Random Cluster Dynamics at $q = 2$ is Rapidly Mixing

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The random cluster model [Fortuin, Kasteleyn 1969]

Parameters $0 \leq p \leq 1$ (edge weight), $q \geq 0$ (cluster weight).

Given graph $G = (V, E)$, the measure on subgraph $r \subseteq E$ is defined as

$$\pi_{RC}(r) \propto p^{|r|} (1 - p)^{|E \setminus r|} q^{\kappa(r)},$$

where $\kappa(r)$ is the number of connected components in $(V, r)$.

$$(1 - p)^4 q^4 \quad p^2 (1 - p)^2 q^2 \quad p^4 q$$
The partition function is defined as

\[ Z_{RC}(p, q) = \sum_{r \subseteq E} p^{|r|} (1 - p)^{|E \setminus r|} q^{\kappa(r)}. \]

Equivalent to the Tutte polynomial \( Z_{Tutte}(x, y) \):

\[ q = (x - 1)(y - 1) \quad \quad p = 1 - \frac{1}{y} \]
The random cluster model \cite{Fortuin, Kasteleyn 1969}

\[ \pi_{RC}(r) \propto p^{|r|} (1 - p)^{|E\setminus r|} q^{|K(r)|} \]

The motivation is to unify:

- Ising model
- Potts model
- Bond percolation
- Electrical network
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- **Electrical network**
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Glauber Dynamics

Glauber Dynamics (single edge update) $P_{RC}$ (Metropolis):

Current state $x \subseteq E$

1. With prob. $1/2$ do nothing. (Lazy)

2. Otherwise, choose an edge $e$ u.a.r.

3. Move to $y = x \oplus \{e\}$ with prob. $\min \left\{ 1, \frac{\pi_{RC}(y)}{\pi_{RC}(x)} \right\}$.

Detailed balance:

$$\pi(x)P(x, y) = \pi(y)P(x, y) = \min\{\pi(x), \pi(y)\}$$
Glauber Dynamics

Glauber Dynamics (single edge update) \( P_{RC} \) (Metropolis):

\[
P_{RC}(x, y) = \begin{cases} 
\frac{1}{2m} \min \left\{ 1, \frac{\pi_{RC}(y)}{\pi_{RC}(x)} \right\} & \text{if } |x \oplus y| = 1; \\
1 - \frac{1}{2m} \sum_{e \in E} \min \left\{ 1, \frac{\pi_{RC}(x \oplus \{e\})}{\pi_{RC}(x)} \right\} & \text{if } x = y; \\
0 & \text{otherwise.}
\end{cases}
\]

We are interested in the mixing time \( \tau_\varepsilon(P_{RC}) \):

\[
\tau_\varepsilon(P_{RC}) = \min \left\{ t : \|P_{RC}^t(x_0, \cdot) - \pi\|_{TV} \leq \varepsilon \right\}.
\]
A Simple Example

Let $p < 1/2$.

\[
\min \left\{ 1, \frac{\pi_{RC}(x \cup \{e\})}{\pi_{RC}(x)} \right\} = \begin{cases} 
\frac{p}{1-p} & \text{if } e \text{ is not a cut edge} \\
\frac{p}{q(1-p)} & \text{if } e \text{ is a cut edge}
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Brief History

Studied extensively for special graphs, such as the complete graph (mean-field) and the lattice $\mathbb{Z}^2$.

- **Mean-field**: [Blanca, Sinclair 2015]
  - Swendsen-Wang algorithm:
    - [Gore, Jerrum 1999]
    - [Long, Nachimus, Ning, Peres 2011]
    - [Galanis, Štefankovič, Vigoda, 2015]

- $\mathbb{Z}^2$: [Duminil-Copin, Sidoravicius, Tassion 2015]
  - [Blanca, Sinclair 2016]

$q > 2$: Slow mixing for the complete graph.

