p)} \right)^{1/p};
For[bb3 = 1, bb3 <= Delta - 1, bb3++,
   Print["Checking for b3=", bb3];
   Print[Resolve[ (Cproof/.{b3->bb3}) > CC[bb3] ]];
]
### 10.5 Mathematica code for Proof of Lemma 20

```mathematica
SUBS1 = {(v1/(1 - v1))^\[Chi] -> (2 z1)/(1 - z1^2),
          (1 - v1)^b2 \[Chi] -> ((1 - z1^2)/(1 + z1^2)) b2,
          v1 -> (4 z1^2)/(1 + z1^2)^2};

SUBS2 = {(v2/(1 - v2))^\[Chi] -> (2 z2)/(1 - 2 z2^2),
          (1 - v2^2)^b3 \[Chi] -> ((1 - 4 z2^4)/(1 + 4 z2^4))^b3,
          v2 -> (4 z2^2)/(1 + 4 z2^4)};

SUBS = Join[SUBS1, SUBS2];

kappa = delta^b2 * (b2 v1 (psi - (v1/(1 - v1))^\[Chi])
                 + 2 b3 K2 CC[b3]((v2^2)/(1 + v2))(psi - (v2/(1 - v2))^\[Chi]))/
                 (psi - (1 - v1)^(b2 \[Chi]) (1 - v2^2)^b3 \[Chi])/. SUBS

delta0 = 9789/10000; psi0 = 13/10; KK2 = 1069/1000; d0 = 5; alpha = 1 - 10^(-4);
tau[0, 0] = 0; tau[0, 1] = 42/100; tau[1, 0] = 42/100;
tau[0, 2] = 54/100; tau[1, 1] = 59/100; tau[2, 0] = 63/100;
tau[0, 3] = 72/100; tau[1, 2] = 74/100; tau[2, 1] = 76/100; tau[3, 0] = 79/100;
tau[0, 4] = 864/1000; tau[1, 3] = 868/1000; tau[2, 2] = 876/1000;
tau[3, 1] = 886/1000; tau[4, 0] = 901/1000;

For[bb2 = 0, bb2 <= d0, bb2++,
    For[bb3 = 0, bb3 <= d0 - bb2, bb3++,
        Print["b2 =", bb2, ", b3 =", bb3];
        EXP = kappa /. {b2 -> bb2, b3 -> bb3};
        Print[Resolve[Exists[{z1, z2},
                              EXP > tau[bb2, bb3] && 0 <= z1 <= z1up && 0 <= z2 <= z2up]]];
    ];
];
```

10.6 Mathematica Code for Proof of Lemma 21

(** Verification of (78) **) 

\[
\kappa = (\tau_{b2, b3} (\psi - A) + 3 \cdot \delta^{b2} \cdot K3 \cdot b4 \cdot (v3^3)/(1+v3+v3^2) \cdot (\psi - (v3/(1-v3))^\chi))/ (\psi - A(v3^3) \cdot (b4^\chi))
\]

\[
\text{Delta} = 6; \ \text{dd0} = \text{Delta} - 1; \ \text{delta0} = 9789/10000; \ \text{psi0} = 13/10; \ \text{chi0} = 1/2; \ \text{alpha} = 1 - 10^{-(4)};
\]

\[
\tau[0, 0] = 0; \ \tau[0, 1] = 42/100; \ \tau[1, 0] = 42/100;
\]

\[
\tau[0, 2] = 54/100; \ \tau[1, 1] = 59/100; \ \tau[2, 0] = 63/100;
\]

\[
\tau[0, 3] = 72/100; \ \tau[1, 2] = 74/100; \ \tau[2, 1] = 76/100; \ \tau[3, 0] = 79/100;
\]

\[
\tau[0, 4] = 864/1000; \ \tau[1, 3] = 868/1000; \ \tau[2, 2] = 876/1000;
\]

\[
\tau[3, 1] = 886/1000; \ \tau[4, 0] = 901/1000;
\]

\[
\tau[5, 0] = \alpha;
\]

\[
\kappa = \kappa /. \{\delta \to \delta_0, \ \psi \to \psi_0, \ \chi \to \chi_0, \ K3 \to 1160/1000\}
\]

For \(b2 = 0, b2 \leq \text{dd0}, b2++\),

For \(b3 = 1, b3 \leq \text{dd0} - b2, b3++\),

For \(b4 = 1, b4 \leq \text{dd0} - b2 - b3, b4++\),

Print["Running for b2 =", b2, " b3 =", b3, " and b4 =", b4];

\[
\text{EXP} = \kappa /. \{b2 \to b2, b3 \to b3, b4 \to b4\};
\]

\[
\text{B} = b2 + b3 + b4;
\]

