1 Hamiltonian cycles - Dirac’s theorem

Recall that in extremal graph theory, we would like to answer questions of the following sort: ‘What is the maximum/minimum possible parameter $C$ among graphs satisfying a certain property $P$?’ In the last lecture, we see Mantel’s theorem, which answers the above question with the parameter being the number of edges and the property being triangle-free. In this lecture, we will be looking at other interesting parameters and properties.

**Definition 1.** Let $G$ be a graph. A path $P$ (or cycle $C$) in $G$ is said to be simple if and only if all vertices of $P$ (or $C$) are distinct.

**Question:** What is the minimal number of edges in a graph to guarantee the existence of a cycle? In other words, what is maximal number of edges without a cycle?

Notice that a tree $T$ of order $n$ contains no cycle and it has $n - 1$ many edges. On the other hand, a graph $G$ of order $n$ and $e(G) \leq n$ must contain a cycle.

**Theorem 1.** A graph $G$ of order $n \geq 3$ contains a cycle if $e(G) \geq n$.

One key observation is that if the minimum degree of $G$ is at least 2, then it must contain a cycle.

**Definition 2.** Let $G$ be a graph. Define $\delta(G)$ to be the minimum degree of all vertices in $G$:

$$\delta(G) := \min \{ d(v) : v \in V(G) \}.$$  

Suppose $\delta(G) \geq 2$. We may start from an arbitrary vertex, and go to one of its neighbours. Since $\delta(G) \geq 2$ and $G$ is finite, we can always continue this process, until we come back to a vertex that has been visited. This forms a cycle.

With the observation in hand, we can show Theorem 1 by induction. The base case is trivial. For the induction step, if $G$ has no cycle, then it must have a vertex $v$ of degree 1. Consider $G \setminus v$, which has $n - 1$ vertices and at least $n - 1$ edges. Hence by induction hypothesis $G \setminus v$ contains a cycle.

Next let us turn our attention to cycles that visit every vertex. Contradiction.

**Definition 3.** Let $G$ be a graph of order $n$. A Hamiltonian cycle is a simple cycle of order $n$. Also, $G$ is said to be Hamiltonian if it has a Hamiltonian cycle.
In other words, a Hamiltonian cycle visits every vertex exactly once.

**Question:** What is the minimal number of edges to guarantee the existence of a Hamiltonian cycle? In other words, what is the maximal number of edges without a Hamiltonian cycle?

However, this question is not very interesting, as the answer is close to the maximum possible number of edges, \(\binom{n}{2}\). Consider the following family of graphs \(G_n\). Take a \(K_{n-1}\) together with an isolated vertex \(v\). Add one edge between \(K_{n-1}\) and \(v\). There is no Hamiltonian cycle since \(d(v) = 1\) and no cycle can go through \(v\). On the other hand,

\[
e(G_n) = \binom{n-1}{2} + 1 = \binom{n}{2} - (n-2).
\]

Thus, the edge “density” of this family of graphs is

\[
\frac{e(G_n)}{\binom{n}{2}} = 1 - \frac{2(n-2)}{n(n-1)} \to 1
\]

as \(n \to \infty\). This means that even if the graph contains almost all the possible edges, it could still be non-Hamiltonian. In contrast, by Mantel’s Theorem, a triangle-free graph \(G\) has density at most

\[
\frac{|n^2/4|}{\binom{n}{2}} \to \frac{1}{2} \text{ as } n \to \infty.
\]

Note that the degrees of the example we constructed above are distributed very unevenly. There are \(n-1\) vertices with degree at least \(n-2\) and 1 vertex with degree 1. A more interesting question is that can we guarantee the existence of Hamiltonian cycles by lower bounding the minimum degree of the graph.

**Theorem 2** (Dirac 1952). Let \(n \geq 3\). If \(G\) is a graph of order \(n\) and \(\delta(G) \geq n/2\), then \(G\) is Hamiltonian.