$0 \leq q \leq 2$: No known fast mixing bound for general graphs.
Main theorem

Theorem

For the random cluster model with parameters $0 < p < 1$ and $q = 2$,

$$\tau_\epsilon (P_{RC}) \leq 10n^4 m^2 (\ln \pi_{RC}(x_0)^{-1} + \ln \epsilon^{-1}).$$

- For $q > 2$, there exists $p$ such that $P_{RC}$ is slow mixing on complete graphs. [Gore, Jerrum 1999] [Blanca, Sinclair 2015]

- For $q > 2$ and $0 < p < 1$, it is #BIS-hard to approximate $Z_{RC}(p, q)$. [Goldberg, Jerrum 2012]

- For $0 \leq q < 2$, there is no known obstacle.
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The Proof
Equivalent formulations at $q = 2$

The Ising model

Let $\sigma : V \rightarrow \{+, -\}$.

$$\pi^{\text{Ising}}(\sigma) \propto \beta^{\text{mono}(\sigma)} = \beta^{\text{m-cut}(\sigma)}$$

Partition function $Z^{\text{Ising}}(\beta)$

\[
\begin{align*}
\beta^0 & \quad \beta^2 & \quad \beta^4
\end{align*}
\]
Equivalent formulations at \( q = 2 \)

Even subgraphs

Let \( r \subseteq E \) such that every vertex in \( (V, r) \) has an even degree.

\[
\pi_{\text{even}}(r) \propto p^{|r|} (1 - p)^{|E \setminus r|}
\]

Partition function \( Z_{\text{even}}(p) \)

\[
(1 - p)^4 \quad \text{NOT EVEN} \quad p^4
\]
Equivalent formulations at $q = 2$

Let $\beta = \frac{1}{1-p}$.

\[
Z_{\text{Ising}}(\beta) = \beta^{|E|} Z_{\text{RC}}(p, 2) = 2^{|V|} \beta^{|E|} Z_{\text{even}} \left( \frac{p}{2} \right)
\]
Equivalent formulations at $q = 2$

Random-cluster $(p, 2)$

Even subgraphs $p/2$

Ising model $\beta = (1 - p)^{-1}$

[Fortuin, Kasteleyn 1969]

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FPRAS
[Jerrum, Sinclair 93]
This talk
Heng Guo (QMUL)
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Slow mixing
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FPRAS [Jerrum, Sinclair 93]

Even subgraphs $p/2$

[Grimmett, Janson 2009]
Grimmett-Janson coupling

Given a graph $G$, draw a random even subgraph $S \subseteq E$ with $p \leq \frac{1}{2}$:

$$\Pr(S = s) = \pi_{\text{even}}(s).$$

Then we add every edge $e \notin S$ with probability $p' = \frac{p}{1-p}$.

Call this subgraph $R$.

Theorem (Grimmett, Janson 2009)

$$\Pr(R = r) = \pi_{RC;2p,2}(r).$$
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**Theorem (Grimmett, Janson 2009)**

$$\Pr(R = r) = \pi_{\text{RC};2p,2}(r).$$
Proof of Grimmett-Janson

For each $r \subseteq E$,

$$\Pr(R = r) \propto \sum_{s \subseteq r, s \text{ even}} p^{|s|} (1 - p)^{|E \setminus s|} \left( \frac{p}{1 - p} \right)^{|r \setminus s|} \left( 1 - \frac{p}{1 - p} \right)^{|E \setminus r|}$$

$$\propto p^{|r|} (1 - 2p)^{|E \setminus r|} N(r),$$

where $N(r) = 2^{|r| - |V| + \kappa(r)}$ is the number of even subgraphs of $(V, r)$.

Hence,

$$\Pr(R = r) \propto (2p)^{|r|} (1 - 2p)^{|E \setminus r|} 2^{\kappa(r)}.$$
A Markov chain is a random walk on its state graph.

Construct a set $\Gamma$ of canonical paths $\gamma_{xy}$ for all pairs of states $(x, y)$.

The key quantity of $\Gamma$ is its congestion:

$$\rho(\Gamma) := \max_{(z, z') \in \Omega^2, P(z, z') > 0} \frac{L}{\pi(z) P(z, z')} \sum_{x, y \in \Omega^2, \gamma_{xy} \ni (z, z')} w(\gamma_{xy}),$$

where

$$w(\gamma_{xy}) = \pi(x) \pi(y).$$

**Theorem (Sinclair 1992)**

$$\tau_\epsilon(P) \leq \rho(\Gamma)(\ln \pi(x_0)^{-1} + \ln \epsilon^{-1}).$$
Fix $\Gamma = \{\gamma_{xy}\}$ and an integer $k \leq L$.