Print[
  \[
  \text{Resolve[Exists[}\{A, v3\}, \]
  
  EXP > \tau[B, 0] & & 0 <= v3 <= 1/2 & & 0 <= A <= 1\]]
];

]

(** Verification of (80) **) 

\[
\text{SUBS} = \{(v1/(1 - v1))^\chi \to u1, v1 \to (u1^2)/(1+u1^2), (v3/(1-v3))^\chi \to u3, v3 \to (u3^3)/(1+u3^2)\};
\]

\[
\kappa = \delta^{b2} \cdot (b2 \cdot v1 \cdot (\psi - (v1/(1 - v1))^\chi) + 3 \cdot K3 \cdot b4 \cdot (v3^3)/(1+v3+v3^2) \cdot (\psi - (v3/(1 - v3))^\chi))/ (\psi - (1-v1)^2 \cdot (b4^\chi)) / . \text{SUBS}
\]

\[
\text{Delta} = 6; \ \text{dd0} = \text{Delta} - 1; \ \text{delta0} = 9789/10000; \ \text{psi0} = 13/10; \ \text{chi0} = 1/2; \ \text{alpha} = 1 - 10^{-(4)};
\]

\[
\tau[1, 0] = 42/100; \ \tau[2, 0] = 63/100; \ \tau[3, 0] = 79/100; \ \tau[4, 0] = 901/1000; \ \tau[5, 0] = \alpha;
\]

\[
\kappa = \kappa /. \{\delta \to \delta_0, \ \psi \to \psi_0, \ \chi \to \chi_0, \ K3 \to 1160/1000\}
\]

For \(b2 = 0, b2 \leq \text{dd0}, b2++\),

For \(b4 = 1, b4 \leq \text{dd0} - b2, b4++\),

Print["Running for b2 =", b2, " b3 =", b3, " and b4 =", b4];

\[
\text{EXP} = \kappa /. \{b2 \to b2, b3 \to b3, b4 \to b4\};
\]

\[
\text{B} = b2 + b4;
\]

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10.7 Mathematica Code for Proof of Lemma 22

The following code can be executed by copying it in a Mathematica cell.

(** Verification of (81) **)  

\[
\kappa = \frac{(\tau_{Bp, 0} (\psi-A) + 4 \times K4 \times b5 \times (v4^4)/(1+v4+v4^2+v4^3)\times(\psi-(v4/(1-v4))^\chi))}{(\psi - A(1-v4^4)^{(b5\times\chi)})}
\]

Delta=6; dd0=Delta-1; delta0=9789/10000; psi0=13/10; chi0=1/2; alpha=\(1-10^{-4}\);
tau[0,0]=0; tau[1,0]=42/100; tau[2,0]=63/100; tau[3,0]=79/100; tau[4,0]=901/1000; tau[5,0]=alpha;

\[
\kappa = \kappa /. \{\delta -> \delta0, \psi -> \psi0, \chi -> \chi0, K4 -> 1225/1000\}
\]

For[BBp = 0, BBp <= dd0, BBp++,
    For[bb5 = 1, bb5 <= dd0 - BBp, bb5++,
        Print["Running for Bp=", BBp, " and b5=", bb5];
        EXP = kappa /. \{Bp -> BBp, b5 -> bb5\};
        B = BBp + bb5;
        Print[
            Resolve[Exists[\{A, v4\},
                EXP > \tau[B, 0] && 0 <= v4 <= 1/2 && 0 <= A <= 1]]
            ];
        ];
    ];
**Mathematica Code for Proof of Lemma 23**

(\textbf{** Verification of (82) **})

\[
kappa = \frac{\tau_{0,0} (\psi - A) + 5 \times K_5 \times b_6 \times (v^5)/(1 + v^5 + v^5^2 + v^5^3 + v^5^4) \times ((v^5/(1-v^5))^\chi))}{(\psi - A(1-v^5)^{b_6 \times \chi})}
\]

\[
\text{Delta}=6; \text{dd0}=\text{Delta}-1; \text{delta0}=9789/10000; \text{psi0}=13/10; \text{chi0}=1/2; \alpha=1-10^{-(-4)}; \\
\tau[0,0]=0; \tau[1,0]=42/100; \tau[2,0]=63/100; \tau[3,0]=79/100; \tau[4,0]=901/1000; \tau[5,0]=\alpha; \\
\text{Resolve[(1/delta0)^4*dd0>1532/1000]}
\]

\[
\text{kappa} = \text{kappa}/.\{\delta->\delta0, \psi->\psi0, \chi->\chi0, K5->1532/1000\}
\]