Theorem 2 is actually the best possible. Consider the graph \(G\), which is putting together two copies of \(K_{n/2}\). Since \(G\) is disconnected, it is not Hamiltonian. Moreover, \(\delta(G) = n/2 - 1\). Thus Theorem 2 is the best possible for even \(n\). The case of odd \(n\) will be left as an exercise.

**Proof of Theorem 2.** First we claim that \(G\) is connected. Suppose otherwise. Then pick one of the smallest components of \(G\). It must contain at most \(n/2\) many vertices. Hence any vertex in this component has degree at most \(n/2 - 1\). Contradiction.

Now suppose \(G\) is not Hamiltonian. Consider the simple path \(P\) of maximum possible length \(\ell \leq n-1\). That is, \(P = \{x_0, x_1, \ldots, x_\ell\}\) where \(x_ix_{i+1} \in E\) for all \(0 \leq i \leq \ell -1\). Since \(P\) is a maximal path, the neighbors of \(x_0\) and \(x_\ell\) must be all inside \(P\). Let

\[
A = \Gamma(x_1), \quad B = \{x_{i+1} : x_i \in \Gamma(x_\ell)\}.
\]

Since \(\delta(G) \geq n/2\), \(|A|, |B| \geq n/2\). On the other hand, it is easy to see that \(x_0 \notin A\) and \(x_0 \notin B\). Hence \(A \cup B \subseteq \{x_1, \ldots, x_\ell\}\). Thus \(|A \cup B| \leq \ell \leq n-1\). It implies that \(A \cap B \neq \emptyset\) (as otherwise \(|A \cup B| \geq n/2 + n/2 = n\)).
Suppose $x_t \in A \cap B$ for some $t$. Then consider the following cycle $C$
\[ x_1 - x_t - x_{t+1} - x_{t+2} - \cdots - x_{\ell-1} - x_{\ell} - x_{\ell-1} - x_{\ell-2} - \cdots - x_2 - x_1. \]

A picture can be found in Figure 1. Clearly, $C$ has length $\ell$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The path $P$ and cycle $C$}
\end{figure}

If $\ell = n$, then $C$ is a Hamiltonian cycle. Contradiction.

If $\ell < n$, we then construct a simple path $P'$ of length $\geq \ell + 1$. As $\ell < n$, there exists at least one vertex $v \notin C$. However, $G$ is connected. Hence there exists a simple path from $v$ to some vertex $x_r$ in $C$. Construct the path $P'$ as follows: start from $v$, to $x_r$, and then traverse $C$ to $x_{r-1}$. The length of $P'$ is at least $\ell + 1$. It contradicts to the maximality of $P$.}

\section{Forbidding a path of length $k$}

The way we prove Dirac’s theorem is useful to answer the following question.

\textbf{Question:} What is the maximal number of edges in a graph of order $n$ without a simple path of length $k$?

Let try to guess the answer first. An easy way to avoid paths of length $k$ is when every component has size at most $k$. Then to maximize the number of edges, we put all possible edges in each component. Thus our construction $G$ is $n/k$ many copies of cliques $K_k$ (assuming $k \mid n$). In this case,

\[ e(G) = \frac{n}{k} \binom{k}{2} = \frac{n}{k} \cdot \frac{k(k-1)}{2} = \frac{(k-1)n}{2}. \]

We will show that this is indeed the best possible.

\begin{theorem}
Let $G$ be a graph of order $n$ and there is no path of length $k$ in $n$. Then

\[ e(G) \leq \frac{(k-1)n}{2}. \]
\end{theorem}

The proof of Theorem 3 relies on the following lemma, which is a similar result to Dirac’s theorem.
Lemma 4. Let $G$ be a connected graph of order $n$ and $\delta(G) \geq k/2$ for some integer $k < n$. Then $G$ contains a simple path of length $k$.

Proof. Suppose that $G$ contains no path of length $k$. Let $P = \{x_0, x_1, \cdots, x_\ell\}$ be a path of maximum length $\ell < k$.