1. Draw the initial and final states $I$ and $F$ independently according to $\pi(\cdot)$.
2. A random path $\gamma_{IF} \in \Gamma$.

$$
\mu(\gamma_{IF}) = w(\gamma_{IF}) = \pi(I)\pi(F)
$$

3. Let $Z_k$ be the $k$th state of $\gamma_{IF}$.

(Assume all paths in $\Gamma$ have the same length $L$.)

The congestion $\rho(\Gamma)$ is polynomial related with $\max_k \frac{\Pr(Z_k = z)}{\pi(z)}$. 
Let $q = 1$. Then $\pi_{RC}(\cdot)$ is a product measure.
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$$\Pr(Z_k = z) = 1$$
From paths to flows

Instead of one path from \( x \) to \( y \), we can have a random path from \( x \) to \( y \).

Flow \( \Gamma \) is a collection of paths equipped with weights \( w(\cdot) \) such that

\[
\sum_{\gamma \text{ is from } x \text{ to } y} w(\gamma) = \pi(x)\pi(y).
\]

\( Z_k \) is defined similarly.

1. Random initial and final states \( I \) and \( F \)
2. A random path \( \gamma \) from \( I \) to \( F \) according to \( w(\cdot) \).
3. \( Z_k \) is the \( k \)th state of \( \gamma \).

We will look at \( \frac{\Pr(Z_k = z)}{\pi(z)} \).
Lifting canonical paths

In an **ideal** world . . .

- Suppose we have canonical paths $\Gamma_{\text{even}}$ for **even** subgraphs with low congestion.

- Then use **Grimmett-Janson** to lift $\Gamma_{\text{even}}$ to a flow for **random cluster**.

\[ I = W_0 \longrightarrow W_1 \longrightarrow W_2 \longrightarrow \cdots \longrightarrow W_{L-1} \longrightarrow W_L = F \]

- $w(\zeta) = w(\gamma) \Pr(\gamma \rightarrow \zeta)$
If $W_k$ deviates from $\pi_{\text{even}}(\cdot)$ by at most polynomial, then so does $Z_k$ from $\pi_{\text{RC}}(\cdot)$.

$$\frac{\text{Pr}(W_k = w)}{\pi_{\text{even}}(W)} \leq n^{O(1)} \rho(\Gamma)$$

$$\text{Pr}(Z_k = z) = \sum_{w \subseteq z, w \text{ even}} \text{Pr}(W_k = w) \left( \frac{p}{1 - p} \right)^{|z \setminus w|} \left( \frac{1 - 2p}{1 - p} \right)^{|E \setminus z|}$$

$$\leq n^{O(1)} \rho(\Gamma) \sum_{w \subseteq z, w \text{ even}} \pi_{\text{even}}(w) \left( \frac{p}{1 - p} \right)^{|z \setminus w|} \left( \frac{1 - 2p}{1 - p} \right)^{|E \setminus z|}$$

$$= n^{O(1)} \rho(\Gamma) \pi_{\text{RC}}(Z) \quad \text{(by GJ)}$$
Ideal lifting

If $W_k$ deviates from $\pi_{\text{even}}(\cdot)$ by at most polynomial, then so does $Z_k$ from $\pi_{RC}(\cdot)$.

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$$Pr(Z_k = z) = \sum_{w \subseteq z, w \text{ even}} Pr(W_k = w) \left( \frac{p}{1 - p} \right)^{|z \setminus w|} \left( \frac{1 - 2p}{1 - p} \right)^{|E \setminus z|}$$

$$\leq n^{O(1)} \rho(\Gamma) \sum_{w \subseteq z, w \text{ even}} \pi_{\text{even}}(w) \left( \frac{p}{1 - p} \right)^{|z \setminus w|} \left( \frac{1 - 2p}{1 - p} \right)^{|E \setminus z|}$$

$$= n^{O(1)} \rho(\Gamma) \pi_{RC}(Z)$$

(by GJ)
Two issues:

1. We do not have good canonical paths for even subgraphs — \textbf{Jerrum-Sinclair} chain moves among all subgraphs!

2. \textbf{Grimmett-Janson} adds independent edges — $Z_i$ and $Z_{i+1}$ are not adjacent states! They may differ by a lot of edges.
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2. Grimmett-Janson adds independent edges — $Z_i$ and $Z_{i+1}$ are not adjacent states! They may differ by a lot of edges.
Issue 1: need canonical paths for even subgraphs.