\text{For[BBp = 0, BBp <= dd0, BBp++,}
\text{For[bb6 = 1, bb6 <= dd0-BBp, bb6++,}
\text{Print["Running for Bp", BBp, " and b6", bb6];}
\text{EXP = kappa /. \{Bp -> BBp, b6 -> bb6\};}
\text{B = BBp + bb6;}
\text{Print[}
\text{Resolve[Exists\{A, v5\},}
\text{EXP > tau[B, 0] \&\& 0 <= v5 <= 1/2 \&\& 0 <= A <= 1\}]
\text{];}
\text{]};
\text{]};
10.9 Mathematica Code for Proof of Lemma 24

(** Verification of (83) **) 

\[ f = (1 - t) (\psi - (t/(1 - t))^\chi) /. \{\psi \rightarrow 13/10, \chi \rightarrow 1/2, t \rightarrow \text{Exp}[y]\} \]

\[ f2 = \text{FullSimplify[D}[f, \{y, 2\}] \]

\[ \text{Resolve}[\exists y, f2 > 0 \&\& y \leq \text{Log}[1/2]] \]

10.10 Mathematica Code for Proof of Lemma 25

(** Verification of (98) and (99) **) 

\[ \psi = 13/10; \delta = 9789/10000; K2 = 1069/1000; K4 = 1225/1000; \]

\[ \text{AA}[1] = \{A1 \rightarrow 1, A2 \rightarrow 1/\delta^5, A \rightarrow 2K2\}; \]

\[ \text{AA}[2] = \{A1 \rightarrow 2, A2 \rightarrow 2K2/\delta^5, A \rightarrow 4*1120/1000\}; \]

\[ \text{AA}[3] = \{A1 \rightarrow 1, A2 \rightarrow 1/\delta^15, A \rightarrow 5/2\}; \]

\[ \text{AA}[4] = \{A1 \rightarrow 2K2/\delta^5, A2 \rightarrow 5/2, A \rightarrow 4K4\}; \]

\[ \text{LHS} = (4 x1 x2)/(((1 + x1^2) (1 + x2^2) - 4 x1 x2); \]

\[ \text{RHS} = \psi - (A1*(1 - x1^2)^2/(1 + x1^2)^2 (\psi - (2 x1)/(1 - x1^2)) + \\
A2*(1 - x2^2)^2/(1 + x2^2)^2 (\psi - (2 x2)/(1 - x2^2)))/ \\
(A (1 - (4 x1 x2))/((1 + x1^2) (1 + x2^2))) \]

\[\text{EXPR} = \text{RHS}^2 - \text{LHS} \]

\[ xup = \sqrt{2} - 1; \]

\[ \text{For}[i=1, i<= 4, i++, \]

\[ \text{Print["Running for A1=", A1/.\text{AA}[i], " A2=", A2/.\text{AA}[i], " A=", A/.\text{AA}[i]]; \]

\[ \text{RHS0} = \text{RHS}/.\text{AA}[i]; \text{EXPR0} = \text{EXPR} /. \text{AA}[i]; \]

\[ \text{Print[\text{Resolve[\exists\{x1, x2\}, RHS0 < 0 \&\& 0 <= x1 <= xup \&\& 0 <= x2 <= xup\]}];} \]

\[ \text{Print[\text{Resolve[\exists\{x1, x2\}, EXPR0 < 0 \&\& 0 <= x1 <= xup \&\& 0 <= x2 <= xup\}]}]; \]

10.11 Mathematica Code for Proof of Lemma 26

(** Verification of (116) **) 

\[ q = 27/25; \text{Delta} = 6; \]

\[ h = (1-t) (\psi - (t/(1 - t))^\chi) /. \{\psi \rightarrow 13/10, \chi \rightarrow 1/2, t \rightarrow (1-y)^{(1/2)}\}; \]

\[ g2 = ((1-y)/y)^{\chi}; \]

\[ \text{FUN} = \text{Simplify}[(q - 1) y*(D[g2, y])^2 + g2 * ((D[g2, y]) + y*(D[g2, \{y, 2\}])]; \]

\[ \text{Resolve}[\exists y, \text{FUN} > 0 \&\& 3/4 <= y <= 1 - 1/(2^\text{Delta-1} + 1)^2)]