Since $P$ is a maximal path, the neighbours of $x_0$ and $x_\ell$ must be all inside $P$. Let $A = \Gamma(x_1)$, $B = \{x_{i+1} : x_i \in \Gamma(x_\ell)\}$.

Since $\delta(G) \geq k/2$, $|A|, |B| \geq k/2$. On the other hand, it is easy to see that $x_0 \notin A$ and $x_0 \notin B$. Hence $A \cup B \subseteq \{x_1, \cdots, x_\ell\}$. Thus $|A \cup B| \leq \ell < k$. It implies that $|A| \cap |B| \neq \emptyset$ (as otherwise $|A \cup B| \geq k/2 + k/2 = k$).

Suppose $x_t \in A \cap B$ for some $t$. Then consider the following cycle $C$

$$x_1 - x_t - x_{t+1} - x_{t+2} - \cdots - x_{\ell-1} - x_\ell - x_{t-1} - x_{t-2} - \cdots - x_2 - x_1.$$ (Recall Figure 1.) Clearly, $C$ has length $\ell$.

Since $\ell < k < n$, we then construct a simple path $P'$ of length $\geq \ell + 1$. As $\ell < n$, there exists at least one vertex $v \notin C$. However, $G$ is connected. Hence there exists a simple path from $v$ to some vertex $x_r$ in $C$. Construct the path $P'$ as follows: start from $v$, to $x_r$, and then traverse $C$ to $x_{r-1}$. The length of $P'$ is at least $\ell + 1$. It contradicts to the maximality of $P$. \hfill $\square$

With Lemma 4 in hand, we are now ready to prove Theorem 2.

Proof of Theorem 2. If $k = 1$, then there is no possible edge in $G$ and $e(G) = 0$.

Otherwise $k \geq 2$, we do an induction on $n$ (for each fixed integer $k \geq 2$). The base case is when $n \leq k$ and is trivial. This is because

$$e(G) \leq \frac{n}{2} \leq \frac{(k-1)n}{2}.$$ 

For the induction step, we want to show the theorem for a graph $G$ of order $n > k$ assuming it holds for any graph of order $< n$. If $G$ is disconnected, then let $G_0$ be a component of order $n_0 > 0$ and $G_1$ be the rest of the graph. Clearly

$$e(G) \leq e(G_0) + e(G_1).$$ 

Moreover, $G_0$ is of order $n_0 < n$ and $G_1$ has $n - n_0 < n$ many vertices. By induction hypothesis,

$$e(G_0) \leq \frac{k-1}{2} \cdot n_0,$$

$$e(G_1) \leq \frac{k-1}{2} \cdot (n - n_0).$$
Combine all of the above:

\[ e(G) \leq \frac{k - 1}{2} (n_0 + n - n_0) = \frac{k - 1}{2} n. \]

Otherwise, \( G \) is connected. If \( \delta(G) \geq k/2 \), then by Lemma 4 there exists a path of length \( k \). Contradiction

Therefore \( \delta(G) < k/2 \). It implies that there exists a vertex \( v \in V(G) \) such that

\[ d(v) \leq \lceil k/2 \rceil - 1 \leq \frac{k - 1}{2}. \]

Now consider the graph \( G' = G \setminus v \). \( G' \) has \( n - 1 \) many vertices and hence we can apply the induction hypothesis:

\[ e(G') \leq \frac{k - 1}{2} (n - 1). \]

Thus,

\[ e(G) = e(G') + d(v) \leq \frac{k - 1}{2} (n - 1) + \frac{k - 1}{2} = \frac{k - 1}{2} n. \]

We note that the edge density of graphs without a path of length \( k \) is at most

\[ \frac{(k-1)^n}{\binom{n}{2}} = \frac{k - 1}{n - 1} \to 0 \quad \text{as } n \to \infty. \]

2 Turán numbers and Turán densities

Let us fit the examples we have seen so far into a general theory.