- Construct paths $\Gamma_{\text{even}} = \{\gamma_{xy}\}$ where $x$ and $y$ are both even subgraphs.
  - $x \oplus y$ is also even.
    - $x \oplus y$ can be covered by edge-disjoint cycles.
  - Pick a canonical ordering of edges. Unwind each cycle:
    - $W_0 = x$, $W_i = W_{i-1} \oplus e_i$
  - Enlarge the state space to all even and near-even subgraphs.
    - Every path is in the augmented space.
- $\Gamma_{\text{even}}$ has low congestion — same reason as [Jerrum, Sinclair 1993].
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$Z_1$

$Z_2$

$x \oplus y$

$y = Z_6$

$Z_3$
Issue 1: need canonical paths for even subgraphs.

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$\Gamma_{even}$ has low congestion — combinatorial encoding \cite{Jerrum, Sinclair 1993}.

For any $\gamma_{xy} \ni (z, z')$, let $u = x \oplus y \oplus z$. This mapping is injective.
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\[ \pi(x)\pi(y) = \pi(z)\pi(u) \]
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\[
\sum_{\gamma_{xy} \ni (z, z')} \pi(x) \pi(y) \leq \pi(z) \sum_{u} \pi(u) \leq \pi(z)
\]
Issue 1: need canonical paths for even subgraphs.

Flow lifted from $\Gamma_{\text{even}}$ is valid for $\pi_{RC}(\cdot)$:

- $Z_0$ and $Z_L$ are independently obtained from $W_0$ and $W_L$ with the Grimmett-Janson coupling. ($W_0$ and $W_L$ are both even.)

- Intermediate $W_i$'s can be near-even.

A generalization of Grimmett-Janson:

- Give each near-even subgraph a penalty of $1/n^2$.

- Add independent edges with prob. $\frac{p}{1-p}$ as before. Call the resulting measure $\hat{\pi}(\cdot)$.

- $\frac{\hat{\pi}(x)}{\pi_{RC}(x)} = \Theta(1)$. 

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A generalization of Grimmett-Janson:

- Give each near-even subgraph a penalty of $1/n^2$.

- Add independent edges with prob. $1 - \frac{p}{1-p}$ as before. Call the resulting measure $\hat{\pi}(\cdot)$.

$$\frac{\hat{\pi}(x)}{\pi_{RC}(x)} = \Theta(1).$$
Issue 2: $Z_i$ and $Z_{i+1}$ differ by more than 1 edge.

- An easy fix: insert intermediate states to change edges one by one in $Z_i \oplus Z_{i+1}$, which has a product measure on $E \setminus (W_i \cup W_{i+1})$.

- The distribution of $Z_i^j$ is the same as that of $Z_i$ ($j < m$).

- Total length is $mL$. 
Better patch 2

Issue 2: $Z_i$ and $Z_{i+1}$ differ by more than 1 edge.

- Lift $W_{i+1}$ to $Z_{i+1}$ conditional on $Z_i$ such that $Z_{i+1}$ and $Z_i$ are adjacent and the marginal of $Z_{i+1}$ is correct.
- The marginal distributions of $Z_0$ and $Z_L$ are correct, but their joint distribution is not — $Z_0$ and $Z_L$ are correlated.
- Append a tail on the path after $Z_L$ to re-randomize edges that are not in $W_L$. This removes the correlation.
- Total length is at most $L + m$. 
Putting everything together

\[ Z_0 \quad Z_{L+m} \]
Putting everything together

\[ W_0 = f_0 \]

\[ Z_0 = \mathbb{P}(W_0) \]

\[ W_1 = f_1 \]

\[ Z_1 = \mathbb{P}(W_1) \]

\[ W_2 = f_2 \]

\[ Z_2 = \mathbb{P}(W_2) \]

\[ Z_L \leftrightarrow m \]

\[ W_L \]

Grimmett-Janson

Heng Guo (QMUL)
Putting everything together

\[ Z_0 \]

\[ Z_1 = Z_0 \]

\[ Z_2 = Z_1 \]

\[ W_0 \rightarrow W_1 \rightarrow W_2 \rightarrow \cdots \rightarrow W_L \]

Grimmett-Janson

Grimmett-Janson

Grimmett-Janson

\[ Z_{L+m} \]
Putting everything together

\[ W_1 = W_0 \cup \{e\} \quad \Rightarrow \quad Z_1 = Z_0 \cup \{e\} \]
Putting everything together

$W_1 = W_0 \cup \{e\}$ \quad \Rightarrow \quad $Z_1 = Z_0 \cup \{e\}$

$W_2 = W_1 \setminus \{e'\}$ \quad \Rightarrow \quad $Z_2 = \begin{cases} Z_1 & \text{prob. } p' \\ Z_1 \setminus \{e'\} & \text{prob. } 1 - p' \end{cases}$
Putting everything together

\[ W_0 \rightarrow W_1 \rightarrow W_2 \rightarrow \ldots \rightarrow W_L \]

\[ Z_0 \rightarrow Z_1 \rightarrow Z_2 \rightarrow \ldots \rightarrow Z_L \]

\[ Z_L \rightarrow Z_{L+m} \]

- \( W_1 = W_0 \cup \{e\} \) \quad \Rightarrow \quad \( Z_1 = Z_0 \cup \{e\} \)

- \( W_2 = W_1 \setminus \{e'\} \) \quad \Rightarrow \quad \( Z_2 = \begin{cases} Z_1 & \text{prob. } p' \\ Z_1 \setminus \{e'\} & \text{prob. } 1 - p' \end{cases} \)
Putting everything together

\[ Z_0 \rightarrow Z_1 \rightarrow Z_2 \rightarrow \cdots \rightarrow Z_L \rightarrow Z_{L+m} \]

\[ W_0 \rightarrow W_1 \rightarrow W_2 \rightarrow \cdots \rightarrow W_L \]

\[ W_1 = W_0 \cup \{e\} \Rightarrow Z_1 = Z_0 \cup \{e\} \]

\[ W_2 = W_1 \setminus \{e'\} \Rightarrow Z_2 = \begin{cases} Z_1 & \text{prob. } p' \\ Z_1 \setminus \{e'\} & \text{prob. } 1 - p' \end{cases} \]
The Aftermath
Consequence — Swendsen-Wang algorithm is rapidly mixing

A global Markov chain to sample Ising configurations.

[Swendsen, Wang 1987]

Current configuration $\sigma$

1. Mark all monochromatic edges under $\sigma$ as $M$
2. Remove each edge in $M$ with probability $\beta^{-1}$
3. Assign a random spin to each component of $(V, M)$

Practically very fast for the Ising model, but difficult to analyze.

Open problem since 90s.
Consequence — Swendsen-Wang algorithm is rapidly mixing

\[ \tau_\epsilon (P_{SW}) \leq \tau_\epsilon (P_{RC}) \]

Theorem (Ullrich 2014)

Combine with our theorem:
The Swendsen-Wang algorithm is rapidly mixing at \( q = 2 \), namely, for the ferromagnetic Ising model at any temperature.

- The Swendsen-Wang algorithm is conjectured to have a \( n^{1/4} \) mixing time.
Consequence — Swendsen-Wang algorithm is rapidly mixing

Theorem (Ullrich 2014)

\[ \tau_\epsilon(P_{SW}) \leq \tau_\epsilon(P_{RC}) \]

Combine with our theorem:

The Swendsen-Wang algorithm is rapidly mixing at \( q = 2 \), namely, for the ferromagnetic Ising model at any temperature.

- The Swendsen-Wang algorithm is conjectured to have a \( n^{1/4} \) mixing time.
Tutte polynomial [Goldberg, Jerrum 08,12,14]

\[ q = (x - 1)(y - 1) \]

Tractable
FPRAS

NP-hard
(most \#P-hard)

\#PM-equivalent

\#BIS-hard

Open

\[ 0 \leq q < 1 \]

\[ 1 < q < 2 \]
Theorem

At $q = 2$, $\tau_\epsilon(P_{RC}) \leq 10n^4 m^2 (\ln \pi_{RC}(x_0)^{-1} + \ln \epsilon^{-1})$.

- $q = 2$ tighter mixing time bound?
- $1 < q < 2$ (monotone)
- $0 \leq q < 1$ (e.g. Tutte(2,1) = #Forests)
Theorem

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Thank You!

Paper available: arxiv.org/abs/1605.00139