**Definition 4.** Let \( F \) be an unlabelled graph. We say that a graph \( G \) is \( F \)-free if \( G \) does not contain any isomorphic copy of \( F \) as a subgraph.

Notice that here we do mean subgraph rather than induced subgraph. For example, \( K_5 \) is not \( C_4 \)-free because it contains a lot of cycles of length 4. However, the induced graph of \( K_5 \) on any 4 vertices is a \( K_4 \neq C_4 \).

**Definition 5.** Let \( F \) be an unlabelled graph, and let \( n \geq 2 \) be an integer. Define the Turán number of \( F \) to be

\[ \text{ex}(n, F) := \max\{ e(G) : G \text{ is an } F\text{-free graph of order } n \}. \]

Determining \( \text{ex}(n, F) \) is one of the basic problems of extremal graph theory. Mantel’s theorem tells us that \( \text{ex}(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor \), and Theorem 3 shows that \( \text{ex}(n, P_k) \leq \frac{(k-1)n}{2} \).

We also look at the “edge” density of \( F \)-free graphs. In particular, it is natural to consider the following limit:

\[ \lim_{n \to \infty} \frac{\text{ex}(n, F)}{\binom{n}{2}}. \]

Let us first show that the limit above does exist for any graph \( F \).
Lemma 5. Let $F$ be a graph. Then for any integer $n \geq 3$,

$$\frac{ex(n, F)}{\binom{n}{2}} \leq \frac{ex(n - 1, F)}{\binom{n-1}{2}}.$$ 

Proof. Let $G$ be an $F$-free graph of order $n$ such that $e(G) = ex(n, F)$. Let $v_0 \in V(G)$ of the minimum degree, i.e. $d(v_0) = \delta(G)$. Thus by the handshaking lemma,

$$2e(G) = \sum_{v \in V} d(v) \geq nd(v_0).$$

Let $G' = G - v$. Thus $G'$ is an $F$-free graph of order $n - 1$. By Definition 5,

$$e(G') \leq ex(n - 1, F).$$

On the other hand,

$$e(G) = e(G') + d(v).$$

Hence

$$e(G) \leq ex(n - 1, F) + \frac{2e(G)}{n}.$$

It implies that

$$ex(n, F) = e(G) \leq \frac{n}{n - 2} ex(n - 1, F).$$

Rearranging the terms yields

$$\frac{ex(n, F)}{ex(n - 1, F)} \leq \frac{n}{n - 2} = \frac{\binom{n}{2}}{\binom{n-1}{2}},$$

or equivalently,

$$\frac{ex(n, F)}{\binom{n}{2}} \leq \frac{ex(n - 1, F)}{\binom{n-1}{2}}.$$ 

Lemma 5 implies that the sequence

$$\left( \frac{ex(n, F)}{\binom{n}{2}} \right)_{n=2}^{\infty}$$

is monotone non-increasing. It is also a sequence of positive real numbers. Hence its limit exists. Define

$$\pi(F) := \lim_{n \to \infty} \frac{ex(n, F)}{\binom{n}{2}}. \quad (1)$$
This limit $\pi(F)$ is also called the *Turán density* of $F$.

As we have seen, Mantel's theorem implies that $\pi(K_3) = \pi(C_3) = \frac{1}{2}$. Moreover, Theorem 3 implies that

$$\pi(P_k) \leq \frac{\text{ex}(n, P_k)}{\binom{n}{2}} \leq \frac{(k - 1)n/2}{(n - 1)n/2} = \frac{k - 1}{n - 1} \to 0 \text{ as } n \to \infty.$$ 

It implies that $0 \leq \pi(P_k) \leq 0$, and thus $\pi(P_k) = 0$. Later, we will see the Erdős-Stone theorem, which gives us precise answer of $\pi(F)$ for any $F$. A consequence of the Erdős-Stone theorem is that $\pi(F) = 0$ if and only if $F$ is bipartite.